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両側 Jones projection により生成される因子環の部分因子環の指数

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1. Introduction

The index theory for finite factors was introduced by Jones in [3]. In the paper, the following sequence $\{e_i; i=1,2,...\}$ of projections plays an important role:

(a) $e_i e_{i\pm1} e_i = \lambda e_i$ for some $\lambda \leqq 1$

(b) $e_i e_j = e_j e_i$ for $|i-j| \geqq 2$

(c) the von Neumann algebra P generated by $\{e_i; i=1,2,...\}$ is a hyperfinite II_1-factor,

(d) $tr(we_i) = \lambda tr(w)$ if w is a word on $1, e_1, e_2, ... e_{i-1}$, where tr is the canonical trace of P and 1 is the identity operator.

If Q is a subfactor of P generated by $\{e_i; i=2,3,...\}$, then the index $[P:Q]$ of Q in P is $1/\lambda$. In the case of $\lambda > 1/4$, Q has the trivial relative commutant in P and $[P:Q] = 4\cos^2(\pi/m)$ for some $m = 3,4,...$. Hence by his basic construction, we have the family $\{e_i; i=...,-2,-1,0,1,2,...\}$ of projections with the properties (a),(b),(c') and (d');

(c') $\{e_i; i=0,\pm1,\pm2,...\}$ generates a hyperfinite II_1 factor M

(d') $tr(we_i) = \lambda tr(w)$ for the trace tr of M if w is a word on 1 and $\{e_j; j<i\}$ (cf.[5]).

We shall call this family $\{e_i; i=0,\pm1,\pm2,\ldots\}$ the _two sided Jones' projections for_ λ. The main purpose of this note is to show the following theorem.

Theorem . Let $\{e_i; i=0,\pm1,\pm2,\ldots\}$ be the two sided Jones' projections for $\lambda=(1/4)\sec^2(\pi/m)$ for some m $(m=3,4,\ldots)$. If M (resp. N) is the von Neumann algebra generated by $\{e_i; i=0,\pm1,\pm2,\ldots\}$ (resp. $\{e_i; i=\pm1, \pm2,\ldots\}$), then N is a subfactor of M with the index

$$[M:N] = (m/4)\mathrm{cosec}^2(\pi/m),$$

and the relative commutant of N in M is trivial, that is, $N'\cap M = \mathbb{C}1$.

2. Notations and Preliminaries

Let B be a subfactor of a II_1-factor A. Then Jones defined in [3] the index $[A:B]$ of B in A using the coupling constants of A and B due to Murray and von Neumann ([4]) and he (also, Pimsner-Popa in [5]) gives some methods to get the number $[A:B]$. In [6], Wenzl gets another method to compute $[A:B]$ in the case where those factors are σ-weak closures of the union of increasing sequences of finite dimensional algebras, which satisfy some good conditions.

In this note, we shall use results in [6] and give a proof of Theorem.

(2.1) Let A be a finite dimensional von Neumann algebra. Then

A is decomposed into the direct sum $\Sigma_{i=1}^{m} + A_i$ of the a(i) by a(i) matrix algebra A_i. The vector a=(a(i)) is called the <u>dimension vector</u> of A following after Wenzl[6]. Each trace ϕ on the algebra A is determined by a column vector w =(w(i)) which satisfies $\phi(x)=\Sigma_{i=1}^{m} w(i)Tr(x_i)$ for $x\varepsilon A$, where $x=\Sigma + x_i (x_i \varepsilon A_i)$ and Tr is the usual nonnormalized trace on the matrix algebra. The column vector w is called the <u>weight vector</u> of the trace ϕ. Let B be a von Neumann subalgebra of A with the direct summund $B = \Sigma_{i=1}^{n} + B_i$ of the b(i) by b(i) matrix algebras B_i. The inclusion of B in A is specified up to conjugacy by an n by m matrix $[g_{i,j}]$, where $g_{i,j}$ is the number of simple components of a simple A_j module viewed as an B_i module. The matrix $[g_{i,j}]$ is called the <u>inclusion matrix</u> of B in A which we denote by $[B \to A]$. Let b = (b(i)) be the dimension vector of B and v the weight vector of the restriction of ϕ to B, then

(e) $b[B \to A] = a$ and $[B \to A]w = v$.

(2.2) Let $\{e_i; i=0,\pm1,\pm2,...\}$ be two sided Jones' projections for $\lambda(\lambda \leqq 1)$. A reduced word is a word on e_i's of minimal length for the rules (a),(b) and $e_i^2 \leftrightarrow e_i$. If a reduced word is further reduced by cyclic permutations, it is said totally reduced ([3]).

<u>Lemma</u>.1 The von Neumann algebra N generated by $\{e_i; i=\pm1,\pm2,...\}$ is a subfactor of the hyperfinite II_1 factor M generated by $\{e_i; i=0,\pm1,\pm2,...\}$.

<u>Proof</u>. By the theory of the basic construction, M is a hyperfinite II_1-factor. Let ϕ be a faithful normal normalized trace on N. It is sufficient to prove that ϕ is the restriction of the

trace tr of M to N. Let A(resp.B) be the von Neumann algebra generated by $\{e_i; i=1,2,\ldots\}$(resp.$\{e_i; i=-1,-2,\ldots\}$). Then N is the σ-weak closure of linear combinations of $\{ab; a$(resp.b) is a reduced word in A(resp.B)$\}$. Since $ab=ba$ for $a\varepsilon A$ and $b\varepsilon B$, it is sufficient to prove that $\phi(wv) = tr(wv)$ for totally reduced words $w\varepsilon A$ and $v\varepsilon B$. We use a similar technique as in [3] or [6]. Let $w\varepsilon A$ and $v\varepsilon B$ be totally reduced words. Then there is an infinite sequence of totally reduced words $\{w_i\}$ in A such that $w_i=w$, $w_i w_k = w_k w_i$ for all k,i, and $tr(\Pi_{j=1}^{m} w_{k_j}) = tr(w)^m$ for all m, and $\{k_i, k_j\}$ with $k_j \neq k_i (i\neq j)$. If g is a finite permutation of positive integers, there is a unitary u_g in A such that $u_g w_i u_g^* = w_{g(i)}$ for all i by [2]. Put $p_i = w_i v$ for all i, then $\{p_i\}$ is a sequence of projections. The group S of finite permutations acts on the von Neumann algebra generated by the sequence $\{p_i\}$ by $g(p_i) = p_{g(i)}$ for all i and $g\varepsilon S$. The action is induced by $\{u_g; g\varepsilon S\}$ in A. Since ϕ is a trace on N, ϕ is invariant under the action. The action is ergodic. Hence $\phi(wv) = tr(wv)$.

