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Character $\mathcal{D}$-modules on a reductive group

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Though the problem we are going to discuss is ancient, our motivation is rather new. One of the most important problem of group representations is clearly the classification and the description of the irreducible characters of a group you happen to be interested in. Here let all of us assume to be interested in a real reductive group. You probably know much about the subject and you can rightfully assert those great masterpieces created by I. M. Gelfand, Harish-Chandra, T. Hirai, R. P. Langlands and ... are enough for its understanding. There may seem to be no rooms for new comers with new ideas.

But we also know a great monument by G. Lusztig which, however, is centered around a reductive group over a finite field. Since Lusztig himself has clearly been stimulated by those ancestors' work when he has got the idea of the classification [L1] and that of the so-called "character sheaves" [L2], it never seems for us to be waste of time to retrospect Lusztig's philosophy from our point of view, i.e., the Riemann-Hilbert correspondence between perverse sheaves and regular holonomic $\mathcal{D}$-modules on our group. The name "character $\mathcal{D}$-modules" comes simply from this correspondence. Here thus occurs an involutive base change principle $\mathbb{R} \mapsto \mathbb{F}_q \mapsto \mathbb{R}$! As a bonus, we shall get a new approach (actually only half a way) to Barbasch-Vogan's mysterious "unipotent representations" [BV].

This is a report of work in progress jointly with M. Kashiwara.

§1. A Gauss-Manin connection associated to the Grothendieck-Springer-Steinberg diagram

1.1. *The Grothendieck-Springer-Steinberg diagram.*

Let G be a connected complex reductive algebraic group and $B \supset T$ a Borel subgroup and a maximal torus, fixed once and for all. We then have a notable simultaneous resolution diagram

$$T \xleftarrow{\ \theta\ } \tilde{G} \xrightarrow{\ p\ } G \ .$$

Here the notations are as follows.

$$\tilde{G} := \{(x, gB) \in G \times X \mid x \in gBg^{-1}\} \qquad (X := G/B),$$

$$p := pr_G|\tilde{G} \quad \text{(proper)}.$$

We have an isomorphism

$$G \times^B B \xrightarrow{\sim} \tilde{G}, \qquad ((g, b) \longmapsto (gbg^{-1}, gB)),$$

where $G \times^B B$ is the fiber bundle associated to the B-principal bundle $G \longrightarrow X$ with the inner B-action on B. The projection

$$G \times B \longrightarrow B \longrightarrow T = B/U$$

(U ≔ unipotent radical of B) gives rise to the smooth map

$$\theta : \tilde{G} \simeq G \times^B B \longrightarrow T .$$

1.2. *A Gauss-Manin connection* N_λ.

Let $\mathfrak{t} := \mathrm{Lie}\, T$ be the Lie algebra of T and $\mathfrak{t}^*$ the linear dual of $\mathfrak{t}$. An element $\lambda \in \mathfrak{t}^*$ defines the multi-valued holomorphic function t^λ on T^{an} (≔ underlying analytic manifold of T) and we have the rank-one integrable connection

$$\mathcal{O}_\lambda := \mathcal{D}_T t^\lambda \simeq \mathcal{D}_T / \textstyle\sum_{h\in\mathfrak{t}} \mathcal{D}_T (L_h - \lambda(h))$$

where, in general, $\mathcal{D}_Y$ denotes the sheaf of algebraic linear differential operators on an algebraic manifold Y and $L_h \in \Gamma(T, \mathcal{D}_T)$ is the (left) invariant differential operator defined by $h \in \mathfrak{t}$.

We consider a complex of $\mathcal{D}_G$-modules

$$\int_p \theta^* \mathcal{O}_\lambda \quad \text{on } G.$$

Theorem 1.2.1. $\mathcal{H}^i \int_p \theta^* \mathcal{O}_\lambda = 0 \quad (i \neq 0)$ *and*

$$N_\lambda = \mathcal{H}^0 \int_p \theta^* \mathcal{O}_\lambda$$

is a regular holonomic $\mathcal{D}_G$*-modules in the algebraic sense.*

1.3. *The integral Weyl group* $W(\lambda)$ *acting on* N_λ.

We know the categorical equivalence by Kashiwara-Mebkhout, the Riemann-Hilbert correspondence

$$\mathrm{Sol} : D^b_{rh}(\mathcal{D}_Y) \xrightarrow{\sim} D^b_c(\mathbb{C}_Y)^o$$

defined by $\mathrm{Sol}\, M := R\mathcal{H}om_{\mathcal{D}_Y}(M, \mathcal{O}_{Y^{an}})$. Here $D^b_{rh}(\mathcal{D}_Y)$ is the derived category of bounded complexes of $\mathcal{D}_Y$-modules whose cohomologies are regular holonomic in the algebraic sense (Y is an algebraic manifold) and $D^b_c(\mathbb{C}_Y)$ is that of bounded complexes of $\mathbb{C}_{Y^{an}}$-modules whose cohomologies are algebraically constructible (Y^{an} is the underlying complex manifold of Y).

We first consider the special case (Deligne's theorem). We have an equivalence

$$\begin{aligned} &\{\text{regular connections of rank one on } T\} \\ &\simeq \{\text{local systems of rank one on } T^{an}\} \\ &\simeq \pi_1(T)^\vee := \mathrm{Hom}(\pi_1(T), \mathbb{C}^\times), \qquad (\pi_1(T) := \pi_1(T^{an})). \end{aligned}$$

This correspondence is realized as follows. Since $\mathrm{Sol}\, \mathcal{O}_\lambda \simeq \mathbb{C}t^\lambda$, the monodromy representation of $\mathbb{C}t^\lambda$ is given by the one dimensional representation $1^\lambda \in \pi_1(T)^\vee$ defined by

$$1^\lambda(\gamma) := e^{2\pi i\langle\lambda,\gamma\rangle} \qquad (\gamma \in \pi_1(T)).$$

Here we take the identification

$$\pi_1(T) \simeq \mathrm{Hom}_{alg\ gp}(\mathbb{C}^\times, T) \hookrightarrow \mathfrak{t}$$

and the natural pairing

$$\langle\ ,\ \rangle : \mathfrak{t}^* \times \mathfrak{t} \longrightarrow \mathbb{C}.$$

We then have a short exact sequence

$$0 \longrightarrow P \longrightarrow \mathfrak{t}^* \longrightarrow \pi_1(T)^\vee \longrightarrow 0 \qquad (\lambda \longmapsto 1^\lambda)$$

where $P := \{\lambda \in \mathfrak{t}^* \mid \langle\lambda, \gamma\rangle \in \mathbb{Z} \quad (\gamma \in \pi_1(T))\}$.

