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Algorithmic Construction of the Recursion Operators of Toda and Landau-Lifshitz Equation†

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Abstract

A new approach to the construction of recursion operators of completely integrable system is exhibited. It is explicitly applied to derive the hierarchy of equations of motion of the celebrated Toda lattice as well as the well known Landau-Lifshitz equation.

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A. Recursion Operator for the Toda Lattice

The equations of motion for the Hamiltonian System of the Toda lattice with Hamiltonian

$$H = \sum_n \left\{ \frac{1}{2} p_n^2 + e^{x_{n+1}-x_n} \right\} \tag{1}$$

are:

$$x_{n,t} = p_n, \quad p_{n,t} = e^{x_{n+1}-x_n} - e^{x_n - x_{n-1}} \tag{2}$$

and the shift operator of the Lax pair acting on the vector

$$\psi_n = \begin{bmatrix} t_n \\ \epsilon_n \end{bmatrix} \tag{3}$$

is given by (Takhtadzhan and Fadeev (1979))

$$L_n(\lambda)\psi_n = \psi_{n+1} \tag{4}$$

where

$$L_n(\lambda) = \begin{bmatrix} p_n + \lambda & -e^{x_n} \\ e^{-x_n} & 0 \end{bmatrix} \tag{5}$$

After introducing

$$v_n \equiv \epsilon_{n+1} \tag{6}$$

(4) yields the following second order difference equation for v_n:

$$\lambda v_n = e^{x_{n+1}-x_n} v_{n+1} + (-p_n)v_n + v_{n-1} \tag{7}$$

Define now

$$a_n \equiv e^{x_{n+1}-x_n} \quad , \quad b_n \equiv -p_n \tag{8}$$

then (7) is written as:

$$a_n v_{n+1} + b_n v_n + v_{n-1} = \lambda v_n \tag{9}$$

The time evolution of the auxilliary vector ψ_n is expressed in terms of v_n's as

$$v_{n,t} = (A_n v_{n+1} - B_n v_n) a_n \tag{10}$$

and the compatibility of (9), (10) gives:

$$[a_{n,t} + \lambda a_n(A_{n+1} - A_n) - a_n(b_{n+1}A_{n+1} - b_nA_n) - a_n(a_{n+1}B_{n+1} - a_{n-1}B_{n-1})]\, v_{n+1}$$
$$+ [b_{n,t} + \lambda(a_nB_n - a_{n-1}B_{n-1}) + b_n(a_{n-1}B_{n-1} - a_nB_n) + a_{n-1}A_{n-1} - a_nA_{n+1}]\, v_n = 0 \quad (11)$$

hence both coefficients of v_n and v_{n+1} should vanish i.e.

$$a_{n,t} + \lambda a_n(A_{n+1} - A_n) - a_n(b_{n+1}A_{n+1} - b_nA_n) - a_n(a_{n+1}B_{n+1} - a_{n-1}B_{n-1}) = 0 \quad (12)$$

and

$$b_{n,t} + \lambda(a_nB_n - a_{n-1}B_{n-1}) + b_n(a_{n-1}B_{n-1} - a_nB_n) + a_{n-1}A_{n-1} - a_nA_{n+1} = 0 \quad (13)$$

One may postulate

$$A_n = \sum_{j=0}^{N} A_n^{(j)}\lambda^j \quad , B_n = \sum_{j=0}^{N} B_n^{(j)}\lambda^j \quad (14)$$