(2.3) The factor M is the σ-weak closure of the union of the increasing sequence of the following von Neumann algebras$\{M_k; k=1,2,\ldots\}$:

$$M_1=\mathbb{C}1, \quad M_{2m}=\{e_j; |j| \leqq m-1\}'', \quad M_{2m+1}=\{M_{2m}, e_{2m}\}''.$$

The subfactor N of M is generated by the following increasing sequence of $\{N_k; k=1,2,\ldots\}$:

$$N_1=N_2=C1, \quad N_{2m}=\{e_j; 0\neq |j| \leqq m-1\}'', \quad N_{2m+1}=\{N_{2m}, e_{2m}\}''.$$

The algebras M_k and N_k are all finite dimensional([2]). We denote by a_k(resp.b_k) the dimension vector of M_k(resp.N_k). In the case where M_k is the direct sum of d_k matrix algebras, we say d_k the <u>dimension of the dimension vector</u> a_k.

(2.4) Every N_k is a subalgebra of M_k. Let $E(B)$ be the conditional expectation of M onto the von Neumann subalgebra B of M conditioned by $tr(xE(B)(y))= tr(xy)$ for $x\varepsilon B$ and $y\varepsilon M$.

<u>Lemma</u>.2 $E(N_{k+1})E(M_k)=E(N_k)$ and $E(N)E(M_k)=E(N_k)$ for all k.

<u>Proof</u>. Since $E(N_{k+1})E(M_k)=E(N_k)$ if and only if $E(N_{k+1})E(M_K)= E(N_{k+1})E(N_k)E(M_k)$, it is sufficient to prove that $tr(yE(N_{k+1})(x)) =tr(yE(N_k)(x))$, for $x\varepsilon M_k$, $y\varepsilon N_{k+1}$. Every reduced word $y\varepsilon N_{2m+1}$ has a form $y=vw_1e_mw_2$, where v is a reduced form on $\{e_i;i=-m+1$ $-m+2,...,-1\}$ and $w_i(i=1,2)$ is a reduced word on $\{e_i;i=1,2,...,m-1\}$. Let w be a reduced word in M_{2m}, then

$$tr(yE(N_{2m+1})(w))=tr(yw)=\lambda tr(w_2wvw_1)=\lambda tr(E(2m)(w)vw_1w_2)$$
$$=tr(w_2E(N_{2m})(w)w_1e_m)=tr(yE(N_{2m})(w)).$$

Since each algebra is generated by reduced words, $E(N_{2m+1})E(M_{2m}) =E(N_{2m})$. Similarly $E(N_{2m})E(M_{2m+1})=E(N_{2m-1})$. Since $E(_{k+1})E(M_k) =E(N_{k+i})E(M_{k+i-1})E(M_k)=E(N_{k+i-1})E(M_k)=...=E(M_k)$, $E(N)E(M_k)=E(M_k)$ for all k.

(2.5) Let (A_k) and (B_k) be sequences of finite dimensional von Neumann algebras such that B_k A_k for all k. Following after [6], we write $(A_k)_k$ $(B_k)_k$ if $(A_k)_k$ (resp. $(B_k)_k$) generates a II_1-factor A (resp. a subfactor B of A) and satisfies the property of Lemma 2. So, by (c'), Lemma 1 and Lemma 2, we have

(N_k) (M_k). Such the sequence (M_k) is said to be <u>periodic</u> with period r if there is a number m such that $[M_{n+r} \to M_{n+r+i}] = [M_n \to M_{n+i}]$ for $n \geqq m$ $(i=1,2,\ldots)$ and the matrix $[M_n \to M_{n+k}]$ is primitive for $n \geqq m$. The sequences $(M_k)_k$ $(N_k)_k$ is <u>periodic</u> if both (M_k) and (N_k) are periodic with same period r and $[N_{n+r} \to M_{n+r}] = [N_n \to M_n]$ for a large enough n ([6]). In section 6, we show the periodicity of $(N_k)_k$ $(M_k)_k$.

3. Bratteli diagram for (M_k) and path maps

For convenience' sake, throughout the bellow, we put

(3.1) for a positive integer k, $p=[\frac{k}{2}]$ and $q=k-p$.

In this section, we shall get, for the sequence (M_k) in (2.3), the components of the inclusion matrix $[M_q \to M_k]$, which we need to obtain the inclusion matrix $[N_k \to M_k]$. Let $A_k=\{1,e_1,\ldots,e_k\}''$. Then M_k is *-isomorphic to A_{k-1} for $k \geqq 2$. On the other hand there is a unitary u in M_{2m} which satisfies $ue_iu^* = e_{-i}$ and $ue_{-i}u^* = e_i$ for all $i=0,1,\ldots,m-1$ ([2]). Hence $[M_k \to M_{k+1}]=[A_{k-1} \to A_k]$ for all $k \geqq 2$. It is clear that $[M_1 \to M_2]$ is the 1 by 2 matrix $[1,1]$. In [3], Jones gets the Bratteli diagram ([1]) for the sequence (A_k), and so we get the Bratteli diagram for (M_k). The dimension vector a_k of M_k, the dimension d_k of a_k and the weight vector w_k of the restriction of tr on M_k are as follows:

(3.2) If $\lambda \leqq 1/4$, then

$$d_k= p+1, \quad a_k(i)= \begin{cases} \binom{k}{p+1-i} - \binom{k}{p-i} & \text{if } i=1,2,\ldots,d_k-1 \end{cases}$$

