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Σ_2 Collection and Maximal Sets

C. T. Chong

National University of Singapore

The subject of *reverse recursion theory* studies the following basic question (*): What axioms of Peano arithmetic are required, or sufficient, to prove theorems in recursion theory ? This question (perhaps first raised by Stephen Simpson) is a natural offshoot of a related, more general question: Which set existence axioms of second order arithmetic are required, or sufficient, to prove theorems in ordinary mathematics (Simpson [1985]) ? While it was only in recent years that investigations were carried out on (*), some of the answers obtained have nevertheless been very interesting—not only because they provide a better understanding of the fundamental constructions in recursion theory, but also because many of the techniques used to obtain the answers were inspired by those introduced in α recursion theory. Indeed in many cases the original techniques appear to fit snugly into the new situation, giving the impression of a technical development that is historically correct. Our purpose here is to study the question on the existence of maximal sets, prove a general nonexistence result of these sets for a wide class of models of $P^- + B\Sigma_2$, and to point out the connection of the proof techniques with those in α recursion theory.

Let P^- be the set of axioms of Peano arithmetic minus the induction scheme. These consist of universal closures of the following:

$$x' \neq 0$$
$$(x' = y') \rightarrow (x = y)$$
$$x \neq 0 \rightarrow 0' \leq x$$
$$x < y \leftrightarrow (\exists t)(x + t' = y)$$
$$x < y \vee x = y \vee x > y$$

$$x + y = y + x; \qquad x \cdot y = y \cdot x$$
$$x + (y + z) = (x + y) + z$$
$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$
$$x + 0 = x; \qquad x \cdot 0 = 0; \qquad x^0 = 0'$$
$$x + y' = (x + y)'; \qquad x \cdot y' = (x \cdot y) + x$$
$$x^{y'} = x^y \cdot x$$
$$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$$
$$x + y = x + z \rightarrow y = z$$

The induction scheme is arranged into a hierarchy of increasing complexity strength. For each $n < \omega$, let $I\Sigma_n$ be the Σ_n *induction scheme* which says that for every Σ_n formula φ,

$$[(\varphi(0) \;\&\; (\forall x)(\varphi(x) \rightarrow \varphi(x')) \rightarrow (\forall x)\varphi(x)].$$

Clearly we have Peano Arithmetic $= P^- + \{I\Sigma_n \,|n < \omega\}$.

A scheme which is closely related to the Σ_n induction scheme is the Σ_n *least member scheme.* This states that if φ is Σ_n and is nonempty, then there is a least member a satisfying φ. And finally, we have the Σ_n *collection scheme*: if φ is Σ_n, then

$$(\forall y < x)(\exists w)\varphi(y, w)$$

implies there is a b such that

$$(\forall y < x)(\exists w < b)\varphi(y, w).$$

In other words, on every initial segment of a model of P^-, the existence of a witness for every member in the initial segment implies the existence of a uniform bound where witnesses may be found.

Define $B\Pi_n$, $I\Pi_n$ and $L\Pi_n$ similarly for Π_n formulas.

The next theorem provides a classification of the relative strengths of these arithmetical schema:

Proposition (Kirby and Paris [1978]). *In every model of* $P^- + I\Sigma_0$, *we have*

$$I\Sigma_{n+1} \rightarrow B\Sigma_{n+1} \rightarrow I\Sigma_n$$

$$I\Sigma_n \leftrightarrow I\Pi_n \leftrightarrow L\Sigma_n \leftrightarrow L\Pi_n$$

$$B\Pi_n \leftrightarrow B\Sigma_{n+1}$$

Arrows do not reverse except where indicated.

It is not difficult to verify that all the basic notions of recursion theory can be formalized in $P^- + I\Sigma_0$. For example, n-tuples can be coded by single elements in models of $P^- + I\Sigma_0$. Indeed, given $\mathcal{M} \models P^- + I\Sigma_0$, one has the following definition:

DEFINITION. *$H \subset \mathcal{M}$ is $\mathcal{M}$-finite if H has a code in $\mathcal{M}$.*

In particular, finite sets are not the only $\mathcal{M}$-finite sets. In any initial segment of $\mathcal{M}$, the Δ_0 sets are all $\mathcal{M}$-finite. Using this notion of $\mathcal{M}$-finiteness, we may define, in a model $\mathcal{M}$ of $P^- + I\Sigma_0$, a set to be recursively enumerable (r.e.) if it is $\Sigma_1(\mathcal{M})$, and is recursive if its complement is r.e. as well. The notion of reduction can also be introduced:

DEFINITION. *Let X and Y be subsets of $\mathcal{M} \models P^- + I\Sigma_0$. X is pointwise recursive in Y (or weakly recursive in Y) if there is an r.e. set Φ of quadruples such that for all x,*

$$x \in X \longleftrightarrow (\exists H)(\exists K)[(x,1,H,K) \in \Phi \;\&\; H \subset Y \;\&\; K \cap Y = \emptyset],$$

and

$$x \notin X \longleftrightarrow (\exists H)(\exists K)[(x,0,H,K) \in \Phi \;\&\; H \subset Y \;\&\; K \cap Y = \emptyset].$$

(H, K are $\mathcal{M}$-finite sets.)

The notation $X \leq_w Y$ is used to express the relation *pointwise recursive in*. It is not difficult to see that if $\mathcal{M}$ is the standard model of arithmetic, then $\leq_w$ is a transitive relation. In general, however, the transitivity of $\leq_w$ is not automatic.

Let $\mathcal{M}$ be a model of $P^- + I\Sigma_0$. Let $\mathcal{R}$ be denote the collection of all r.e. sets in $\mathcal{M}$. One may verify that $\mathcal{R}$ forms a lattice, with $\emptyset$ and $\mathcal{M}$ forming respectively the least and greatest element in the lattice. Let $\mathcal{R}^*$ be obtained from $\mathcal{R}$ by identifying those r.e. sets with $\mathcal{M}$-finite difference.

DEFINITION. *An r.e. set M is maximal in $\mathcal{R}^*$ if there is no r.e. set lying strictly between M and $\mathcal{M}$, modulo $\mathcal{M}$-finite sets.*

Maximal sets were first constructed by Friedberg [1957] for the standard model $\mathcal{N}$. It has since become a subject of intense study for recursion theorists (see Soare [1987] for an exposition). Our interest here is to examine the strength of the statement 'there exists a maximal set' *vis à vis* fragments of the induction scheme. More specifically,

THEOREM 1. *(a) There is a maximal set in every model of $P^- + I\Sigma_2$.*

(b) There is a model of $P^- + B\Sigma_2 + \neg I\Sigma_2$ with no maximal set.

(c) There is a model of $P^- + I\Sigma_0 + \neg I\Sigma_1$ with a maximal set.

Our original proof for (a) covered only the case of $P^- + I\Sigma_3$. Slaman pointed out that the argument worked for $I\Sigma_2$ as well. We will not discuss the proofs of (a) and (c) (see Chong [to appear]), but will instead take up (b).

To obtain a model as specified by (b), one is reminded of the ordinal $\aleph_\omega^L$ in which Lerman and Simpson [1973] showed that there is no maximal set. A key property that wes used in that paper was that every constructible subset of ω is $\aleph_\omega^L$-finite. Thus the first step towards establishing (b) is to perhaps identify a model $\mathcal{M}$ of $P^- + B\Sigma_2 + \neg I\Sigma_2$ with a similar property. This is supplied by a result of Mytilinaios and Slaman [1988]:

LEMMA 1. *There is a model $\mathcal{M}_0$ of $P^- + B\Sigma_2 + \neg I\Sigma_2$ such that every set of natural numbers is the standard part of an $\mathcal{M}_0$-finite set.*
Proof: Starting with $V_{\omega+\omega}$, the collection of all sets of rank less than $\omega+\omega$, form the ultrapower V^* of $V_{\omega+\omega}$ over a nonprincipal ultrafilter. There is an embedding j of $V_{\omega+\omega}$ into V^*. The structure $j(\mathcal{N})$ is then a model of full Peano arithmetic with the additional property that every set of natural numbers is the standard part of a $j(\mathcal{N})$-finite set. Now take a nonstandard number a in $j(\mathcal{N})$, and let $\mathcal{M}_0$ be the union of the H_n's defined below:

$$\begin{aligned} H_0 &= \{b | b < a\}; \\ H_{n+1} &= \Sigma_1^{n+1}\text{-Hull}(\{b | (\exists c > b)(c \in H_n)\}). \end{aligned}$$

Here $\Sigma_1^{n+1}(H_n)$ means taking the Skolem hull of H_n in $j(\mathcal{N})$ with respect to the first $n+1$ Σ_1 functions. Then $\mathcal{M}_0$ is a Σ_1 elementary substructure of $j(\mathcal{N})$. An argument of Kirby and Paris [1978] shows that $\mathcal{M}_0$ is a model of $B\Sigma_2$ but not of $I\Sigma_2$. Furthermore, in $\mathcal{M}_0$ every set of natural numbers is the standard part of an $\mathcal{M}_0$-finite set.