So after substitution of (15) into (12), (13) and equating coefficients of λ^j, one obtains the following equations:

$$a_n(A_{n+1}^{(N)} - A_n^{(N)}) = 0 \quad , a_nB_n^{(N)} - a_{n-1}B_{n-1}^{(N)} = 0 \quad (15)$$

$$a_{n,t} = a_n(b_{n+1}A_{n+1}^{(0)} - b_nA_n^{(0)}) + a_n(a_{n+1}B_{n+1}^{(0)} - a_{n-1}B_{n-1}^{(0)}) \quad (16)$$

$$b_{n,t} = b_n(a_nB_n^{(o)} - a_{n-1}B_{n-1}^{(o)}) + a_nA_{n+1}^{(o)} - a_{n-1}A_{n-1}^{(o)} \quad (17)$$

$$a_n(A_{n+1}^{(j-1)} - A_n^{(j-1)}) = a_n(b_{n+1}A_{n+1}^{(j)} - b_nA_n^{(j)}) + a_n(a_{n+1}B_{n+1}^{(j)} - a_{n-1}B_{n-1}^{(j)}) \quad (18)$$

$$a_nB_n^{(j-1)} - a_{n-1}B_{n-1}^{(j-1)} = b_n(a_nB_n^{(j)} - a_{n-1}B_{n-1}^{(j)}) + a_nA_{n+1}^{(j)} - a_{n-1}A_{n-1}^{(j)} \quad (19)$$

for $j = 1, \ldots, n$.

Upon introducing the operators Δ, Δ^+ (cf. Soliani et al., (1983))

$$\Delta u_n \equiv u_{n+1} - u_n$$
$$\Delta^+ u_n \equiv u_{n-1} - u_n \quad (20)$$

one may write (17), (18) in matrix form

$$\begin{bmatrix} a_{n,t} \\ b_{n,t} \end{bmatrix} = \begin{bmatrix} a_n(\Delta - \Delta^+)a_n & a_n \Delta b_n \\ -b_n \Delta^+ a_n & a_n \Delta - \Delta^+ a_n \end{bmatrix} \begin{bmatrix} B_n^{(0)} \\ A_n^{(0)} \end{bmatrix}$$

$$\equiv \Omega \begin{bmatrix} B_n^{(0)} \\ A_n^{(0)} \end{bmatrix} \tag{21}$$

and (18), (19) expressed as:

$$\Theta \begin{bmatrix} B_n^{(j-1)} \\ A_n^{(j-1)} \end{bmatrix} = \Omega \begin{bmatrix} B_n^{(j)} \\ A_n^{(j)} \end{bmatrix} \tag{22}$$

where

$$\Theta \equiv \begin{bmatrix} 0 & a_n \Delta \\ -\Delta^+ a_n & 0 \end{bmatrix} \tag{23}$$

Note that this operator was present by Soliani et al., (1983). The recursion relation takes the form

$$\begin{bmatrix} B_n^{(j-1)} \\ A_n^{(j-1)} \end{bmatrix} = \Theta^{-1}\Omega \begin{bmatrix} B_n^{(j)} \\ A_n^{(j)} \end{bmatrix} \equiv \Psi \begin{bmatrix} B_n^{(j)} \\ A_n^{(j)} \end{bmatrix} \tag{24}$$

From (25) one obtains recursively:

$$\begin{bmatrix} B_n^{(0)} \\ A_n^{(0)} \end{bmatrix} = \Psi^N \begin{bmatrix} B_n^{(N)} \\ A_n^{(N)} \end{bmatrix} \tag{25}$$

and since a solution of (15) is

$$A_n^{(N)} = c \ , \ \ B_n^{(N)} = 0 \tag{26}$$

where c is an arbitrary constant, the hierarchy of the Toda lattice is given by:

$$\begin{bmatrix} a_{n,t} \\ b_{n,t} \end{bmatrix} = \Omega\Psi^N \begin{bmatrix} 0 \\ c \end{bmatrix} \tag{27}$$