$1 \qquad\qquad\qquad \text{if} \quad i=d_k$

$w_k(i)=\lambda^{p+1-i} \; P_{k-1-2p+2i}(\lambda),$

where P_j is the polynomial defined in [2] by $P_1(x)=P_2(x)=1$ and $P_{n+1}(x)=P_n(x)-xP_{n-1}(x)$.

$$[M_k \to M_{k+1}] = [\ \delta_{i,j} + \delta_{i+1,j}\ ]_{i,j}, \text{ for Kronecker's } \delta_{i,j}.$$

where $i=1,2,\dots,[\frac{k+1}{2}]+1$ and $j = \begin{cases} 1,2,\dots,[\frac{k+1}{2}]+1 & \text{if } k \text{ is even} \\ 1,2,\dots,\frac{k+3}{2} & \text{if } k \text{ is odd.} \end{cases}$

(3.3) If $\lambda > 1/4$, then $\lambda = (1/4)\sec^2(\pi/n+2)$ for some $n=1,2,\dots$. The Blatteri diagram for $M_1 \subset M_2 \subset \dots \subset M_n$ has the same form as in the case of $\lambda \leqq 1/4$ and the diagram for $M_{n+2i-1} \subset M_{n+2i}$ (resp. $M_{n+2i} \subset M_{n+2i-1}$) is same as one for $M_{n-1} \subset M_n$ (resp. the reverse form of one for $M_{n-1} \subset M_n$), for all $i=0,1,2,\dots$. Hence $\{d_k, a_k, t_k\}$ follows after the movement of the diagram. For examle,

$$d_k = \begin{cases} p+1 & \text{if } k < n-1, \\ [\frac{n}{2}]+1 & \text{if } k \geqq n-1 \text{ and } n \text{ is odd,} \\ \frac{n}{2} & \text{if } k \geqq n-1,\ k \text{ is odd and } n \text{ is even,} \\ \frac{n}{2}+1 & \text{if } k \geqq n-1,\ k \text{ is even and } n \text{ is even.} \end{cases}$$

Now we consider the Bratteli diagram for (M_k) as a graph Λ, the set of vertices of which is the set of points where $a_k(i)$ $(k=1,2,\dots, i=1,2,\dots,d_k)$ stand. We denote the vertex in Λ corresponding to $a_k(i)$ by the same notation $a_k(i)$. We denote by $[a_k(i) \to a_{k+1}(j)]$ the edge from $a_k(i)$ to $a_{k+1}(j)$. A <u>path</u> on Λ is a sequence $\xi = (\xi_r)$ of edges such that $\xi_r =$

$[a_{k(r)}(i_r) \rightarrow a_{k(r)+1}(j_r)]$ for some i_r, j_r and $k(r)$ such that $k(r+1) = k(r)+1$. The set of all paths in Λ with the starting point $a_k(i)$ and the ending point $a_r(j)$ is called a <u>polygon from the vertex</u> $a_k(i)$ <u>to the vertex</u> $a_r(j)$ and denoted by $[a_k(i) \rightarrow a_r(j)]$. Also the set of all paths in Λ with $a_k(i)$ as the starting point and for some j $a_r(j)$ as the ending point is called a <u>path map from the vertex</u> $a_k(i)$ <u>to the floor</u> a_r and denoted by $(a_k(i) \rightarrow a_r)$. Let Ξ_m be the set of paths on Λ consisting of m edges. For a ξ in Ξ_1 and y in Ξ_m let $\xi\, y = \{\xi\, \eta ; \eta \varepsilon y\}$. Let $x \varepsilon \Xi_m$ be a polygon. If there are polygons y and z in Ξ_{m-1} such that as sets of paths x is either the union of ξ y and η z or the union of y ξ and z η for some ξ and η in Ξ_1, we say x is the <u>direct sum of</u> y <u>and</u> z and we write $x = y \oplus z$ or $y = x \ominus z$.

<u>Remark</u>.3 The i-th coodinate $a_k(i)$ of the dimension vector a_k represents a cardinal number of different paths in the polygon $[a_1(1) \rightarrow a_k(i)]$. In the below, we consider $a_k(i)$ as the polygon $[a_1(1) \rightarrow a_k(i)]$ and the dimension vector a_k as the path map $[a_1(1) \rightarrow a_k]$. Also, for path map $x = (x(1),\ldots,x(m))$, we denote by the same notation x the path map $(x(1),\ldots,x(m),0,\ldots,0)$.

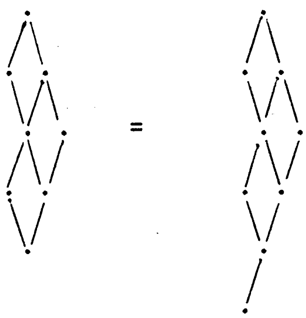
Under such the identification, we define the direct sum of path maps Let $x = (x(1),\ldots,x(h))$, $y = (y(1),\ldots,y(m))$ and $z = (z(1),\ldots,z(n))$ be path maps. If $h = \max\{h,m,n\}$ and $x(i) = y(i) + z(i)$ for every polygons $\{x(i),y(i), z(i)\}$, we say x is the <u>direct sum</u> of y and z, and we write $x = y + z$.

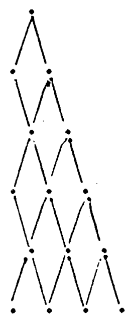
<u>Remark</u>.4 If we use the method of path model in [4], a polygon corresponds a matrix algebra and a path map corresponds a multi-matrix algebra.