We say that a set A in a model $\mathcal{M}$ is regular if $A|a$ is $\mathcal{M}$-finite for every a. The next result is well-known:

LEMMA 2. *Every r.e. set A in a model of $P^- + I\Sigma_1$ is regular.*

LEMMA 3. *Let $\mathcal{M}_0$ be as in Lemma 1. There is a function $f \leq_w \emptyset'$ such that f maps $\mathcal{N}$ cofinally into $\mathcal{M}_0$.*
Proof: Define $f(n)$ to be the supremum of H_n in the proof of Lemma 1.

An effective version of Lemma 3 yields the following approximation for the function f:

LEMMA 4. *There is a total recursive function f' such that*

(a) For all $n \in \mathcal{N}$, $\lim_s f'(s,n) = f(n)$;

(b) For all nonstandard n, $\lim_s f'(s,n)$ does not exist;

(c) $f'(s,n) \leq f'(t,m)$ for all $s \leq t$ and $n \leq m$.

Thus the model $\mathcal{M}_0$ is seen to be endowed with properties reminiscent of the ordinal $\aleph_\omega^L$: Every set of natural numbers is the standard part of an $\mathcal{M}_0$-finite set, and there is a Σ_2 cofinal function from $\mathcal{N}$ into $\mathcal{M}_0$. In Lerman and Simpson [1973], analog of these properties in $\aleph_\omega^L$ were sufficient to show that no maximal sets exist. The idea was to split $\aleph_\omega^L$ into the union $\{A_n\}$ of ω many pairwise disjoint simultaneous r.e . sets. By choosing those n's for which A_n has nonempty intersection with a given Π_1 set X, one gets an $\aleph_\omega^L$-finite subset K of ω, with the propertry that $X \cap A_n \neq \emptyset$ for each $n \in K$. One can now easily split K into two disjoint infinite $\aleph_\omega^L$-finite sets K_1 and K_2, so that the corresponding r.e. sets $\cup\{A_n | n \in K_1\}$ and $\cup\{A_n | n \in K_2\}$ split X into two non-$\aleph_\omega^L$-finite pieces.

Now for models of fragments of arithmetic such as $\mathcal{M}_0$, a recursive splitting of the universe into ω pieces is not possible (by the Overspill Lemma), and so a different strategy is required. The intuition remains the same: Given a Π_1 set X, devise a method of recursively guessing (correctly) ω many elements of X, without 'touching' ω many other elements of X.

LEMMA 5. *Let $\mathcal{M}_0$ be the model of Lemma 1. If M is r.e. with complement non-$\mathcal{M}_0$-finite, then M is contained in an r.e. set B such that neither $B \setminus M$ nor $\mathcal{M}_0 \setminus B$ are $\mathcal{M}_0$-finite.*

Proof: By Lemma 2, M is regular so that $\mathcal{M}_0 \setminus M$ is not bounded. Now for each $n \in \mathcal{N}$, there is a standard $m > n$ such that every member of $M|f(n)$ is enumerated by stage $f(m)$. Let $g(n)$ be the the least such m.

The set K of pairs $(n, g(n))$ is the standard part of an $\mathcal{M}_0$-finite set K^*. Assume without loss of generality that $\bar{M} \cap [f(n), f(n+1)) \neq \emptyset$ for each $n \in \mathcal{N}$. Choose $m_{g(n)}$ to be the least member of $\bar{M}$ greater than or equal to $f(g(n))$. We then have the situation where at any stage s, if the value of $m_{g(n+1)}$ is correctly guessed (recursively with the help of the function f), then so are all the values of $m_{g(n')}$ for all $n' < n$.

The next step is to ensure that when computing approximations to $m_{g(n)}$, there is no possibility of mistaken identity. In other words, we need a recursive guessing function such that at any stage s, if x 'appears' to be $m_{g(n)}$, then x is not $m_{g(n')}$ for any $n' < n$. This is obtained via the function h whose existence is asserted below:

SUBLEMMA. *There is a recursive function h taking each triple (s, n', n) into 2 such that for standard $n' < n$, $\lim_s h(s, n', n) = h(n', n)$ exists, and such that if $h(s, n', n) = h(n', n)$ then the number which appears to be $m_{g(n)}$ is not equal to $m_{g(n')}$.*

This technical lemma evolves from Chong and Lerman [1976] which studies the existence problem of hyperhypersimple sets in $\aleph_\omega^L$. The key point is that whilst it is not possible to select recursively from a given Π_1 set a Π_1 subset of order type ω, the existence of functions like h allows one to devise a good approximation to this set.