The first system of equations ($N = 0$, $c = -1$)

$$\begin{bmatrix} a_{n,t} \\ b_{n,t} \end{bmatrix} = \begin{bmatrix} -a_n \Delta b_n \\ \Delta^+ a_n \end{bmatrix} = \begin{bmatrix} a_n(b_n - b_{n+1}) \\ a_{n-1} - a_n \end{bmatrix} \tag{28}$$

is equivalent to (2), using (8). The second system ($N = 1$) is:

$$\begin{bmatrix} a_{n,t} \\ b_{n,t} \end{bmatrix} = \Omega\Theta^{-1}\Omega \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{29}$$

and, after noting that

$$\Theta^{-1} = \begin{bmatrix} 0 & a_n^{-1}(\Delta^+)^{-1} \\ \Delta^{-1}a_n^{-1} & 0 \end{bmatrix} \tag{30}$$

where

$$(\Delta^{-1}u)_n \equiv -\sum_{j=n}^{+\infty} u_j \tag{31}$$

(29) becomes:

$$\begin{bmatrix} a_{n,t} \\ b_{n,t} \end{bmatrix} = \begin{bmatrix} a_n(a_{n+1} - 2a_n + a_{n-1}) - a_n(b_{n+1}^2 - b_n^2) \\ -b_n(a_{n-1} - a_n) - a_n(b_{n+1} - b_n) + a_{n-1}b_{n-1} - a_n b_n \end{bmatrix} \tag{32}$$

B. Landau-Lifshitz Equation

The Landau-Lifshitz equation (LL) is given by

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times J\mathbf{S} \tag{1}$$

where J is the diagonal matrix

$$J = \text{ diag } (J_1, J_2, J_3) \tag{2}$$

and $\mathbf{S}$ is the classical unit spin $\mathbf{S} = (S_1, S_2, S_3)$, i.e.,

$$\mathbf{S} \cdot \mathbf{S} = 1. \tag{3}$$

It is well known that (1) is completely integrable and Sklyanin (1979) and others presented its Lax-pair. Since the LL equation is the continuum limit of the equation of motion of the quantum non-isotropic Heisenberg Hamiltonian (the so-called XYZ), it is not surprising that the Lax pair is expressed in terms of Jacobi elliptic functions. The algebraic structure of (1) was studied in detail by Date, Jimbo, Kashiwara and Miwa (1983) who derived its quasi-periodic solutions as well. Furthermore, Fuchssteiner (1984) studied its master-symmetries.

Consider the equation for the auxilliary vector ψ given by

$$\psi_x = -i \left(\sum_{j=1}^{3} S_j W_j \sigma_j \right) \psi \equiv -iL\psi \tag{4}$$

while L may be viewed as the shift operator associated with the Lax pair. The operators σ_j are the Pauli spin operators given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$

and the Jacobi elliptic functions W_j are given by Sklyanin as:

$$\begin{aligned} W_1 &= \rho \frac{1}{sn(u,k)} \\ W_2 &= \rho \frac{dn(u,k)}{sn(u,k)} \\ W_3 &= \rho \frac{cn(u,k)}{sn(u,k)} \end{aligned} \tag{6}$$

with the modulus k given by

$$k = \left\{ \frac{J_2 - J_1}{J_3 - J_1} \right\}^{1/2} \qquad 0 < k < 1 \tag{7}$$

and the arbitrary normalization parameter ρ as well as the parameters α, β are defined by

$$W_1^2 - W_3^2 = \frac{1}{4}(J_3 - J_1) \equiv \alpha \tag{8a}$$

$$W_2^2 - W_3^2 = \frac{1}{4}(J_3 - J_2) \equiv \beta \tag{8b}$$

Formally, one may express the time evolution of the auxilliary vector ψ as

$$\psi_t = -iV\psi \tag{9}$$

and the structure of the operator L suggests that V has similar form, i.e. one may postulate

$$\psi_t = -i\left\{\sum_{j=1}^{3} W_j V_j \sigma_j\right\}\psi \tag{10}$$

with the compatibility condition

$$L_t - V_x - i[L, V] = 0 \tag{11}$$

that takes the form

$$\sum_{j=1}^{3} S_{j,t} W_j \sigma_j - \sum_{j=1}^{3} V_{j,x} W_j \sigma_j - i\left[\sum_{j=1}^{3} S_j W_j \sigma_j, \sum_{j=1}^{3} V_j W_j \sigma_j\right] = 0 \tag{12}$$

Equating coefficients of σ_j for $j = 1, 2, 3$, one obtains

$$S_{1,t} = \frac{2W_2W_3}{W_1}(S_3V_2 - S_2V_3) + V_{1,x} \tag{13}$$

as well as other cyclic permutations.