Example

(1) The polygon $a_6(1) = (a_1(1) \rightarrow a_6(1))$ and the path map $a_6 = (a_1(1) \rightarrow a_6)$ are as follows in the case of either $\lambda \leqq 1/4$ or $n \geqq 6$:

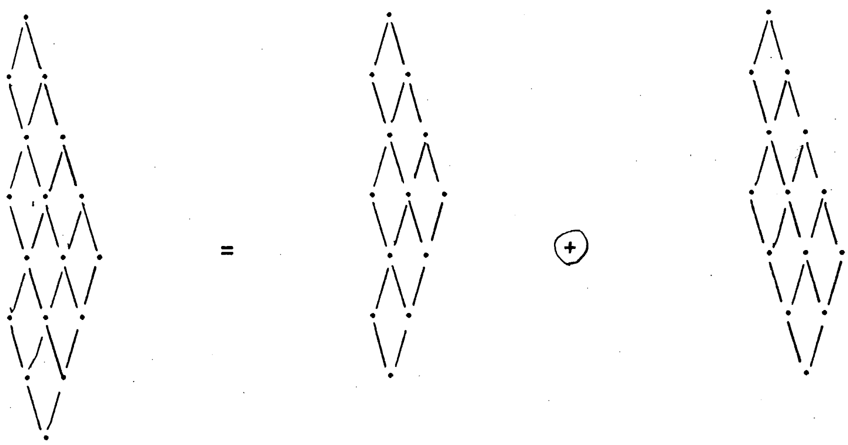
$a_6(1)$ a_6



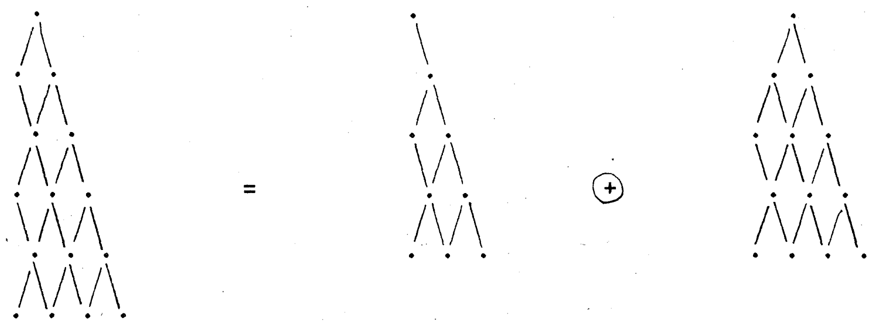


(2) Let $x \varepsilon \Xi_7$, $y \varepsilon \Xi_6$ and $z \ \varepsilon \ \Xi_6$ be polygons, then $x = y + z$ are as follows:

x y z



(3) Direct sum of path maps.



Now we discuss the inclusion matrix $[M_q \to M_k]$. It is obvious that the (i,j)-component of $[M_q \to M_k]$ means the cardinal number of $[a_q(i) \to a_k(j)]$. Hence the i-th row vector x_i of $[M_q \to M_k]$ is considerd as the path map $[a_q(i) \to a_k]$.

Under the identification of vectors and path maps, we define the polynomials $f_i(m)$ of path maps on Λ by

$$f_i(0) = a_i, \quad f_i(1) = a_{i+1} \text{ and } f_i(m+1) = f_{i+1}(m) - f_i(m-1).$$

TThen for all positive integers i and m, $f_i(2m)$ (resp. $f_i(2m+1)$) is a polynomial on path maps $\{a_{i+2j};\ j=0,1,2,\dots,m\}$ (resp. $\{a_{i+2j+1}; j=0,\ 1,2,\dots,m\}$ with positive integers as coefficients.

<u>Lemma</u>.5 Let x_i be the i-th row vector of the inclusion matrix $[M_q \to M_k]$, for a triplet (k,p,q) in (3.1). Then, the path map x_i is as follows for all i $(i=1,2,\dots,d_q)$;

$$x_i = \begin{cases} f_p(2i-2) & \text{if } q \text{ is even} \\ f_p(2i-1) & \text{if } q \text{ is odd,} \end{cases}$$

under the idenyification for vectors that $(y(1),\dots,y(m),0,\dots,0) = (y(1),\dots,y(m))$ for $y(j) \neq 0$ $(j=1,\dots,m)$.

<u>Proof</u>. Since the path map x_1 is $(a_q(1) \to a_k)$, it is clear by the shape of graph Λ that

$$x_1 = \begin{cases} a_{p+1} = f_p(1) & \text{if } q \text{ is odd} \\ a_p \quad = f_p(0) & \text{if } q \text{ is even.} \end{cases}$$

Suppose the statements are true for all $j \leqq i$. As a path map, we have

$$x_{i+1} = [a_q(i+1) \to a_k] = \begin{cases} [a_{2i}(i+1) \to a_{p+2i}] & \text{if } q \text{ is even} \\ [a_{2i+1}(i+1) \to a_{p+2i+1}] & \text{if } q \text{ is odd,} \end{cases}$$

by sliding up the line combining $a_q(1)$ and $a_q(i+1)$ as possible. Then the assumptions of the induction means that

$$[a_{2(i-1)}(i) \to a_{p+2i-2}] = f_p(2i-2)$$

and

$$[a_{2(i-1)+1}(i) \to a_{p+2(i-1)+1}] = f_p(2i-1).$$

Since

$$[a_{2i}(i) \to a_{p+2i}] + [a_{2i}(i+1) \to a_{p+2i}] = [a_{2i-1}(i) \to a_{p+2i}],$$

we have

$$[a_{2i}(i+1) \to a_{p+2i}] = [a_{2i-1}(i) \to a_{p+2i}] - [a_{2i}(i) \to a_{p+2i}]$$
$$= [a_{2(i-1)+1}(i) \to a_{p+1+2(i-1)}] - [a_{2(i-1)}(i) \to a_{p+2(i-1)}]$$
$$= f_{p+1}(2i-1) - f_p(2i-2) = f_p(2i).$$

On the other hand,

$[a_{2i+1}(i) \to a_{p+2i+1}] + [a_{2i+1}(i+1) \to a_{p+2i+1}] = [a_{2i}(i+1) \to a_{p+2i+1}]$.
Hence

$$[a_{2i+1}(i+1) \to a_{p+2i+1}] = [a_{2i}(i+1) \to a_{p+1+2i}] - [a_{2(i-1)+1}(i) \to a_{p+2(i-1)+1}]$$
$$= f_{p+1}(2i) - f_p(2i-1) = f_p(2i+1).$$

Thus $x_{i+1} = f_p(2i)$ if q is even and $x_{i+1} = f_p(2(i+1)-1)$ if q is odd.