To complete the proof of Lemma 5, one now uses the function h to 'fill up' the complement of M to arrive at the r.e. set B whose complement contains the set of all $m_{g(n)}$'s for n odd. This is done by setting B to be M together with those x's which appear to be $m_{g(n)}$ (n odd) at some stage s where $h(s, n', n) = h(n', n)$ for all $n' < n$. This ensures that B contains all $m_{g(n)}$ for n odd, and excludes all $m_{g(n)}$ for n even.

Lemma 5 implies Theorem 1 (b). A consequence of this theorem is the following result which is of methodological interest:

COROLLARY. *There is no finite injury construction of a maximal set.*
Proof: Mytilinaios [to appear] showed that every finite injury argument can be carried out in models of $P^- + I\Sigma_1$. Theorem 1 (b) says that $B\Sigma_2$, hence (by the Proposition) $I\Sigma_1$, is not sufficient to do the maximal set construction.

One can generalize Theorem 1 (b) to cover a much wider class of models of $P^- + B\Sigma_2$. To do this we begin with a lemma which is a refinement of Smoryński [1984]:

LEMMA 6. *Let $\mathcal{M} \models P^- + I\Sigma_2$. If $K \subset \mathcal{N}$ is the standard part of a Π_2 or Σ_2 subset of $\mathcal{M}$, then K is the standard part of an $\mathcal{M}$-finite set.*
Proof: Let $\mathcal{M}$ be given as in the hypothesis and suppose that $\varphi(x,a)$ is Π_2 over $\mathcal{M}$ with parameter a. An analog of Lemma 2 says that in a model of $P^- + I\Sigma_2$ every Σ_2 (hence Π_2) set is regular. Let b be a nonstandard number in $\mathcal{M}$. Then the initial segment of b intersected with the set of numbers which satisfy $\varphi(x,a)$ is $\mathcal{M}$-finite. The standard part of this intersection is K. A similar argument applies to Σ_2 subsets. This proves the lemma.

DEFINITION. *A function p on a model of $P^- + B\Sigma_2$ is an $\mathcal{N}$-function if p is total on $\mathcal{N}$ and maps standard numbers to standard numbers.*

LEMMA 7. *Let $\mathcal{M} \models P^- + I\Sigma_2$. There is an $\mathcal{M}' \subset \mathcal{M}$ such that $\mathcal{M}'$ is a model of $P^- + B\Sigma_2$ but not of $I\Sigma_2$, with the additional property that every standard part of a $\mathcal{N}$-function which is Σ_2 definable is the standard part of an $\mathcal{M}'$-finite set.*
Proof: In $\mathcal{M}$ build the sequence $\{H_n\}$ as in the proof of Lemma 1. Then $\mathcal{M}' = \bigcup_n H_n$ is a Σ_1 elementary substructure of $\mathcal{M}$, with the additional property that there is a function $f \leq_w \emptyset'$ mapping $\mathcal{N}$ cofinally into $\mathcal{M}'$. An analog of Lemma 4 then provides a recursive approximation f' such

that for all $n \in \mathcal{N}$, $lim_s f'(s,n) = f(n)$. Let K be the standard part of a Σ_2 definable $\mathcal{N}$-function p over $\mathcal{M}'$, defined by

$$(i,j) \in p \longleftrightarrow \mathcal{M}' \models (\exists x)(\forall y)\varphi(x,y,a,i,j),$$

where φ is Δ_0 and a is a parameter. We claim that K is the standard part of an $\mathcal{M}'$-finite set.

Let Q be a set of triples such that

$$(i,(m,j)) \in Q \longleftrightarrow \mathcal{M}' \models (\exists s)(\exists x)(\forall t \geq s)(\forall y)[\varphi(x,y,a,i,j) \;\&\; f'(t,m) = f'(s,m) \;\&\; x \leq f'(s,m)].$$

Then Q is Σ_2 definable. Let $c_0 \in \mathcal{M}'$ be nonstandard, and set $Q_{c_0} = Q|c_0$. Let K_0 be the standard part of K_{c_0}. Let ψ be the Σ_2 formula used to define Q. Since $\mathcal{M}'$ is a Σ_1 elementary substructure of $\mathcal{M}$, we have for $(i,(m,j)) \in \mathcal{M}'$, $\mathcal{M}' \models \psi$ implies $\mathcal{M} \models \psi$. This means that members of K_{c_0} continue to satisfy the same formula in $\mathcal{M}$.