It is convenient to introduce the parametrization

$$\lambda \equiv \frac{1}{2}W_1W_2W_3, \qquad \mu \equiv W_3^2 \tag{14}$$

with the immediate identity

$$\lambda^2 = \frac{1}{4}\mu(\mu + \alpha)(\mu + \beta) \tag{15}$$

where α, β have been defined by (8a), (8b). Thus, (13) and its cyclic permutations take the form

$$S_{1,t} = \frac{\mu(\mu+\beta)}{\lambda}(S_3V_2 - S_2V_3) + V_{1,x} \tag{16a}$$

$$S_{2,t} = \frac{(\mu+\alpha)\mu}{\lambda}(S_1V_3 - S_3V_1) + V_{2,x} \tag{16b}$$

$$S_{3,t} = \frac{(\mu+\beta)(\mu+\alpha)}{\lambda}(S_2V_1 - S_1V_2) + V_{3,x} \tag{16c}$$

One may formally represent the operators V_k by the finite expansions

$$V_1 = \frac{\mu(\mu+\beta)}{\lambda}\sum_{j=0}^{n}\mu^{n-j}a_1^{(j)} + \sum_{j=0}^{n}\mu^{n-j}b_1^{(j)} \tag{17a}$$

$$V_2 = \frac{(\mu+\alpha)\mu}{\lambda}\sum_{j=0}^{n}\mu^{n-j}a_2^{(j)} + \sum_{j=0}^{n}\mu^{n-j}b_2^{(j)} \tag{17b}$$

$$V_3 = \frac{(\mu+\beta)(\mu+\alpha)}{\lambda}\sum_{j=0}^{n}\mu^{n-j}a_3^{(j)} + \sum_{j=0}^{n}\mu^{n-j}b_3^{(j)} \tag{17c}$$

In other words, determination of the operators $a_k^{(j)}$, $b_l^{(m)}$ is equivalent to a determination of V. Upon substitution of (17) in (16a) one obtains

$$S_{1,t} = \frac{\mu(\mu+\beta)}{\lambda}\sum_{j=0}^{n}\mu^{n-j}a_{1,x}^{(j)} + \sum_{j=0}^{n}\mu^{n-j}b_{1,x}^{(j)}$$
$$-\frac{\mu(\mu+\beta)}{\lambda}\left[S_2\left(\frac{(\mu+\alpha)(\mu+\beta)}{\lambda}\sum_{j=0}^{n}\mu^{n-j}a_3^{(j)} + \sum_{j=0}^{n}\mu^{n-j}b_3^{(j)}\right)\right.$$
$$\left.-S_3\left(\frac{\mu(\mu+\alpha)}{\lambda}\sum_{j=0}^{n}\mu^{n-j}a_2^{(j)} + \sum_{j=0}^{n}\mu^{n-j}b_2^{(j)}\right)\right] \tag{18}$$

namely

$$S_{1,t} = \frac{\mu(\mu+\beta)}{\lambda} \sum_{j=0}^{n} \mu^{n-j}(a_{1,x}^{(j)} - S_2 b_3^{(j)} + S_3 b_2^{(j)}) + \sum_{j=0}^{n} \mu^{n-j} b_{1,x}^{(j)}$$

$$-4(\mu+\beta) S_2 \sum_{j=0}^{n} \mu^{n-j} a_3^{(j)} + 4\mu S_3 \sum_{j=0}^{n} \mu^{n-j} a_2^{(j)} \tag{19}$$