Let $W := N_G(T)/T$ be the Weyl group which acts on T, $\mathfrak{t}$, $\mathfrak{t}^*$, P, $\pi_1(T)^\vee$ etc., $\cdots$. Set

$$W(\lambda) := \{w \in W \mid w1^\lambda = 1^\lambda\} = \{w \in W \mid \lambda - w\lambda \in P\},$$

the integral Weyl subgroup. Let

$$\tilde{W} := W \ltimes \pi_1(T) \supset \tilde{W}(\lambda) := W(\lambda) \ltimes \pi_1(T)$$

be the corresponding modified affine Weyl groups (semidirect product). As was seen in [Ho], we have the following:

Lemma 1.3.1. *Let* G_{rs} *be the open subset of* G *consisting of regular semisimple elements and set* $T_{rs} := T \cap G_{rs}$, $\tilde{G}_{rs} := p^{-1}G_{rs}$.

i) $\tilde{G}_{rs} \simeq G/T \times T_{rs}$ *and hence* $\pi_1(\tilde{G}_{rs}) \simeq \pi_1(T_{rs}) \twoheadrightarrow \pi_1(T)$.

ii) *Let* K *be the kernel of* $\pi_1(\tilde{G}_{rs}) \twoheadrightarrow \pi_1(T)$. *Then by the short exact sequence*

$$1 \longrightarrow \pi_1(\tilde{G}_{rs}) \longrightarrow \pi_1(G_{rs}) \longrightarrow W \longrightarrow 1,$$

we have

$$1 \longrightarrow K \longrightarrow \pi_1(G_{rs}) \longrightarrow \tilde{W} \longrightarrow 1.$$

iii) *Consider the local system* $\mathbb{C}_\lambda := \mathrm{Sol}\ \mathcal{O}_\lambda \simeq \mathbb{C}t^\lambda$ *on* T. *Then the monodromy representation of the local system*

$$p_*\theta^{-1}\mathbb{C}_\lambda|G_{rs} \qquad \textit{on} \quad G_{rs}$$

is given by the induced representation

$$\mathrm{Ind}_{\pi_1(T)}^{\tilde{W}} 1^\lambda \simeq \mathrm{Ind}_{\tilde{W}(\lambda)}^{\tilde{W}} (\mathbb{C}[W(\lambda)] \otimes 1^\lambda)$$

$$\simeq \bigoplus_{\chi \in W(\lambda)^{\vee}} V_{\chi}^{*} \otimes \mathrm{Ind}_{\widetilde{W}(\lambda)}^{\widetilde{W}} (V_{\chi} \otimes 1^{\lambda})$$

through the factorization $\pi_1(G_{rs}) \twoheadrightarrow \widetilde{W}$. *Here* $\mathbb{C}[W(\lambda)]$ *is the group algebra,* V_{χ} *is an irreducible* $W(\lambda)$*-module for a class* $\chi \in W(\lambda)^{\vee}$ *and* V_{χ} *is its dual. Since* $W(\lambda)$ *fixes* $1^{\lambda} \in \pi_1(T)^{\vee}$, $V_{\chi} \otimes 1^{\lambda}$ *defines a representation of* $\widetilde{W}(\lambda) = W \ltimes \pi_1(T)$. *Note that* $\mathrm{Ind}_{\widetilde{W}(\lambda)}^{\widetilde{W}} (V_{\chi} \otimes 1^{\lambda})$ *is irreducible by Mackey's criterion.*

iv) *The induced space* $\mathrm{Ind}_{\pi_1(T)}^{\widetilde{W}} 1^{\lambda}$ *has* $(W(\lambda), \widetilde{W})$*-action in such a way that* $W(\lambda)$ *acts on* V_{χ}^{*} *parts in the irreducible decomposition in* iii). *Hence the local system* $p_{*}\theta^{-1}\mathbb{C}_{\lambda}|G_{rs}$ *has the* $W(\lambda)$*-action and the simple decomposition*

$$p_{*}\theta^{-1}\mathbb{C}_{\lambda}|G_{rs} \simeq \bigoplus_{\chi \in W(\lambda)^{\vee}} V_{\chi}^{*} \otimes F(\chi, \lambda)$$

where $F(\chi, \lambda)$ *is the local system whose monodromy representation is* $\mathrm{Ind}_{\widetilde{W}(\lambda)}^{\widetilde{W}} (V_{\chi} \otimes 1^{\lambda})$.

v) *Let* IC *be the intersection cohomology functor. Then*

$$Rp_{*}\theta^{-1}\mathbb{C}_{\lambda}[n] \simeq IC(p_{*}\theta^{-1}\mathbb{C}_{\lambda}|G_{rs})$$

$$\simeq \bigoplus_{\chi \in W(\lambda)^{\vee}} V_{\chi}^{*} \otimes IC(F(\chi, \lambda))$$

where $n = \dim G$. *In particular, the perverse sheaf* $Rp_{*}\theta^{-1}\mathbb{C}_{\lambda}$ *acquires the* $W(\lambda)$*-action.*

By the Riemann-Hilbert correspondence, we have

$$\mathrm{Sol}\ N_{\lambda} \simeq Rp_{*}\theta^{-1}\mathbb{C}_{\lambda}$$

and by Lemma 1.3.1,v), the $W(\lambda)$-action on the $\mathcal{D}_G$-module N_{λ}.

For the later convenience, we *change* the $W(\lambda)$-action on N_λ by multiplying the signature representation sgn of $W(\lambda)$. Thus we have the following.

Theorem 1.3.2. *The* $(W(\lambda), \mathcal{D}_G)$*-module* N_λ *decomposes as*

$$N_\lambda \simeq \bigoplus_{\chi \in W(\lambda)^\vee} V_\chi \otimes N(\chi, \lambda)$$

where $N(\chi, \lambda)$ *is a regular holonomic simple* $\mathcal{D}_G$*-module such that*

$$\mathrm{Sol}\ N(\chi, \lambda)[n] \simeq IC(F(\chi \otimes \mathrm{sgn}, \lambda)).$$

§2. Character $\mathcal{D}$-modules

2.1. *The Harish-Chandra equations.*

Let $\mathfrak{g} = \mathrm{Lie}\ G$ be the Lie algebra of G, $U(\mathfrak{g})$ the universal enveloping algebra and Z the center of $U(\mathfrak{g})$. The Harish-Chandra isomorphism given by the fixed pair $B \supset T$ is an algebra isomorphism

$$Z \xrightarrow{\sim} U(\mathfrak{t})^{\cdot W} \simeq \mathbb{C}[\mathfrak{t}^*]^{\cdot W}$$

where the superscript $.W$ means the invariants by the dot action defined by

$$w.\lambda := w(\lambda + \rho) - \rho \qquad (w \in W,\ \ \lambda \in \mathfrak{t}^*)$$

(ρ is the half sum of the positive roots). Hence it gives rise to the map

$$\begin{array}{ccc} t^* & \longrightarrow & t^*/.W \simeq Z^\vee := \mathrm{Hom}_{\mathbb{C}\text{-alg}}(Z, \mathbb{C}) \\ \Psi & & \Psi \\ \lambda & \longmapsto & \chi_\lambda \end{array}$$

such that $\chi_\lambda = \chi_{\lambda'} \iff w.\lambda = \lambda'$ for some $w \in W$.