4. Bratteli diagram for (N_k)

Let (N_k) be the sequence in (2.3). Let $N_k(+) = \{e_i \varepsilon N_k; j \geqq 1\}''$ and $N_k(-) = \{e_j \varepsilon N_k; j \leqq -1\}''$. Then N_k is generated by the commuting pair $N_k(+)$ and $N_k(-)$. For a triplet $\{k,p,q\}$ in (3.1), $N_k(+)$ is isomorphic to M_q and $N_k(-)$ is isomorphic to M_p. Two dimension vectors and weight vectors of a finite dimensional von Neumann algebra are respectively conjugate by an inner automorphism. We may take a dimension vector b_k of N_k and the weight vector u_k for the restriction of the trace tr of M to N_k as

$$(4.1) \qquad b_k = (a_p(1)a_q, a_p(2)a_q, \ldots, a_p(d_p)a_q)$$

and

$$(4.2) \qquad {}^t u_k = (w_p(1)\,{}^t w_q, t_p(2)\,{}^t w_q, \ldots, t_p(d_p)\,{}^t w_q),$$

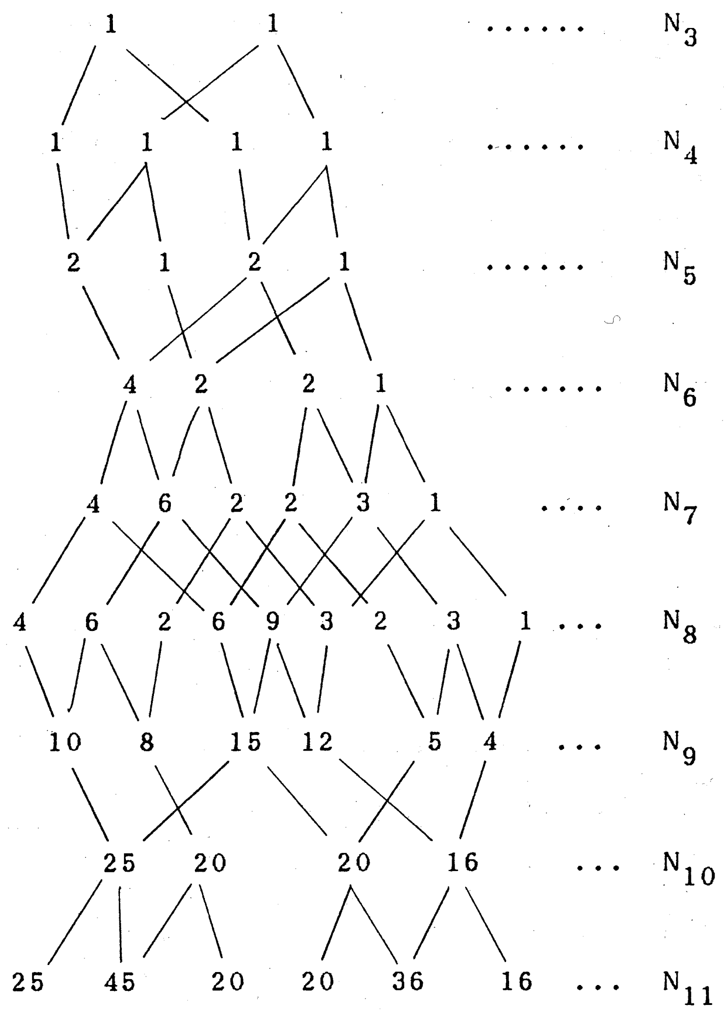
where ${}^t y$ denotes the transposed vector of the vector y.
Since we obtained the inclusion matrices for (M_k) in 3,

$$(4.3) \qquad [N_k \to N_{k+1}] = \begin{cases} I_p \quad [M_p \to M_{p+1}] & \text{if } k \text{ is odd} \\ [M_p \to M_{p+1}] \quad I_q & \text{if } k \text{ is even,} \end{cases}$$

where I_k denotes the d_k by d_k identity matrix. It is easy to check that $[N_k \to N_{k+1}]$ satisfies the property (e) for b_k and u_k. The Blatteri diagram for (N_k) comes from the diagram for (M_k) following after the above information.

In the case of $\lambda = (1/4)\sec^2(\pi/n+2)$ for some n (n=1,2,...), the diagram for $N_1 = N_2$ N_3 ... N_{2n} has the same form as in the case of $\lambda \leqq 1/4$, the diagram for $N_{2n+4i-2}$ $N_{2n+4i-1}$ (resp. $N_{2n+4i-1}$ N_{2n+1}) is similar to one for N_{2n-2} N_{2n-1} (resp. N_{2n-1} N_{2n}) and the diagram for N_{2n+4i} $N_{2n+4i+1}$ (resp. $N_{2n+4i+1}$ $N_{2n+4i+2}$) has the reverse form of order changed one for N_{2n-1} N_{2n} (resp. N_{2n-2} N_{2n}).

<u>Example</u>. In the case of n=4, the diagram is as follows;



5. Inclusion matrix of N_k in M_k.

Let $\{k,p,q\}$ be a triplet in (3.1). Let $x_i(j)$ be the (i,j)-component of $[M_q \to M_k]$ and x_i the i-th column vector of $[M_q \to M_k]$. Here we consider $x(i,j)$ and x_i as a polygon and a path map in Ξ_p. By Lemma 5, the polygon $x_i(j)$ can be decomposed into the direct sum of polygons $\{a_{p+j}(i);\ j = 0,1,\ldots,\ i = 1,2,\ldots,d_p\}$. Then we define the matrix $[a_p \to x_i] = [h(j,k)]$ such that $h(j,k)$ is the number that $a_p(j)$ is contained in $x_i(k)$. We call the matrix $[a_p \to x_i]$ the <u>inclusion matrix of the path map</u> a_p <u>in the path map</u> x_i.

<u>Remark</u>. 6 Let x, y and z be path maps on Λ such that $[x \to y]$ and $[x \to z]$ are defined. Then, by the definition of the direct sum of path maps and the inclusion matrix for path maps, the matrix $[x \to (y \oplus z)]$ is defined and

$$[x \to (y \oplus z)] = [x \to y] \oplus [x \to z].$$

By this property and Lemma 5, the inclusion matrix $[a_p \to x_i]$ of the path map a_p in the path map x_i is defined from the inclusion matrices $[M_p \to M_r]$ $(r \geqq p)$ by the natural method.