or

$$S_{1,t} = \frac{\mu(\mu+\beta)}{\lambda} \sum_{j=0}^{n} \mu^{n-j}(a_{1,x}^{(j)} - S_2 b_3^{(j)} + S_3 b_2^{(j)}) + \sum_{j=0}^{n} \mu^{n-j}[b_{1,x}^{(j)} - 4\beta S_2 a_3^{(j)}]$$

$$-4 \sum_{j=-1}^{n-1} \mu^{n-j}[S_2 a_3^{(j+1)} - S_3 a_2^{(j+1)}] \tag{20}$$

Similarly, the other two equations are given by

$$S_{2,t} = \frac{(\mu+\alpha)\mu}{\lambda} \sum_{j=0}^{n} \mu^{n-j}(a_{2,x}^{(j)} - S_3 b_1^{(j)} + S_1 b_3^{(j)}) + \sum_{j=0}^{n} \mu^{n-j}(b_{2,x}^{(j)} + 4\alpha S_3 a_1^{(j)})$$

$$-4 \sum_{j=-1}^{n-1} \mu^{n-j}(S_3 a_1^{(j+1)} - S_1 a_3^{(j+1)}) \tag{20b}$$

$$S_{3,t} = \frac{(\mu+\beta)(\mu+\alpha)}{\lambda} \sum_{j=0}^{n} \mu^{n-j}(a_{3,x}^{(j)} - S_1 b_2^{(j)} + S_2 b_1^{(j)})$$

$$+\sum_{j=0}^{n} \mu^{n-j}(b_{3,x}^{(j)} - 4\alpha S_1 a_2^{(j)} + 4\beta S_2 a_1^{(j)}) - 4 \sum_{j=-1}^{n-1} \mu^{n-j}(S_1 a_2^{(j+1)} - S_2 a_1^{(j+1)}) \tag{20c}$$

Equating coefficients of μ^j and $\lambda^{-1}\mu^j$ independently one obtains

$$\mathbf{S} \times \mathbf{a}^{(0)} = 0 \tag{21}$$

$$\mathbf{S} \times \mathbf{b}^{(j)} = \mathbf{a}_x^{(j)} \quad ; j = 0, 1, \ldots, n \tag{22}$$

$$\mathbf{S} \times \mathbf{a}^{(j+1)} = \frac{1}{4}\mathbf{b}_x^{(j)} - (A\mathbf{S}) \times \mathbf{a}^{(j)} \quad ; j = 0, 1, \ldots, n-1 \tag{23}$$

$$\mathbf{S}_t = \mathbf{b}_x^{(n)} - 4(A\mathbf{S}) \times \mathbf{a}^{(n)} \tag{24}$$

where A is diagonal matrix given by

$$A = \text{ diag } (\alpha, \beta, 0) \tag{25}$$

First, one solves (22) for $b^{(j)}$

$$\mathbf{b}^{(j)} = -\mathbf{S} \times \mathbf{a}_x^{(j)} + g_j \mathbf{S} \tag{26}$$

where g_j is a scalar function of x to be determined by requiring the solvability condition for (23):

$$\left\{\mathbf{b}_x^{(j)} - (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\} \cdot \mathbf{S} = 0 \tag{27}$$

This condition gives:

$$g_{j,x} = \left\{\mathbf{S}_x \times \mathbf{a}_x^{(j)} + (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\} \cdot \mathbf{S} \tag{28}$$

i.e.

$$g_j = \partial^{-1} \left(\left\{\mathbf{S}_x \times \mathbf{a}_x^{(j)} + (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\} \cdot \mathbf{S}\right) \tag{29}$$

where ∂^{-1} indicates antiderivative with respect to x.