For a fixed $\lambda \in t^*$, we consider the $\mathcal{D}_G$-module

$$M_\lambda := \mathcal{D}_G/\mathcal{I}_\lambda = \mathcal{D}_G u_\lambda$$

where

$$\mathcal{I}_\lambda := \Sigma_{z\in Z}\, \mathcal{D}_G(L_z - \chi_\lambda(z)) + \Sigma_{a\in\mathfrak{g}}\, \mathcal{D}_G(L_a + R_a),$$

$$u_\lambda := 1 \mod \mathcal{I}_\lambda.$$

Here L_z (resp. R_z) denotes the left (resp. right) invariant differential operator on G corresponding to $z \in U(\mathfrak{g})$.

This $\mathcal{D}_G$-module M_λ is known to be the defining equations of invariant eigendistributions. However, as was first noticed by Hirai [Hi], the irreducible characters with infinitesimal character χ_λ generally do not span the space of the invariant eigendistributions. Our first motivation is to construct the $\mathcal{D}_G$-module whose distribution solutions are spanned by the irreducible characters. For this, we want to connect M_λ with N_λ.

2.2. *The $\mathcal{D}_G$-module $\tilde{M}_\lambda$ defining characters.*

We need the following well-known lemma.

Lemma 2.2.1. *Let* $T \xleftarrow{\theta} \tilde{G} \xrightarrow{p} G$ *be the Grothendieck-Springer-Steinberg diagram and* ω_G *a non-zero G-invariant highest form (unique up to scalar multiplications). There then* *exists a unique nowhere vanishing* G-*invariant highest form* $\omega_{\tilde{G}}$ *such that*

$$p^*\omega_G = (D\circ\theta)\omega_{\tilde{G}}$$

where D *is the function on* T *given by*

$$D(t) := \Pi_{\alpha>0}(1 - t^{-\alpha}) = t^{-\rho}\Delta(t),$$

$$\Delta(t) := \Pi_{\alpha>0}(t^{\alpha/2} - t^{-\alpha/2})$$

(*the products run through the positive roots* $\alpha > 0$).

Using these forms ω_G, $\omega_{\tilde{G}}$, we can define a global section $v_\lambda \in \Gamma(G, N_\lambda)$ by

$$v_\lambda := \omega_{\tilde{G}} \otimes 1 \otimes p^*\omega_G^{-1} \otimes t^\lambda \in \Gamma(G, N_\lambda) = \Gamma(\tilde{G}, \mathcal{D}_{G\leftarrow\tilde{G}} \otimes_{\mathcal{D}_{\tilde{G}}} \theta^*\mathcal{O}_\lambda).$$

(Note that by definition

$$\mathcal{D}_{G\leftarrow\tilde{G}} := \Omega_{\tilde{G}} \otimes_{\mathcal{O}_{\tilde{G}}} p^*(\mathcal{D}_G \otimes_{\mathcal{O}_G} \Omega_G^{-1})$$

where $\Omega_{\tilde{G}}$ (resp. Ω_G) is the canonical line bundle of $\tilde{G}$ (resp. G), which is a right $\mathcal{D}$-module.)

By Lemma 2.2.1 combined with Harish-Chandra's formula for $M_\lambda|G_{rs}$ and by the fact that N_λ is a minimal extension (Theorem 1.3.2), it is easily seen that

$$\mathrm{Ann}\ u_\lambda \subset \mathrm{Ann}\ v_\lambda$$

where Ann denotes the annihilator in $\mathcal{D}_G$. Hence we can define a $\mathcal{D}_G$-module homomorphism

$$\varphi : M_\lambda = \mathcal{D}_G u_\lambda \longrightarrow N_\lambda$$

by $\varphi(Pu_\lambda) := Pv_\lambda$ $(P \in \mathcal{D}_G)$.

Definition 2.2.2.

$$\tilde{M}_\lambda := \operatorname{Im} \varphi = \mathcal{D}_G v_\lambda \subset N_\lambda .$$

Theorem 2.2.3. i) *$\tilde{M}_\lambda$ is regular holonomic.*

ii) *Let* $W_{\lambda+\rho} := \{w \in W \mid w(\lambda + \rho) = \lambda + \rho\} \subset W(\lambda)$ *and* $N_\lambda^{W_{\lambda+\rho}}$ *the $\mathcal{D}_G$-submodule of $W_{\lambda+\rho}$-invariants by the $W(\lambda)$-action in* 1.3. *Then*

$$\tilde{M}_\lambda = N_\lambda^{W_{\lambda+\rho}} \simeq \bigoplus_{\chi \in W(\lambda)^\vee} V_\chi^{W_{\lambda+\rho}} \otimes N(\chi, \lambda).$$

iii) *If* $G_{\mathbb{R}}$ *is a connected real form of* G, *then the space of distribution solutions to* $\tilde{M}_\lambda$,

$$\operatorname{Hom}_{\mathcal{D}_G}(\tilde{M}_\lambda, \mathcal{D}ist_{G_{\mathbb{R}}}),$$

is spanned by the irreducible characters of $G_{\mathbb{R}}$ *with infinitesimal character* χ_λ ($\mathcal{D}ist_{G_{\mathbb{R}}}$ *is the sheaf of Schwartz distributions on* $G_{\mathbb{R}}$).

We make a few comments for the proof. i) is clear since N_λ has the same property. ii) follows from the analysis of the $W(\lambda)$-action on N_λ and Theorem 1.3.2. iii) follows from Fomin-Shapovalov-Nishiyama's results [N1], i.e., the solution space turns out to be the space of "constant coefficient invariant eigendistributions".

Remark 2.2.4. By ii), $\tilde{M}_\lambda$ is acted by the Hecke algebra $H(W(\lambda), W_{\lambda+\rho})$ which gives rise to Nishiyama's action [N2] on the *character span* $\mathrm{Hom}_{\mathcal{D}_G}(\tilde{M}_\lambda, \mathcal{D}ist_{G_{\mathbb{R}}})$.