<u>Lemma</u>. 7 Let $\lambda = (1/4)\sec^2(\pi/n+2)$ and $p \geqq n-1$.

(1) If n is odd and p is even, then

$$[a_p \to f_p(m)](i,j) = \begin{cases} 1, & -[\frac{m}{2}] \leqq i-j \leqq [\frac{m+1}{2}],\ [\frac{m}{2}]+2 \leqq i+j \leqq 2[\frac{n}{2}]-[\frac{m-1}{2}] \end{cases}$$

$$\left.\right\} 0, \quad \text{otherwise.}$$

If n is odd and p is odd, then

$$[a_p \to f_p(m)](i,j) = \begin{cases} 1, & -[\frac{m+1}{2}] \leqq i-j \leqq [\frac{m}{2}],\ 1+[\frac{m-1}{2}] \leqq i+j \leqq 2[\frac{n}{2}]-[\frac{m}{2}] \\ 0, & \text{otherwise.} \end{cases}$$

(2) If n is even and p is odd, then

$$[a_p \to f_p(m)](i,j) = \begin{cases} 1, & -[\frac{m+1}{2}] \leqq i-j \leqq [\frac{m}{2}],\quad 1+[\frac{m+1}{2}] \leqq i+j \leqq 2[\frac{n}{2}]-[\frac{m}{2}] \\ 0, & \text{otherwise.} \end{cases}$$

If n is even and p is even, then

$$[a_p \to f_p(m)](i,j) = \begin{cases} 1, & -[\frac{m}{2}] \leqq i-j \leqq [\frac{m+1}{2}],\quad [\frac{m}{2}]+2 \leqq i+j \leqq 2[\frac{n}{2}]-[\frac{m+1}{2}] \\ 0, & \text{otherwise.} \end{cases}$$

Proof. It is sufficient to prove the statement for $p=n-1$ and $p=n$, because $f_p(m)$ is the polinomial on $\{a_{p+j}; j=[\frac{m}{2}],\ j$ is odd(resp. even)$\}$ if m is odd (resp. even) and $[a_p \to a_{p+j}] = [a_{p+2} \to a_{p+2+j}]$ for all $p \geqq n-1$ and j. Since $f_p(1) = a_{p+1}$, it is clear that $[a_p \to f_p(1)]$ satisfies the conditions for all n and p. For a given n, assume that the statements hold for $p=n-1$, n and $m=1,2,\ldots,k$. Then we can give a proof of the statements for $p=n-1$, n and $m=k+1$ by the relation;

$$[a_p \to f_p(k+1)] = [a_p \to a_{p+1}][a_{p+1} \to f_{p+1}(k)] - [a_p \to f_p(k-1)]$$

and

$$[a_{n+1} \to f_{n+1}(k)] = [a_{n-1} \to f_{n-1}(k)].$$

<u>Lemma</u>.8 Let $\lambda = (1/4)\sec^2(\pi/n+2)$ and x_i the i-th column vector of $[M_q \to M_k]$. Assume $q \geqq n$.

(1) If n is odd, then $[a_p \to x_i]$ is a $(1+[\frac{n}{2}]$ square matrix with the following form:

(1.1) If $p=q$ is an odd number, then

$$[a_p \to x_i](j,l) = \begin{cases} 1, & 1-i \leqq l-j \leqq i < j+l \leqq n+2-i \\ 0, & \text{otherwise.} \end{cases}$$

(1.2) If $p+1 = q$ is even, then

$$[a_p \to x_i](j,l) = \begin{cases} 1, & |l-j| < i \leqq j+l \leqq n+2-i \\ 0, & \text{otherwise.} \end{cases}$$

(1.3) If $p=q$ is even, then

$$[a_p \to x_i](j,l) = \begin{cases} 1, & |l-j| < i < j+l \leqq n+3-i \\ 0, & \text{otherwise.} \end{cases}$$

(1.4) If $p+1 = q$ is odd, then

$$[a_p \to x_i](j,l) = \begin{cases} 1, & -i \leqq l-j < i < j+l \leqq n+2-i \\ 0, & \text{otherwise.} \end{cases}$$

(2) Let n is even.

(2.1) If $p = q$ is odd, then $[a_p \to x_i]$ is an $n/2$ by $1+(n/2)$ matrix with

$$[a_p \to x_i](j,l) = \begin{cases} 1, & 1-i \leqq l-j \leqq i < j+l \leqq n+2-i \\ 0, & \text{otherwise.} \end{cases}$$

(2.2) If $p+1=q$ is even, then $[a_p \to x_i]$ is an $n/2$ square matrix with

$$[a_p \to x_i](j,l) = \begin{cases} 1, & |l-j| < i \leqq j+l \leqq n+2-i \\ 0, & \text{otherwise.} \end{cases}$$

(2.3) If $p = q$ is even, then $[a_p \to x_i]$ is a $1+(n/2)$ square matrix with

$$[a_p \to x_i](j,l) = \begin{cases} 1, & |l-j| < i < j+l \leqq n+3-i \\ 0, & \text{otherwise} \end{cases}$$

(2.4) If $p+1 = q$ is odd, then $[a_p \to x_i]$ is a $1+(n/2)$ by $n/2$ matrix with

$$[a_p \to x_i](j,l) = \begin{cases} 1, & -i \leqq l-j < i < j+l \leqq n+2-i \\ 0, & \text{otherwise.} \end{cases}$$

<u>Proof</u>: Let n be odd. Then $d_j = d_{n-1}$ for all $j \geqq n-1$. Since $d_{n-1} = [\frac{n}{2}]+1$, $[M_q \to M_k]$ is a $1+[\frac{n}{2}]$ square matrix. It means that a_j $(j \geqq n-1)$ and each x_i are path maps consisting of $1+[\frac{n}{2}]$ polygons in Ξ_{p+1}. Similarly, if n is even, then a_j is a path map with

$[\frac{n}{2}]$ (resp. $[\frac{n}{2}]+1$) polygons for odd (resp. even) $j \geqq n-1$. Hence x_i is a path map with $[\frac{n}{2}]$ (resp. $[\frac{n}{2}+1)$ polygons if k is odd(resp. even).Therefore by Lemma 5 and Lemma 7, the statements hold.

