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INDECOMPOSABLE POSITIVE MAPS IN MATRIX ALGEBRAS

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We prove that Choi's map in M_3 cannot be written as the sum of a 2-positive map and a 2-copositive map. We also provide other examples of positive maps in M_n which cannot be written as the sum of an n-positive map and a 2-copositive map.

§1. Introduction.

Let M_n be the matrix algebra of order n and $\mathcal{P}(M_n)$ be the set of all positive linear maps in M_n. One of the basic problems about the structure of $\mathcal{P}(M_n)$ is whether there exists some small number of simpler convex cones in $\mathcal{P}(M_n)$ with which every positive map can be written as a sum. Two convex cones were proposed as candidates, the cone of completely positive maps and the cone of completely copositive maps. With these cones the program was successful at least for the algebra M_2 ([9],[14]). That this is not the case for higher dimensional algebras was shown by Choi [3] by an example of an indecomposable positive map in M_3. Woronowicz [14] also showed the existence of such indecomposable maps.

In this paper we provide examples of positive maps in M_n which may be considered as proper extensions of Choi's map, and we prove even stronger indecomposability of such maps (Theorem 1 and 4). On the other hand, we believe that from the above strategy towards the structure theory of the set $\mathcal{P}(M_n)$ what is important is not merely the existence of extremal positive maps but the existence and behavior of those positive maps which are neither 2-positive nor 2-copositive. An example of such an " atomic " positive map in M_4 has been recently observed by Robertson [8]. We shall show that Choi's map in M_3 is also atomic. Although this map is known to be extremal by [4], it is not easy to show the property and thus our direct proof of the atomic property (Theorem 5) would be of own interest.

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§2. Examples of positive maps in M_n.

Throughout this paper we denote by $\{ e_{ij} \}_1^n$ the canonical matrix units in the algebra M_n for which we always assume that $n \geq 3$. The algebra $M_k(M_n)$ means the block matrix algebra of order k over M_n. Let τ be a linear map in M_n. The map τ is said to be k-positive (resp. k-copositive) if the k-multiplicity map $\tau(k)$ (resp. k-comultiplicity map $\tau^c(k)$),

$$\tau(k): [\, a_{ij} \,]_1^k \in M_k(M_n) \longrightarrow [\, \tau(a_{ij}) \,]_1^k$$

$$(\text{resp. } \tau^c(k): [\, a_{ij} \,]_1^k \in M_k(M_n) \longrightarrow [\, \tau(a_{ji}) \,]_1^k)$$

is positive. If $\tau(k)$ is positive for every k, then τ is said to be completely positive. It is, however, known that in M_n positivity of the maps saturates at the order n, that is, every n-positive map is completely positive. Completely copositive maps are defined in a similar way and the saturation of copositivity in M_n also occurs.

Let ε be the projection of norm one of M_n to the diagonal part and let s be the shift unitary in M_n such that

$$s = [\ \delta_{i\ j+1}\ ]$$

where indexes are understood to be mod n.

A prototype of Choi's map is then written as

$$\Phi(x) = 2\varepsilon(x) + \varepsilon(sxs^*) - x\ , \quad x \in M_3.$$

The map Φ cannot be written as a sum of a completely positive (i.e. 3-positive) map and a completely copositive (i.e. 3-copositive) map. The following map may be regarded as an extension of Φ to general matrix algebras.

Therem 1. Define the map τ_1 in M_n by

$$\tau_1(x) = (n-1)\varepsilon(x) + \varepsilon(sxs^*) - x.$$

Then, τ_1 is a positive map.

For the proof we need two lemmas.

Lemma 2. Let a be a positive invertible operator on a Hilbert space and let ξ_0 be the unit vector associated with a one dimensional projection p. Then $a \geq p$ if and only if

$(a^{-1}\xi_0, \xi_0) \le 1.$

The result is rather a standard one, and hence we only mention the equality,

$$\sup\{ (p\xi,\xi)/(a\xi,\xi) \mid \|\xi\| = 1 \} = (a^{-1}\xi_0, \xi_0).$$

The next lemma plays a key role in our discussions.