2.3. *Character $\mathcal{D}$-modules.*

Let $G_{\mathbb{R}}$ be a connected real form of G. Since $\tilde{M}_\lambda$ is regular holonomic,

$$\mathrm{Hom}_{\mathcal{D}_G}(\tilde{M}_\lambda, \mathcal{D}ist_{G_{\mathbb{R}}}) = \mathrm{Hom}_{\mathcal{D}_G}(\tilde{M}_\lambda, \mathcal{B}_{G_{\mathbb{R}}})$$

where $\mathcal{B}_{G_{\mathbb{R}}} = \mathcal{H}^n_{G_{\mathbb{R}}}(\mathcal{O}_{G^{an}}) \otimes or_{G_{\mathbb{R}}}$ is the sheaf of Sato hyperfunctions ($or_{G_{\mathbb{R}}}$ is the orientation sheaf of $G_{\mathbb{R}}$). Hence the character span decomposes as

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}_G}(\tilde{M}_\lambda, \mathcal{D}ist_{G_{\mathbb{R}}}) &= H^0(G_{\mathbb{R}}, R\mathcal{H}om_{\mathcal{D}_G}(\tilde{M}_\lambda, R\Gamma_{G_{\mathbb{R}}}(\mathcal{O}_{G^{an}})[n]) \\ &\simeq H^n(G_{\mathbb{R}}, R\Gamma_{G_{\mathbb{R}}}(\mathrm{Sol}\ \tilde{M}_\lambda)) \\ &\simeq \bigoplus_\chi (V_\chi^{W_{\lambda+\rho}})^* \otimes H^n(G_{\mathbb{R}}, R\Gamma_{G_{\mathbb{R}}}(\mathrm{Sol}\ N(\chi, \lambda)),\end{aligned}$$

where we fix an orientation of $G_{\mathbb{R}}$. Note that by 1.3,

$$\mathrm{Sol}\ N(\chi, \lambda) \simeq IC(F(\chi \otimes \mathrm{sgn}, \lambda))[-n].$$

Definition 2.3.1. For $\lambda \in t^*$, we call a simple $\mathcal{D}_G$-module $N(\chi, \lambda)$ satisfying

$$(\chi | W_{\lambda+\rho} : \mathbb{1}) \neq 0 \quad \text{and} \quad H^n(G_{\mathbb{R}}, R\Gamma_{G_{\mathbb{R}}}(\mathrm{Sol}\ N(\chi, \lambda))) \neq 0$$

a *character $\mathcal{D}_G$-module* with infinitesimal character χ_λ.

Thus character $\mathcal{D}_G$-modules are exactly those simple $\mathcal{D}_G$-modules contributing to irreducible characters of $G_{\mathbb{R}}$. In this aspect, this concept may be considered as a real analogue to Lusztig's "character sheaves" over a finite field [L2]. Solutions to such $N(\chi, \lambda)$ should then be called "almost-characters".

2.4. *Character cycles.*

As in 2.3, for the character span, we have

$$\mathrm{Hom}_{\mathcal{D}_G}(\tilde{M}_\lambda, \mathcal{B}_{G_{\mathbb{R}}}) \simeq H^0(G_{\mathbb{R}}, R\Gamma_{G_{\mathbb{R}}}(\mathrm{Sol}\ \tilde{M}_\lambda)[n])$$

$$\simeq H^0(G_{\mathbb{R}}, R\Gamma_{G_{\mathbb{R}}}((Rp_*\theta^{-1}\mathbb{C}_\lambda)_{W_{\lambda+\rho}})[n])$$

where the subscript $W_{\lambda+\rho}$ means the $W_{\lambda+\rho}$-covariants ($\simeq$ the $W_{\lambda+\rho}$-invariants in the present case). Set $\tilde{G}_{\mathbb{R}} := p^{-1}G_{\mathbb{R}}$. Then the above space is isomorphic to

$$H^0(\tilde{G}_{\mathbb{R}}, R\Gamma_{\tilde{G}_{\mathbb{R}}}(\theta^{-1}\mathbb{C}_\lambda)[n])_{W_{\lambda+\rho}}$$

$$\simeq H^0(\tilde{G}_{\mathbb{R}}, \theta^{-1}\mathbb{C}_\lambda \otimes \omega_{\tilde{G}_{\mathbb{R}}}[-n])_{W_{\lambda+\rho}}$$

($\omega_{\tilde{G}_{\mathbb{R}}}$ is the dualizing complex of $\tilde{G}_{\mathbb{R}}$)

$$\simeq H^{-n}(\tilde{G}_{\mathbb{R}}, \theta^{-1}\mathbb{C}_\lambda \otimes \omega_{\tilde{G}_{\mathbb{R}}})_{W_{\lambda+\rho}}.$$

But then this is by definition isomorphic to

$$H_n^{BM}(\tilde{G}_{\mathbb{R}}, \theta^{-1}\mathbb{C}_{-\lambda})_{W_{\lambda+\rho}}$$

$$\simeq (H_c^n(\tilde{G}_{\mathbb{R}}, \theta^{-1}\mathbb{C}_\lambda)^{W_{\lambda+\rho}})^*$$

where, in general, $H_n^{BM}(Y, F)$ denotes the n-th Borel-Moore homology group with twisted coefficients in a local system F.

The last isomorphism follows from the Verdier duality.

Definition 2.4.1. Let Θ be an irreducible character of $G_{\mathbb{R}}$. Through the above canonical isomorphism

$$\mathrm{Hom}_{\mathcal{D}_{\mathbb{G}}}(\tilde{M}_\lambda, \mathcal{D}ist_{G_{\mathbb{R}}}) \simeq H_n^{BM}(\tilde{G}_{\mathbb{R}}, \theta^{-1}\mathbb{C}_{-\lambda})_{W_{\lambda+\rho}},$$

the corresponding n-th cycle $c(\Theta) \in H_n^{BM}(\tilde{G}_{\mathbb{R}}, \theta^{-1}\mathbb{C}_{-\lambda})_{W_{\lambda+\rho}}$ is called the *character cycle* of Θ.

Describing the transition matrix between the character cycles and the geometric cycles seems to be an extremely interesting but very delicate problem (see [K]). For example, assume λ to be integral with $\lambda + \rho$ regular ($\Longrightarrow$ $\mathbb{C}_\lambda \simeq \mathbb{C}$ and $W_{\lambda+\rho} = \{e\}$). Then RHS $\simeq H_n^{BM}(\tilde{G}_{\mathbb{R}}, \mathbb{C})$ is spanned by the fundamental cycles of $\tilde{G}_{\mathbb{R}}$ and even in this case we do not know much about the transition matrix between this basis and the character cycle basis.