Lemma. 9 For the weight vector w_k of the restriction of tr to M_k, we have

$$[a_p \to x_i]w_k = w_q(i)w_p \qquad (i = 1,2,\ldots,d_q).$$

Proof. We denote the matrix $[[a_p \to a_{p+i}],0,\ldots,0]$ by the same notation $[a_p \to a_{p+i}]$, where 0 is the row vector with all components 0. Then by the Bratteli diagram for (M_k), we have for all i $(i=0,1,\ldots)$

$$[a_p \to a_{p+i}]w_k = \lambda^{n(i)}w_p \quad \text{for} \quad n(i) = [\frac{q}{2}] - [\frac{i}{2}].$$

Since x_i is given by the polynomials f_i on $\{a_{p+i};\ j=0,1,\ldots\}$ by Lemma 5, we have the statement by Lemma 6, (3.2) and the relation between the polynomial f_j's and P_j's, because

$$w_k(i) = \lambda^{p+1-i}P_{k-1-2p+2i}(\lambda),$$

where P_j is the polynomial defined in [2] by $P_1(x) = P_2(x) = 1$ and $P_{n+1}(x) = P_n(x) - xP_{n-1}(x)$.

Let G_k be the d_pd_q by d_k matrix, the$(d_q(j-1)+i)$-th column vector of which is the j-th column vector of the matrix $[a_p \to x_i]$, where $i = 1,2,\ldots,d_q$, $j= 1,2,\ldots, d_p$. That is,the transposed matrix ${}^tG_k$ of G_k is as follows;

$${}^{t}G_k = [G[1]_1, G[2]_1, \ldots, G[d_q]_1, G[1]_2, \ldots, G[d_q]_2, \ldots, G[1]_{d/p}, \ldots G[d_q]_{d/p}],$$

where $G[i]_j$ is the transposed vector of the j-th column vector of $[a_p \to x_i]$.

Lemma. 10 The matrix G_k satisfies the following;

$$b_k G_k = a_k, \quad G_k w_k = u_k \quad \text{and} \quad G_k[M_k \to M_{k+1}] = [N_k \to N_{k+1}]G_{k+1},$$

where a_k, b_k are dimension vectors of M_k, N_k and w_k, u_k are weight vectors of M_k, N_k.

Proof. Since $a_q[M_q \to M_k] = a_k$, we have, by the relation (4.1),

$$b_k G_k = \Sigma_i\ a_q(i) a_p[a_p \to x_i] = \Sigma_i\ a_q(i) x_i = a_k,$$

where i runs over $\{1, 2, \ldots, d_q\}$.

Lemma 7 implies that $G_k w_k = u_k$, combining the definition of G_k and (4.2).

If $\lambda > 1/4$ and $k \geqq 2n$, by Lemma 8, we have $G_k[M_k \to M_{k+1}] = [N_k \to N_{k+1}]G_{k+1}$. For another case, we need a similar lemma as Lemma 8. In the below we does not need such cases. Hence we omit to give a proof of such cases.

Thus we can get a method of inclusion of N_k in M_k. Hence we denote G_k by $[N_k \to M_k]$.

6. Periodicity of $(N_k) \subset (M_k)$ in the case of $\lambda > 1/4$.

In this section, we assume that $\lambda = (1/4)\sec^2\pi/(n+2)$ for some n $(n = 1,2,\ldots)$.

Lemma. 11 The sequence (M_k) is periodic with period 2 and the sequence (N_k) is periodic with period 4.

Proof. Combining the discussions in (2.5) and section 3 with results in [2] or [6], we have that the sequence (M_k) is periodic with period 2.
The fact implies that (N_k) is periodic with period 4, by Lemma 1 and the Bratteli diagram for (N_k).

Lemma. 12 Let x_i (resp. y_i) be the i-th column vector of $[M_q \to M_k]$ (resp. $[M_{q+2} \to M_{k+4}]$). If $q \geqq n$, then

$$[a_p \to x_i] = [a_{p+2} \to y_i] \qquad (i=1,2,\ldots d_q).$$

Proof. First we remark that both $[M_q \to M_k]$ and $[M_{q+2} \to M_{k+4}]$ are d_q by d_k matrices, because (M_k) is periodic with period 2 and $[Mq+2 \to M_{k+4}] = [M_q \to M_k][M_k \to M_{k+2}]$. Since $p = [\frac{k}{2}]$ and $q = k-p$, we have $p+2 = [\frac{k+4}{2}]$ and $q+2 = (k+4 - (p+2)$, that is, $\{k+4, p+2, q+2\}$ satisfies (3.1). Hence $x_i = f_p(2i-2)$ (resp. $x_i = f_p(2i-1)$) if and only if $y_i = f_{p+2}(2i-2)$ (resp. $f_{p+2}(2i-1)$). By the definition, $f_j(2m)$ (resp. $f_j(2m+1)$) is a linear combination on $\{a_j, a_{j+2}, \ldots, a_{j+2m}\}$ (resp. $\{a_{j+1}, a_{j+3}, \ldots, a_{j+2m+1}\}$) with integer coefficients. Therefore, by Remark 6, we have $[a_p \to x_i] = [a_{p+2} \to y_i]$, because (M_k) is periodic with period 2.

Lemma. 13 The sequence $(N_k) \subset (M_k)$ is periodic.

Proof. We already proved that both (M_k) and (N_k) are periodic with same period 4. Hence it is sufficient to prove that

$$[N_k \to M_k] = [N_{k+4} \to M_{k+4}] \quad \text{for} \quad k \geqq 2n.$$

By the form of the matrix $[N_k \to M_k] = G_k$, it is nothing else but Lemma 12. Thus $(N_k) \subset (M_k)$ is periodic.