Then

$$\mathbf{b}_x^{(j)} = -\mathbf{S} \times \mathbf{a}_{xx}^{(j)} + \left[(4A\mathbf{S}) \times \mathbf{a}^{(j)} \cdot \mathbf{S}\right] \mathbf{S}$$

$$+\left[\partial^{-1}\left(\left\{\mathbf{S}_x \times \mathbf{a}_x^{(j)} + (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\} \cdot \mathbf{S}\right)\right] \mathbf{S}_x \tag{30}$$

Now (23) yields:

$$\mathbf{a}^{(j+1)} = -\frac{1}{4}\mathbf{S} \times \left\{\mathbf{b}_x^{(j)} - (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\} + f_{j+1}\mathbf{S} \tag{31}$$

where the scalar f_{j+1} is to be determined by the requirement

$$\mathbf{a}_x^{(j+1)} \cdot \mathbf{S} = 0 \ , \tag{32}$$

for (22) to be solvable for $\mathbf{b}^{(j)}$. Using (31), (32) yields

$$f_{j+1,x} = \frac{1}{4}\mathbf{S}_x \times \left\{\mathbf{b}_x^{(j)} - (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\} \cdot \mathbf{S} \tag{33}$$

i.e.

$$f_{j+1} = \frac{1}{4}\partial^{-1}\left(\mathbf{S}_x \times \left\{\mathbf{b}_x^{(j)} - (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\} \cdot \mathbf{S}\right) \tag{34}$$

so,

$$\begin{aligned}\mathbf{a}^{(j+1)} &= -\frac{1}{4}\mathbf{S} \times \left\{\mathbf{b}_x^{(j)} - (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\} \\ &+\frac{1}{4}\left[\partial^{-1}\left(\mathbf{S}_x \times \left\{\mathbf{b}_x^{(j)} - (4A\mathbf{S}) \times \mathbf{a}^{(j)}\right\} \cdot \mathbf{S}\right)\right] \mathbf{S}\end{aligned} \tag{35}$$

Finally, introducing the operators:

$$\Theta^{-1}\mathbf{a} \equiv \mathbf{S} \times \mathbf{a} + \left[\partial^{-1}(\mathbf{S}_x \times \mathbf{a} \cdot \mathbf{S})\right] \mathbf{S} \tag{36}$$

and

$$\Omega \mathbf{a} \equiv \frac{1}{4}\{-\mathbf{S}\times\mathbf{a}_{xx} - (4A\mathbf{S})\times\mathbf{a} + [(4A\mathbf{S})\times\mathbf{a}\cdot\mathbf{S}]\,\mathbf{S}$$
$$+\left[\partial^{-1}\left(\{\mathbf{S}_x\times\mathbf{a}_x + (4A\mathbf{S})\times\mathbf{a}\}\cdot\mathbf{S}\right)\right]\mathbf{S}_x\} \tag{37}$$

we can write (35) and (24) as:

$$\mathbf{a}^{(j+1)} = \Theta^{-1}\Omega\mathbf{a}^{(j)} \equiv \Psi\mathbf{a}^{(j)} \tag{38}$$

and

$$\mathbf{S}_t = 4\Omega\mathbf{a}^{(n)} \tag{39}$$

Next, one has to deal with the "starting points" of the recursion, $\mathbf{a}^{(0)}$, $\mathbf{b}^{(0)}$. It is best illustrated by an explicit derivation of the hierarchy (39) for $n = 1$:

From (21), solving for $\mathbf{a}^{(0)}$, one obtains:

$$\mathbf{a}^{(0)} = F_o\mathbf{S} \tag{40}$$

It turns out that F_o is a constant in order to be able to solve (22) for $\mathbf{b}^{(0)}$:

$$\mathbf{b}^{(0)} = -F_o\mathbf{S}\times\mathbf{S}_x + G_o\mathbf{S} \tag{41}$$

where G_o is a new constant in order that (23) be solvable for $\mathbf{a}^{(1)}$:

$$\mathbf{a}^{(1)} = f_1\mathbf{S} + \frac{1}{4}\{G_o(\mathbf{S}\times\mathbf{S}_x) + F_o\left[\mathbf{S}_{xx} - (\mathbf{S}\cdot\mathbf{S}_{xx})\mathbf{S}\right]\}$$

$$+F_o\{(\mathbf{S}\cdot A\mathbf{S})\mathbf{S}-A\mathbf{S}\} \tag{42}$$

Since $\mathbf{a}_x^{(1)}$ has to be normal to $\mathbf{S}$,

$$f_{1,x}-F_o\left\{\frac{1}{4}(\mathbf{S}_x\cdot\mathbf{S}_{xx})-\mathbf{S}_x\cdot A\mathbf{S})\right\}=0 \tag{43}$$

Since $\mathbf{S}$ is a unit vector, i.e. $\mathbf{S}\cdot\mathbf{S}=1$, one has:

$$-\mathbf{S}\cdot\mathbf{S}_{xx}=\mathbf{S}_x\cdot\mathbf{S}_x \tag{44}$$

and

$$\mathbf{S}\cdot\mathbf{S}_{xxx}=\frac{3}{2}(\mathbf{S}\cdot\mathbf{S}_{xx})_x \tag{45}$$

so (43) yields:

$$f_1=F_1+\frac{1}{8}F_o\left[(\mathbf{S}_{xx}-4A\mathbf{S})\cdot\mathbf{S}\right] \tag{46}$$

where F_1 is a constant. Hence

$$\mathbf{a}^{(1)}=\left\{F_1+\frac{1}{8}F_o\left[(\mathbf{S}_{xx}-4A\mathbf{S})\cdot\mathbf{S}\right]\right\}\mathbf{S}+\frac{1}{4}G_o\mathbf{S}\times\mathbf{S}_x$$

$$+\frac{1}{4}F_o(\mathbf{S}_{xx}-(\mathbf{S}\cdot\mathbf{S}_{xx})\mathbf{S})+F_o\left[(\mathbf{S}\cdot A\mathbf{S})\mathbf{S}-A\mathbf{S}\right] \tag{47}$$

and

$$\mathbf{a}_x^{(1)}=\left\{F_1+\frac{1}{8}F_o\left[(\mathbf{S}_{xx}-4A\mathbf{S})\cdot\mathbf{S}\right]\right\}\mathbf{S}_x$$

$$+\frac{F_o}{8}\{(\mathbf{S}_{xx}-4A\mathbf{S})\cdot\mathbf{S}_x+(\mathbf{S}_{xxx}-4A\mathbf{S}_x)\cdot\mathbf{S}\}\mathbf{S}$$

$$+\frac{1}{4}G_o\mathbf{S}\times\mathbf{S}_{xx}+\frac{1}{4}F_o\{\mathbf{S}_{xxx}-(\mathbf{S}_x\cdot\mathbf{S}_{xx})\mathbf{S}-(\mathbf{S}\cdot\mathbf{S}_{xxx})\mathbf{S}-(\mathbf{S}\cdot\mathbf{S}_{xx})\mathbf{S}_x\}$$
$$+F_o\{2(\mathbf{S}_x\cdot A\mathbf{S})\mathbf{S}+(\mathbf{S}\cdot A\mathbf{S})\mathbf{S}_x-A\mathbf{S}_x\} \tag{48}$$

Then, solving (22) for $\mathbf{b}^{(1)}$, one gets

$$\mathbf{b}^{(1)}=g_1\mathbf{S}-\mathbf{S}\times\mathbf{a}_x^{(1)} \tag{49}$$

where g_1 has to satisfy the equation (cf. (28))

$$g_{1,x}=\frac{1}{4}G_o[\mathbf{S}_x\cdot\mathbf{S}_{xx}+4A\mathbf{S}\cdot\mathbf{S}_x]-\frac{1}{4}F_o[\mathbf{S}_x\times(4A\mathbf{S}_x)\cdot\mathbf{S}$$
$$+\mathbf{S}_{xx}\times(4A\mathbf{S})\cdot\mathbf{S}-\mathbf{S}_x\times\mathbf{S}_{xxx}\cdot\mathbf{S}] \tag{50}$$

i.e.