§3. Cell decomposition of $\tilde{M}_\lambda$

3.1. *Notation*.

In Theorem 2.2.3, we have decomposed the $\mathcal{D}_G$-module $\tilde{M}_\lambda$ according to its monodromy. But an irreducible character in general is not a solution to a simple factor of $\tilde{M}_\lambda$ and in order to elucidate this point, we shall consider more subtle decomposition according to cell representations of the integral Weyl group.

For this, we introduce the following notation. Fix $\lambda \in \mathfrak{t}^*$. For an $W(\lambda)$-module V, let $N(V, \lambda)$ denote the regular holonomic $\mathcal{D}_G$-module such that

$$\mathrm{Sol}\ N(V, \lambda)[n] = IC\ F(V \otimes \mathrm{sgn}\ , \lambda)$$

where $F(V \otimes \mathrm{sgn}, \lambda)$ is the local system on G_{rs} corresponding to the $\widetilde{W}$-module

$$\mathrm{Ind}_{\widetilde{W}(\lambda)}^{\widetilde{W}} (V \otimes \mathrm{sgn}) \otimes 1^{\lambda}\ .$$

Thus, for example,

$$N(\chi, \lambda) = N(V_{\chi}, \lambda),$$

$$N_{\lambda} \simeq N(\mathbb{C}[W(\lambda)], \lambda),$$

$$\widetilde{M}_{\lambda} = N_{\lambda}^{W_{\lambda+\rho}} \simeq N(\mathrm{Ind}_{W_{\lambda+\rho}}^{W(\lambda)} \mathbb{1}, \lambda) \simeq N(\mathbb{C}[W(\lambda)]_{W_{\lambda+\rho}}, \lambda).$$

In the above, we consider $\mathbb{C}[W(\lambda)]$ as the left regular representation and denote by $\mathbb{C}[W(\lambda)]_{W_{\lambda+\rho}}$ the space of covarints by the right $W_{\lambda+\rho}$-action.

3.2. *Cells.*

For the integral Weyl group $W(\lambda)$, let $\overline{V}^{L}(w)$ (resp. $\overline{V}^{R}(w)$, $\overline{V}^{LR}(w)) \subset \mathbb{C}[W(\lambda)]$ be the left (resp. right, double) cone ideal for $w \in W(\lambda)$. The Kazhdan-Lusztig preorders $\leq$ then mean

$$w \underset{L}{\leq} y \iff \overline{V}^{L}(w) \subset \overline{V}^{L}(y),$$

$$w \underset{R}{\leq} y \iff \overline{V}^R(w) \subset \overline{V}^R(y),$$
$$w \underset{LR}{\leq} y \iff \overline{V}^{LR}(w) \subset \overline{V}^{LR}(y).$$

The equivalence relations generated by these preorders are denoted respectively by $\underset{L}{\sim}$, $\underset{R}{\sim}$, $\underset{LR}{\sim}$. Thus

$$w \underset{L}{\sim} y \iff \overline{V}^L(w) = \overline{V}^L(y),$$
$$w \underset{R}{\sim} y \iff \overline{V}^R(w) = \overline{V}^R(y),$$
$$w \underset{LR}{\sim} y \iff \overline{V}^{LR}(w) = \overline{V}^{LR}(y).$$

Note that the double cell equivalence $\underset{LR}{\sim}$ is the smallest equivalence relation containing $\underset{L}{\sim}$ and $\underset{R}{\sim}$.

We denote $w < y$ when $w \leq y$ but $w \sim y$ fails. The left cell representation corresponding to $w \in W(\lambda)$ is then defined by

$$V^L(w) := \overline{V}^L(w)/\textstyle\sum_{y \underset{L}{<} w} \overline{V}^L(y) .$$

Hence if $w \underset{L}{\sim} y$, then $V^L(w) \simeq V^L(y)$. Similarly we define the double (resp. right) cell representation $V^{LR}(w)$ (resp. $V^R(w)$), which have similar properties.

If w_0 is the longest element of $W(\lambda)$, then there exists a natural isomorphism

$$V^L(w) \simeq V^L(w_0 w)^* \otimes \mathrm{sgn}.$$

If w_λ is the longest element of $W_{\lambda+\rho}$ $(\subset W(\lambda))$, then

$$\mathrm{Ind}_{W_{\lambda+\rho}}^{W(\lambda)} \mathbb{1} \simeq \mathrm{sgn} \otimes \overline{V}^L(w_\lambda) .$$

Now let

$$\overline{V}^L(w_\lambda) \simeq \bigoplus_{i=1}^k V^L(w_i)$$

($w_1 = w_\lambda$, $w_i \underset{L}{<} w_\lambda$ for $i > 1$) be the (non-canonical) decomposition of the left cone representation $\overline{V}^L(w_\lambda)$ into the left cell representations. Then by the above, we have

$$\mathrm{sgn} \otimes \overline{V}^L(w_\lambda) \simeq \bigoplus_{i=1}^k V^L(w_0 w_i)$$

(note that for a Weyl group module V, $V \simeq V^*$). We thus have the cell decomposition of the $\mathcal{D}_G$-module $\widetilde{M}_\lambda$ as follows.

Theorem 3.2.1. $\widetilde{M}_\lambda \simeq \bigoplus_{i=1}^k N(V^L(w_0 w_i), \lambda)$.

3.3. *Characteristic varieties of the decomposition factors.*

For simplicity, assume λ is integral and hence $W(\lambda) = W$. We consider the Springer correspondence

$$\mathcal{O} : W^{\vee} \longrightarrow N/G := \text{nilpotent orbits in } \mathfrak{g}^*$$

normalized so that $\mathcal{O}(\mathbb{1})$ = regular orbit and $\mathcal{O}(\mathrm{sgn}) = \{0\}$.

For an irreducible class $\chi \in W^{\vee}$, there exists a unique double cell representation $V^{LR}(w)$ which contains V_χ. We then set

$$\mathcal{C}^{LR}(\chi) := \{\chi' \in W^{\vee} \mid V_{\chi'} \subset V^{LR}(w)\}.$$

$\mathcal{C}^{LR}(\chi)$ contains a unique *special* class (the class of a *special* representation) and thus $W^{\vee}$ decomposes as

$$W^{\vee} = \coprod_{\chi:\text{special}} \mathcal{C}^{LR}(\chi)$$

(partition into the families [L1]).