7. Proof of Theorem.

Lemma. 14 If $\lambda = (1/4)\sec^2(\pi/m)$ for some m $(m = 3,4,\ldots)$, then

$$[M:N] = (m/4)\operatorname{cosec}^2(\pi/m).$$

Proof. The factors M and N are generated by the periodic sequences $(N_k) \subset (M_k)$ of finite dimensional algebras. Hence, by [6;Theorem 1.5], for the weight vectors w_k and u_k of the restriction tr to M_k and N_k, we have that $[M:N] = ||u_k||_2^2/||w_k||_2^2$ for a large enough k. By (4.2),

$$||u_k||_2^2 = ||w_p||_2^2 \, ||w_q||_2^2 \quad \text{for a } (k,p,q) \text{ in } (3.1).$$

Put $n = m - 2$. Then we have

$$[M:N] = ||u_k||_2^2 \,/\, ||w_k||_2^2 \quad \text{for all} \quad k \geqq n-1.$$

Since $||w_k||_2^2 \,/\, ||w_{k+1}||_2^2 = 1/\lambda$ for all $k \geqq n-1$,

$$[M:N] = ||w_{n-1}||_2^4 \,/\, ||w_{2(n-1)}||_2^2 = ||w_{n-1}||_2^2 \,/\, \lambda^{n-1}.$$

By (3.3),

$$||w_{n-1}||_2^2 = \Sigma_j \lambda^{2j} P_{n-2j}(\lambda)^2, \text{ where } j \text{ runs over } \{0,1,\dots,[\tfrac{n-1}{2}]\}.$$

On the other hand, by [3],

$$P_k((1/4)\sec^2\theta) = \sin k\theta \,/\, 2^{k-1} \cos^{k-1}\theta \sin\theta \quad \text{for all} \quad k \text{ and } \theta.$$

Hence

$$[M:N] = \Sigma_j \sin^2((n-2j)\pi/(n+2)) \,/\, \sin^2(\pi/(n+2))$$

$$=\Sigma_j \{2-\exp(2(n-2j)/(n+2))\pi i - \exp(2(2j-n)/(n+2))\pi i\} \,/\, 4\sin^2(\pi/(n+2))$$

$$= ((n+2)/4)\operatorname{cosec}^2(\pi/(n+2)) = (m/4) \operatorname{cosec}^2(\pi/m),$$

because $\Sigma_{j=1}^k \exp((j/k)2\pi i) = 0$, for all integer k.

Remark. 15 (1) If $m = 3$ or 4, then $[M:N] = [P:Q]$ for the subfactor $Q = \{e_i;\ i=2,3,\dots\}''$ of the factor $P = \{e_i; i=1,2,\dots\}''$. That is, $[M:N\} = 1$ if $m=3$ and $[M:N] = 2$ if $m = 4$.

(2) If $m \geqq 5$, then $[M:N] \neq [P:Q]$. If $m = 5$, then $[M:N]<4$.

Hence trere is an integer k $(k \geqq 3)$ such that $[M:N] = 4\cos^2(\pi/k)$. H. Choda gets the number k, that is $k = 10$. (Here the author thank to H. Choda for helping her by computing a lot of indices $[M:N]$.) On the other hand, by the proof of Lemma 14,

$$[M:N] = 4\cos^2(\pi/3) + 4\cos^2(\pi/5).$$

This implies the following equation (the equation is proved by an ellementary method, which M.Fujii tells us);

$$\cos^2(\pi/3) + \cos^2(\pi/5) = \cos^2(\pi/10).$$

Lemma. 16 The relative commutant $N' \cap M$ of N in M is trivial.

Proof. Since $[M:N]$ is finite, $N' \cap M$ is finite dimensional. Let c be the dimension vector of $N' \cap M$. Since $(M_k) \supset (N_k)$ is periodic, by [6:Theorem 1.7],

$$||c||_1 \leqq \alpha = \min(||G[i]_j||_1 \ ; k \geqq 2n, i=1,2,\ldots,d_q, j=1,2,\ldots,d_p),$$

where $G[i]_j$ is the vector in the section 5. By Lemma 8, there are many (i,j)'s such that ${}^tG[i]_j = (1,0,\ldots,0)$. It implies $\alpha = 1$. Hence $N' \cap M$ is 1-dimensional, so that $N' \cap M = \mathbb{C}1$.

8. A generalization

Let take and fix a positive integer n. Let

$$L = \{,,,. \ e_{-n-1}, e_{-n}, \ e_1, e_2, e_3, \ldots\}''.$$

In the case of $n = 1$, $L = N$. By a similar proof as Lemma 1, L is a subfactor of M, for all n. Also, L is a subfactor of N and $[N:L] = 4\cos^2(\pi/m)$. Hence

$$[M:L] = (m/4)\mathrm{cosec}^2(\pi/m)\{4\cos^2(\pi/m)\}^{n-1}.$$

Let

$$L_1 = L_2 = \mathbb{C}1, \quad L_{2i-1} = L_{2i} = \{e_i; i=1,2,\ldots,n-1\}'' \quad \text{if} \quad i \leqq n$$

and

$$L_{2i+1} = \{L_{2i}, e_i\}'' \quad L_{2i+2} = \{e_{-i}, L_{2i+1}\}'' \quad \text{if} \quad i \geqq n.$$

The sequence (L_k) is periodic with period 4 and generates L. By a similar method as for $(N_k) \subset (M_k)$, we get the inclusion matrix $[L_k \to M_k]$. For a triplet (k,p,q) in (3.1), we consider the matrix $[a_{p-(n-1)} \to x_i]$ for a large k, where x_i is the same as in section 3, that is the i-th column vector of $[M_q \to M_k]$. Then $(N_k) \subset (M_k)$ is periodic. Let h be the dimension vector of $L' \cap M$.

If q is even, then $x_1 = a_p$, hence $[a_{p-(n-1)} \to x_1] = [a_{p-(n-1)} \to a_p]$.

If $n = 2$, we have $N' \cap M = \mathbb{C}1$, by the form of $[a_k \to a_{k+1}]$ for an odd k.

If $n \geqq 3$, $\{e_{-n+2}, e_{-n+3} \ldots, e_{-1}\}''$ is contained in $L' \cap M$ and isomorphic to M_{n-1}. Hence we have

$$L' \cap M = \{e_{-n+2}, e_{-n+3}, \ldots, e_{-1}\}''.$$

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尚、この結果は、 OCNEANU により、 JONES の問題として WARWICK の研究集会で紹介されているものの解になっている。