Lemma 3. Let $\lambda_1,\lambda_2,\ldots,\lambda_n$ be positive numbers. Then

$$\sum_{i=1}^{n} \frac{\lambda_i}{(n-1)\lambda_i + \lambda_{i-1}} \le 1$$

where we put $\lambda_0 = \lambda_n$.

Proof. Put $a_i = \lambda_{i-1}/\lambda_i$. Then the inequality changes into the form $\sum_{i=1}^{n} \frac{1}{n-1+a_i} \le 1$ where $\{ a_i \}$ are positive numbers with $a_1a_2\cdots a_n = 1$. Thus it suffices to show that

$$\begin{aligned} T = \ & (n-1+a_1)(n-1+a_2)(n-1+a_3)\cdots(n-1+a_n) \\ & - 1\,(n-1+a_2)(n-1+a_3)\cdots(n-1+a_n) \\ & - (n-1+a_1)\,1\,(n-1+a_3)\cdots(n-1+a_n) \\ & \cdots \\ & - (n-1+a_1)(n-1+a_2)(n-1+a_3)\cdots 1 \end{aligned}$$

is non-negative. Let T_k $(1 \le k \le n-1)$ be the sum of those terms of the form $\lambda(i_1,i_2,\ldots,i_k)a_{i_1}a_{i_2}\cdots a_{i_k}$ where $\lambda(i_1,i_2,\ldots,i_k)$ is the coefficient of $a_{i_1}a_{i_2}\cdots a_{i_k}$. Notice that here the

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coefficient $\lambda(i_1, i_2, \ldots, i_k)$ does not depend on the choice of $\{ a_{i_1}, a_{i_2}, \ldots, a_{i_k} \}$. It follows that

$$T_k = \mu_k \sum_{i_1 < i_2 < \cdots < i_k} a_{i_1} a_{i_2} \cdots a_{i_k}$$

where

$$\begin{aligned} \mu_k &= \text{coefficient of } a_1 a_2 \cdots a_k \\ &= (n-1)^{n-k} - (n-k)(n-1)^{n-k-1} \\ &= (k-1)(n-1)^{n-k-1} \geq 0 . \end{aligned}$$

On the other hand, since $a_1 a_2 \cdots a_n = 1$, we have

$$\sum_{i_1 < i_2 < \cdots < i_k} a_{i_1} a_{i_2} \cdots a_{i_k} \geq {}_nC_k .$$

Here we note that ${}_nC_k$ is the value of the left member when

$$a_1 = a_2 = \cdots = a_n = 1.$$

Since the other terms are constant with respect to the a_i's, the minimum of T for valuable a_i's is obtained in the above case. Therefore,

$$T \geq n^n - n \cdot n^{n-1} = 0,$$

and the proof is completed.

Proof of Theorem 1. It suffices to show that $\tau_1(p) \geq 0$ for every one dimensional projection p, that is,

$$(n-1)\varepsilon(p) + \varepsilon(sps^*) \geq p.$$

Let $\xi_0 = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be the unit vector associated with p. Without loss of generality we may assume that $\alpha_i \neq 0$ for every i. Then the matrix

$$a = \begin{bmatrix} (n-1)|\alpha_1|^2 + |\alpha_n|^2 & 0 & \cdots & 0 \\ 0 & (n-1)|\alpha_2|^2 + |\alpha_1|^2 & \cdots & 0 \\ & & \cdots & \\ 0 & 0 & \cdots & (n-1)|\alpha_n|^2 + |\alpha_{n-1}|^2 \end{bmatrix}$$

is clearly positive and invertible. It follows by Lemma 2 that the above inequality reduces to the form

$$\sum_{i=1}^{n} \frac{|\alpha_i|^2}{(n-1)|\alpha_i|^2 + |\alpha_{i-1}|^2} = (a^{-1}\xi_0, \xi_0) \le 1,$$

but this inequality holds by Lemma 3. Thus the map τ_1 is positive.