For $\chi \in W^{\vee}$, when $V_{\chi} \subset V^{LR}(w)$, we similarly set

$$\overline{\mathscr{C}}^{LR}(\chi) := \{\chi' \in W^{\vee} \mid V_{\chi'} \subset \overline{V}^{LR}(w)\}.$$

For $\chi \in W^{\vee}$, let $\chi_0 \in \mathscr{C}^{LR}(\chi)$ be the unique special class. Then we set

$$\mathcal{O}_{sp}(\chi) := \mathcal{O}(\chi_0) \in N/G$$

which is by definition a special nilpotent orbit.

Lemma 3.3.1 ([KT]). *Assume* $\chi' \in \overline{\mathscr{C}}^{LR}(\chi)$. *Then* $\mathcal{O}(\chi') \subset \overline{\mathcal{O}_{sp}(\chi)}$ *where* $\overline{}$ *denotes the closure of a nilpotent orbit.*

For a coherent $\mathscr{D}_G$-module M, let $\mathrm{Ch}\, M \subset T^*G$ denote the characteristic variety of M. Set

$$\mathrm{Ch}_e\, M := \mathrm{Ch}\, M \cap T^*_eG \subset \mathfrak{g}^*$$

where $T^*_eG \simeq \mathfrak{g}^*$ is the cotangent space at the identity $e \in G$ ("wave front set" of M). We then have the following by [HK1].

Lemma 3.3.2. *Let* $N(\chi, \lambda)$ $(\chi \in W^{\vee})$ *be the simple* $\mathscr{D}_G$*-module defined in* 1.3. *Then*

$$\mathrm{Ch}_e\, N(\chi, \lambda) = \overline{\mathcal{O}(\chi)}.$$

Theorem 3.3.3. *In the decomposition of Theorem* 3.2.1,

$$\mathrm{Ch}_e\, N(V^L(w_0w_\lambda), \lambda) \subset \mathrm{Ch}_e\, N(V^L(w_0w_i), \lambda)$$

and each variety is the closure of a special orbit.

We briefly indicate the proof. For $w \in W$, let $\chi_w \in W^{\vee}$ be the unique special class such that $V_{\chi_w} \subset V^{LR}(w)$. Then by 3.3.1,

$$\mathrm{Ch}_e\ N(V^L(w),\ \lambda) = \overline{\mathcal{O}(\chi_w)}.$$

Since $w_\lambda \underset{L}{\geq} w_i$, $w_0 w_\lambda \underset{L}{\leq} w_0 w_i$, which implies the minimality of $\overline{\mathcal{O}(\chi_{w_0 w_\lambda})}$ by 3.3.1. Thus Lemma 3.3.2 implies the theorem.

Remark 3.3.4. Even if λ is not integral, the theorem will be true (except the statement on the speciality of orbits) by using Luszitg's induction $j_{W(\lambda)}^{W}$.

§4. Primitive ideals

4.1. *Review.*

For $\lambda \in t^*$, let $\mathrm{Prim}_\lambda U(\mathfrak{g})$ be the set of primitive ideals of $U(\mathfrak{g})$ with infinitesimal character $\chi_\lambda \in \check{Z}$. Assume $\lambda + \rho$ is anti-dominant and let $L(w)$ be the unique irreducible quotient of the Verma module $U(\mathfrak{g}) \otimes_{U(b)} \mathbb{C}_{w.\lambda}$ where $\mathbb{C}_{w.\lambda}$ is the one dimensional $U(b)$-module with weight $w.\lambda = w(\lambda + \rho) - \rho$ $(w \in W)$. Let $I_w := \mathrm{Ann}\ L(w) \in \mathrm{Prim}_\lambda U(\mathfrak{g})$. Then the map

$$\begin{array}{c} \{w \in W(\lambda) \mid w \text{ is minimal in a left coset in } W(\lambda)/W_{\lambda+\rho}\} \\ \Big\downarrow \quad (w \longmapsto I_w) \\ \mathrm{Prim}_\lambda U(\mathfrak{g}) \end{array}$$

is surjective (Duflo). In particular, if $\lambda + \rho$ is regular, in the surjection

$$W(\lambda) \twoheadrightarrow \mathrm{Prim}_\lambda U(\mathfrak{g})$$

it is notable that

$$I_w = I_y \iff w \underset{L}{\sim} y \quad \text{in} \quad W(\lambda).$$

4.2. *Primitive quotient of* $\tilde{M}_\lambda$.

In our paper [HK2], we have shown that if $\lambda + \rho$ is integral regular, then for the $\mathcal{D}_G$-module

$$M_w := M_\lambda / I_w M_\lambda \simeq N_\lambda / I_w N_\lambda ,$$

we have an isomorphism

$$M_w \simeq N(\overline{V}^L(w), \lambda).$$

In general, the situation is as follows:

$$\begin{array}{ccccccc} M_\lambda & \twoheadrightarrow & \tilde{M}_\lambda = N^{W_{\lambda+\rho}} & \hookrightarrow & N_\lambda \simeq N_{\lambda_0} & \simeq & M_{\lambda_0} \\ \downarrow & & \downarrow & & & & \downarrow \\ M_{\lambda,w} & \twoheadrightarrow & \tilde{M}_{\lambda,w} & & \longrightarrow & & M_{\lambda_0,w} \end{array}.$$

In the above, we first choose $\lambda_0 = \lambda + \mu \in t^*$ such that μ is integral and $\lambda_0 + \rho$ is anti-dominant regular. The isomorphism $N_\lambda \simeq N_{\lambda_0} \simeq M_{\lambda_0}$ follows from the definition and Theorem 2.2.3 ($\lambda_0 + \rho$ is regular). The row sequence

$$\tilde{M}_\lambda \hookrightarrow N_\lambda \simeq N_{\lambda_0} \simeq M_{\lambda_0}$$

reflects the "translation principle" of characters. The others are defined as follows. For w minimal in a coset in $W(\lambda)/W_{\lambda+\rho}$,

$$M_{\lambda,w} := M_\lambda / I_w M_\lambda \ ,$$
$$\tilde{M}_{\lambda,w} := \tilde{M}_\lambda / I_w \tilde{M}_\lambda \ ,$$
$$M_{\lambda_0,w} := M_{\lambda_0} / I_w^0 \ M_\lambda$$

where I_w^0 is the annihilator of $L(w)$ with infinitesimal character χ_{λ_0}. Thus the correspondence

$$\mathrm{Prim}_\lambda U(g) \ni I_w \longmapsto I_w^0 \in \mathrm{Prim}_{\lambda_0} U(g)$$

gives the translation principle of primitive ideals.