Let X be the set of elements less than c_0 in $\mathcal{M}$ satisfying ψ. Then X is Σ_2 definable over $\mathcal{M}$ and so by Lemma 6 is $\mathcal{M}$-finite. As $\mathcal{M}'$ is a Σ_1 elementary substructure of $\mathcal{M}$, X is also $\mathcal{M}'$-finite.

By the definition of ψ, we see that if i, m and j are standard such that $(i,(m,j)) \in X$, then it must be that $(i,(m,j)) \in K_{c_0}$. Furthermore, by the very nature that p is an $\mathcal{N}$-function, for each standard i there is a unique standard j such that $(i,(m,j))$ belongs to X for some m. Hence let K^* be the set of (i,j)'s in X such that j is the least member of $\mathcal{M}'$ satisfying $(i,(m,j)) \in X$ for some $m < c_0$. Then K is the standard part of K^* and K^* is $\mathcal{M}'$-finite. This proves the lemma.

Now Lemmas 3, 4 and 5 continue to hold for models $\mathcal{M}'$ satisfying the conclusions of Lemma 7. In particular, the $\mathcal{N}$-function g in the proof of Lemma 5 is Σ_2, and is therefore the standard part of an $\mathcal{M}'$-finite set.

The same holds true for the $\mathcal{N}$-function h in the Sublemma. It follows that in $\mathcal{M}'$ there are no maximal sets.

Now in Smoryński [1984], there is a proof of Scott's Theorem which gives a characterization of subsets of 2^ω which are *standard systems* of models of Peano arithmetic (i.e . members of 2^ω which are standard parts of 'finite sets' of a given model of Peano arithmetic). This is stated as follows:

LEMMA 8. *Let $\mathcal{X}$ be a countable family of sets of natural numbers, then there is a model $\mathcal{M}$ of Peano arithmetic for which $\mathcal{X}$ is the standard system if and only if:*

(a) $\mathcal{X}$ is closed under Boolean operations;

(b) $\mathcal{X}$ is closed under Turing reducibility;

(c) $\mathcal{X}$ satisfies a weak form of König's Lemma: If $X \in \mathcal{X}$ codes an infinite binary tree, then some Y in $\mathcal{X}$ codes an infinite path through X.

Thus there exist infinitely many different countable subsets of 2^ω which are standard systems of models of Peano arithmetic. Applying Lemma 7 one finds infinitely many countable models $\mathcal{M}'$ of $P^- + B\Sigma_2 + \neg I\Sigma_2$ with pairwise different standard systems for which all standard parts of Π_2 or Σ_2 sets are standard parts of $\mathcal{M}'$-finite sets. Each of these models has no maximal sets. This proves the next result.

THEOREM 2. *There exist infinitely many countable models of $P^- + B\Sigma_2 + \neg I\Sigma_2$ with pairwise different standard systems which have no maximal sets.*

Note that in contrast the model $\mathcal{M}_0$ of Theorem 1(b) is uncountable. A theorem of Guaspari allows one to improve the above result to (uncountable) models of $P^- + B\Sigma_2 + \neg I\Sigma_2$ with standard systems of size $\aleph_1$.

We end this paper with three questions:

(a) Assume $\mathcal{M} \models P^- + B\Sigma_2$. Is it true that if $\mathcal{M}$ has a maximal set then $\mathcal{M} \models I\Sigma_2$? A positive answer to this question would give a complete characterization of the existence of maximal sets over the base theory $P^- + B\Sigma_2$.

(b) Theorem 1 (c) indicates that the existence of maximal sets does not require any assumption stronger than $P^- + I\Sigma_0$, provided that the underlying universe is carefully chosen. In the proof of Theorem 1 (c) (Chong [to appear]), the model chosen has the property that there is a Σ_2 map from $\mathcal{N}$ onto the whole universe. Do all models of $P^- + I\Sigma_0 + \neg I\Sigma_1$ with maximal sets have this property ?

(c) What is the complexity, in the hierarchy of fragments of Peano arithmetic, of various theorems on maximal sets ? In particular, is Soare's theorem (Soare [1974]) on the automorphisms of the lattice of r.e. sets sending maximal sets to maximal sets provable in $P^- + I\Sigma_2$?

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