$$g_{1,x}=\frac{1}{8}G_o[\mathbf{S}_x\cdot\mathbf{S}_x+4A\mathbf{S}\cdot\mathbf{S}]_x$$
$$+\frac{1}{4}F_o[\mathbf{S}_x\times(\mathbf{S}_{xx}-4A\mathbf{S})\cdot\mathbf{S}]_x \tag{51}$$

and because of (45):

$$g_1=G_1-\frac{1}{8}G_o[(\mathbf{S}_{xx}-4A\mathbf{S})\cdot\mathbf{S}]+\frac{1}{4}F_o[\mathbf{S}_x\times(\mathbf{S}_{xx}-4A\mathbf{S})\cdot\mathbf{S}] \tag{52}$$

One may set $n=1$ in equation (24). The resulting evolution equation contains the arbitrary constants G_1, F_1, G_o, F_o. By letting all but one vanish, one obtains the hierarchy of evolution equations as:

$$\text{(i)}\quad \mathbf{S}_t=\mathbf{S}_x \tag{53}$$

$$\text{(ii)}\quad \mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + (4A\mathbf{S}) \times \mathbf{S}$$

which is the same as

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times J\mathbf{S} \tag{54}$$

because of (8a,b) and (25).

$$\text{(iii)}\quad \mathbf{S}_t = \mathbf{S}_{xxx} + \frac{3}{2}\left[(\mathbf{S}_x \cdot \mathbf{S}_x) - J\mathbf{S} \cdot \mathbf{S} + J_3\right]\mathbf{S}_x + 3(\mathbf{S}_x \cdot \mathbf{S}_{xx})\mathbf{S} \tag{55}$$

This equation was obtained by Date, Jimbo, Kashiwara and Miwa (1983)

$$\begin{aligned}\text{(iv)}\quad \mathbf{S}_t = {} & \mathbf{S} \times \mathbf{S}_{xxxx} + \mathbf{S}_x \times \mathbf{S}_{xxx} - \frac{1}{2}\left[3\mathbf{S} \cdot \mathbf{S}_{xx} + \mathbf{S} \cdot J\mathbf{S}\right]\mathbf{S} \times \mathbf{S}_{xx} \\ & + \left[3(\mathbf{S}_x \cdot \mathbf{S}_{xx}) - \mathbf{S}_x \cdot J\mathbf{S}\right]\mathbf{S} \times \mathbf{S}_x - \left[\mathbf{S}_x \times (\mathbf{S}_{xx} + J\mathbf{S}) \cdot \mathbf{S}\right]\mathbf{S}_x \\ & + \frac{1}{2}\left[3(\mathbf{S} \cdot \mathbf{S}_{xx}) + (\mathbf{S} \cdot J\mathbf{S})\right](J\mathbf{S}) \times \mathbf{S} \\ & - \left[\mathbf{S}_x \times (\mathbf{S}_{xx} + J\mathbf{S}) \cdot \mathbf{S}\right]_x \mathbf{S} + (\mathbf{S} \times J\mathbf{S}_x)_x + \mathbf{S}_{xx} \times (J\mathbf{S}).\end{aligned}$$

Detailed account of this work, in particular the bi-Hamiltonian formulation and the connection with the master-symmetry approach, will be published elsewhere.

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Note added in the proof: The first presentation of the Toda recursion operator was given by H. Flaschka and D. W. McLaughlin in "Backlund Transformation", ed.by R.M.Miura, Lecture Notes in Mathematics 515, Springer-Verlag, New york, p.253 (1976).