Theorem 4.2.1. i) *The diagram*

$$\begin{array}{ccc} \tilde{M}_\lambda & \hookrightarrow & M_{\lambda_0} \\ \downarrow & & \downarrow \\ \tilde{M}_{\lambda,w} & \hookrightarrow & M_{\lambda_0,w} \end{array}$$

is isomorphic to the diagram

$$\begin{array}{ccc} N(\mathbb{C}[W(\lambda)]_{W_{\lambda+\rho}}, \lambda) & \hookrightarrow & N(\mathbb{C}[W(\lambda)], \lambda_0) \\ \downarrow & & \downarrow \\ N(\overline{V}_\lambda^L(w), \lambda) & \hookrightarrow & N(\overline{V}^L(w), \lambda_0) \end{array}$$

where $\overline{V}_\lambda^L(w) \subset \mathbb{C}[W(\lambda)]_{W_{\lambda+\rho}}$ *is the image of* $\overline{V}^L(w)$ *by the projection* $\mathbb{C}[W(\lambda)] \twoheadrightarrow \mathbb{C}[W(\lambda)]_{W_{\lambda+\rho}}$.

In other words, the diagram corresponds to the $W(\lambda)$*-module diagram*

$$\begin{array}{ccc} \mathbb{C}[W(\lambda)]_{W_{\lambda+\rho}} & \longleftarrow & \mathbb{C}[W(\lambda)] \\ \uparrow & & \uparrow \\ \overline{V}^{L}_{\lambda}(w) & \longleftarrow & \overline{V}^{L}(w) \end{array}$$

by the functor $N(\ ,\lambda)$.

ii) $\quad Ch_e\, \widetilde{M}_{\lambda,w} = Ch_e\, M_{\lambda_0,w} = \mathrm{Ass}\ I^0_w = \mathrm{Ass}\ I_w$

where Ass *denotes the associated variety of a primitive ideal, which is the closure of a single nilpotent orbit.*

4.3. *Unipotent characters.*

We shall try to make an approach to Barbasch-Vogan's "unipotent representations" [BV] in our view-point.

Retain the situation and the notations as before. If I_λ is maximal in $\mathrm{Prim}_\lambda U(\mathfrak{g})$ ($\Longleftrightarrow$ $\mathrm{Ass}\ I_\lambda$ is minimal), then

$$I_\lambda = I_{w_0 w_\lambda} .$$

Thus by Theorem 4.2.1,

$$\widetilde{M}^{min}_\lambda := \widetilde{M}_\lambda / I_\lambda \widetilde{M}_\lambda \simeq \widetilde{M}_{\lambda, w_0 w_\lambda} \simeq N(\overline{V}^L_\lambda(w_0 w_\lambda), \lambda)$$

and

$$Ch_e\, \widetilde{M}^{min}_\lambda = \overline{\mathcal{O}(\chi_0)}$$

where χ_0 is the unique special class in $V^L(w_0 w_\lambda)$. But then by Theorems 3.2.1 and 3.3.3, for $i > 1$,

$$Ch_e\, N(V^L(w_0 w_\lambda), \lambda) \supsetneqq \overline{\mathcal{O}(\chi_0)} .$$

Actually we have

$$\overline{V}^L_\lambda(w_0 w_\lambda) = V^L(w_0 w_\lambda).$$

Theorem 4.3.1. $\tilde{M}_\lambda^{min} \simeq N(V^L(w_0w_\lambda), \lambda)$.

Barbasch and Vogan have found a distinguished property of the left cell representation $V^L(w_0w_\lambda)$ related to Lusztig's picture [L1] when λ is particularly chosen with respect to a special nilpotent orbit. We recall the situation of [BV]. Let ${}^L\mathfrak{g}$ be the Lie algebra dual to $\mathfrak{g}$ ($\mathfrak{t}^* = {}^L\mathfrak{t}$, {coroots of $(\mathfrak{g}, \mathfrak{t})$} = {roots of $({}^L\mathfrak{g}, {}^L\mathfrak{t})$}). There then exists an order-reversing bijection

$$\mathcal{O} \longmapsto {}^L\mathcal{O}$$

between the set of all special nilpotent orbits of $\mathfrak{g}$ and that of ${}^L\mathfrak{g}$ (the order is the inclusion relation among orbit-closures). In the notation of 3.3, if $\mathcal{O} = \mathcal{O}(\chi)$ with $\chi \in W^\vee$ special, then ${}^L\mathcal{O} = \mathcal{O}_{sp}(\chi \otimes sgn)$, i.e., ${}^L\mathcal{O} = \mathcal{O}(\chi')$ where $\chi' \in \mathfrak{g}^{LR}(\chi \otimes sgn)$ is the unique special class of ${}^LW \simeq W$.

Assume now that λ is integral and I_λ is maximal in $Prim_\lambda U(\mathfrak{g})$ as before. Set $\overline{\mathcal{O}} := Ass\ I_\lambda$ ($\mathcal{O} = \mathcal{O}(\chi_0)$). We make the following:

Assumption 4.3.2. ${}^L\mathcal{O}$ is even.

This assumption 4.3.2 is equivalent to the condition that

$$\lambda_{\mathcal{O}} := \frac{1}{2} h_{{}^L\mathcal{O}} \in {}^L\mathfrak{t} = \mathfrak{t}^* \text{ is integral,}$$

where $h_{{}^L\mathcal{O}} \in {}^L\mathfrak{t}$ is a semisimple element such that $(u, h_{{}^L\mathcal{O}}, v)$ is a corresponding $\mathfrak{sl}_2$-triplet ($u \in {}^L\mathcal{O}$). Then

$$|\lambda + \rho| \geq |\lambda_{\mathcal{O}}|$$

and the equality holds if and only if

$$\lambda_{\mathcal{O}} = w(\lambda + \rho) \quad \text{for some} \quad w \in W$$

([BV; 5.10]).

Under Assumption 4.3.2, the left cell representation $V^L(w_0 w_{\mathcal{O}})$ ($w_{\mathcal{O}}$ is the longest element of $W_{\lambda_{\mathcal{O}}}$) has a surprising feature. To see this, we need Lusztig's extraordinary picture for cells ([L1]).

For a special orbit $\mathcal{O}$, let $A(\mathcal{O})$ be the Lusztig group (the canonical quotient of the component group $Z_G(\mathcal{O})/Z_G(\mathcal{O})^0$). (In [BV], $A(\mathcal{O})$ is denoted by $\overline{A(\mathcal{O})}$.) For a simple $\mathfrak{g}$, $A(\mathcal{O})$ is one of the following finite groups:

$$S_2{}^{\text{power}}, \quad S_i \quad (1 \leq i \leq 5)$$

where S_i is the i-th symmetric group. Set

$$M(\mathcal{O}) := \{([x], \xi) \mid [x] \in \text{Cl } A(\mathcal{O}),\ \ \xi \in Z_{A(\mathcal{O})}(x)^\vee\}$$

where $\text{Cl } A(\mathcal{O})$ is the set of conjugacy classes of $A(\mathcal{O})$ and ξ belongs to the irreducible classes of the centralizer of x in $A(\mathcal{O})$. There are obvious injections:

$$\text{Cl } A(\mathcal{O}) \hookrightarrow M(\mathcal{O}), \qquad [x] \longmapsto ([x], \mathbb{1}),$$

$$A(\mathcal{O})^\vee \hookrightarrow M(\mathcal{O}), \qquad \xi \longmapsto ([e], \xi).$$

Let $\chi_{\mathcal{O}} \in W^\vee$ be the special class corresponding to $\mathcal{O}$. Lusztig [L1] has then defined the injection

$$\mathcal{g}^{LR}(\chi_{\mathcal{O}}) \hookrightarrow M(\mathcal{O})$$

(case by case!) with the following property. If $\mathcal{O}$ is not exceptional, then the image of the above injection contains the image of $\text{Cl } A(\mathcal{O})$. In this case, we thus have the injection

(4.3.3) $\quad \mathrm{Cl}\ A(\mathcal{O}) \hookrightarrow \mathcal{E}^{LR}(\chi_{\mathcal{O}}), \qquad [x] \longmapsto \chi_x$.

In this situation, we have the following.

Lemma 4.3.4 ([BV; 5.28]). *Let* $\mathcal{O}$ *be a special orbit such that* ${}^L\mathcal{O}$ *is even and* $w_{\mathcal{O}}$ *the longest element of* $W_{\lambda_{\mathcal{O}}}$. *Then the left cell representation* $V^L(w_0 w_{\mathcal{O}})$ *decomposes as*

$$V^L(w_0 w_{\mathcal{O}}) = \bigoplus_{[x]\in \mathrm{Cl}A(\mathcal{O})} V_{\chi_x}.$$

where $\chi_x \in W^{\vee}$ $(x \in A(\mathcal{O}))$ *is defined by* 4.3.3.

Applying this lemma to the $\mathcal{D}_G$-module $\tilde{M}^{\min}_{\lambda}$ $(\lambda = \lambda_{\mathcal{O}} - \rho)$, we have the following.

Theorem 4.3.5. *Let the assumption be as above. Then*

$$\tilde{M}^{\min}_{\lambda_{\mathcal{O}}-\rho} \simeq \bigoplus_{[x]\in \mathrm{Cl}A(\mathcal{O})} N(V_{\chi_x}).$$

Remark 4.3.6. Assume a real form $G_{\mathbb{R}} = G_0$ is a connected complex group. Then $G \simeq G_0 \times G_0$, $T \simeq T_0 \times T_0$, $\lambda = (\lambda_1, \lambda_2) \in \mathfrak{t}_0^* \times \mathfrak{t}_0^*$ etc.,$\cdots$, as usual. Applying the above discussion to a special nilpotent orbit $\mathcal{O} \times \mathcal{O}$ in $\mathfrak{g} = \mathfrak{g}_0 \times \mathfrak{g}_0$, we have the following decomposition for the solutions to $\tilde{M}^{\min}_{\lambda_{\mathcal{O}}-\rho}$.

$$\mathrm{Hom}_{\mathcal{D}_G}(\tilde{M}^{\min}_{\lambda_{\mathcal{O}}-\rho}, \mathcal{B}_{G_0}) \simeq \bigoplus_{[x]\in \mathrm{Cl}A(\mathcal{O})} \mathrm{Hom}_{\mathcal{D}_G}(N(V_{\chi_x} \boxtimes V_{\chi_x}), \mathcal{B}_{G_0}).$$

In RHS, the direct factor $\mathrm{Hom}_{\mathcal{D}_G}(N(V_{\chi_x} \boxtimes V_{\chi_x}), \mathcal{B}_{G_0})$ turns out to be one-dimensional, and Barbasch-Vogan's virtual character R_x

($[x] \in \mathrm{Cl}\ A(\mathcal{O})$) belongs to this space. The unipotent character χ_π corresponding to $\pi \in A(\mathcal{O})^\vee$ is given by

$$\chi_\pi = (\#A(\mathcal{O}))^{-1} \sum_{x \in A(\mathcal{O})} \pi(x)\ R_x.$$

Thus the R_x's are "almost-characters" in this case.

Generalization 4.3.7. Let ${}^L\mathcal{O}$ be a nilpotent orbit of ${}^L\mathfrak{g}$ (not necessarily special) and set $\lambda_{\mathcal{O}} := \frac{1}{2} h_{{}^L\mathcal{O}} \in {}^L\mathfrak{t} = \mathfrak{t}^*$ as before. Let $W(\lambda_{\mathcal{O}}) \subset W$ be the integral subgroup. Let

$${}^L\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} {}^L\mathfrak{g}(i)$$

be the Dynkin decomposition associated to an nilpotent element $u \in {}^L\mathcal{O}$. Then $W(\lambda_{\mathcal{O}})$ is the Weyl group of

$${}^L\mathfrak{m} := \bigoplus_{i \in \mathbb{Z}} {}^L\mathfrak{g}(2i)\ .$$

Assume $\mathcal{O}_{{}^L\mathfrak{m}} := \mathcal{O}_{{}^L\mathfrak{m}}(u) \subset {}^L\mathfrak{m} \cap {}^L\mathcal{O}$ is special in ${}^L\mathfrak{m}$. Setting $\lambda := \lambda_{\mathcal{O}} - \rho$, we then have

$$\tilde{M}_\lambda^{min} \simeq N(V^L(w_0 w_\lambda),\ \lambda),$$

$$V^L(w_0 w_\lambda) \simeq \bigoplus_{[x] \in \mathrm{Cl}A(\mathcal{O}_{\mathfrak{m}})} V_{\chi_x} \quad \text{as } W(\lambda_{\mathcal{O}})\text{-modules}$$

where $\mathfrak{m} \subset \mathfrak{g}$ is the dual Lie algebra of ${}^L\mathfrak{m} \subset {}^L\mathfrak{g}$ and $\mathcal{O}_{\mathfrak{m}} = {}^L\mathcal{O}_{{}^L\mathfrak{m}}$ (special in $\mathfrak{m}$).

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