There is also an example of a positive map τ_{n-2} due to Ando [1] which may be considered as another extension of Φ to M_n:

$$\tau_{n-2}(x) = 2\varepsilon(x) + \sum_{i=1}^{n-2} \varepsilon(s^i x s^{*i}) - x .$$

Following an observation due to Y. Nakamura, we suspect that there would be a series of positive maps connecting τ_1 and τ_{n-2}, namely,

$$\tau_k(x) = (n - k)\varepsilon(x) + \sum_{i=1}^{k} \varepsilon(s^i x s^{*i}) - x.$$

However, we have not been able to establish the positivity of these maps, or the indecomposability of τ_{n-2}. Finally, we remark that the map τ_{n-1} defined as

$$\tau_{n-1}(x) = \varepsilon(x) + \sum_{i=1}^{n-1} \varepsilon(s^i x s^{*i}) - x$$

is completely copositive. In fact, as in the case of completely positive maps, completely copositivity of τ_{n-1} is equivalent to positivity of the matrix

$$[\ \tau_{n-1}(\ e_{ji}\ )\ ]_1^n \ = \ 1 - [\ e_{ji}\ ]_1^n ,$$

and as $[\ e_{ji}\ ]_1^n$ is a selfadjoint unitary element, this matrix is positive.

§3 Indecomposable positive maps in M_n.

In this section we study the problem of decomposability of a positive map in M_n, not only as the sum of an n-positive map and an n-copositive map, but also as a sum of maps in any two categories with higher order of positivity than the original one. With this point of view in mind, we call a positive map in M_n an atom if it can not be written as the sum of a 2-positive map and a 2-copositive map. We shall show that there always exists such an atomic positive map in M_n if $n \geq 3$. The mutual independence of atomic maps in an appropriate sense and the number of independent atomic maps would be an interesting next problem for the structure of $\mathcal{P}(M_n)$, which will be discussed elsewhere.

Theorem 4. Let k be the integer such that $k = \frac{n-1}{2}$ if n is odd and $k = \frac{n}{2} - 1$ if n is even. Then the positive map

$$\tau(x) = (\ n - 1\ )\varepsilon(x) + \sum_{i=1}^{k} \varepsilon(\ s^i x s^{*i}\ ) - x$$

is not decomposable into the sum of an n-positive map and a 2-copositive map.

Proof. Suppose that $\tau = \rho_1 + \rho_2$ for a n-positive map ρ_1 and a 2-copositive map ρ_2. Since the matrix $[\ e_{ij}\ ]_1^n$ is positive, the matrix

$$[\ \rho_1(\ e_{ij}\ )]_1^n = [\ a_{ij}\ ]_1^n$$

is positive whereas for the matrix

$$[\ \rho_2(\ e_{ij}\ )]_1^n = [\ b_{ij}\ ]_1^n$$

every 2×2 submatrix is the transpose of a positive matrix. Now since $\tau(\ e_{ii}\ )$ is orthogonal to any projection e_{ll} for $l = i+k+1, i+k+2, \ldots, i+n-1$ (where indexes are understood to be mod n), we have, considering the positivity of a_{ii} and b_{ii},

$$e_{ll}b_{ii} = b_{ii}e_{ll} = 0$$

for $i = 1,2,\ldots,n$ and $l = i+k+1,\ i+k+2,\ \ldots,\ i+n-1$.

It follows that

$$e_{ll}b_{ji} = b_{ij}e_{ll} = 0$$

for every projection e_{ll} for $j = 1,2,\ldots,n$ because

$$\begin{bmatrix} b_{ii} & b_{ji} \\ b_{ij} & b_{jj} \end{bmatrix} \geq 0\ ,$$

and we may assume (cf. [6]) that the above matrix is a sum of matrices of the form

$$\begin{bmatrix} b_i^*b_i & b_i^*b_j \\ b_j^*b_i & b_j^*b_j \end{bmatrix} .$$

Write

$$[\ \tau(\ e_{ij}\ )]_1^n = (\ n-1\ )\begin{bmatrix} e_{11} & 0 & \ldots & 0 \\ 0 & e_{22} & \ldots & 0 \\ & & \ldots & \\ 0 & 0 & \ldots & e_{nn} \end{bmatrix} - [\ e_{ij}\ ]_1^n$$

$$+ \begin{bmatrix} e_{22}+e_{33}+\ldots+e_{k+1k+1} & 0 & \ldots & 0 \\ 0 & e_{33}+e_{44}+\ldots+e_{k+2k+2} & \ldots & 0 \\ & & \ldots & \\ 0 & 0 & \ldots & e_{11}+e_{22}+\ldots+e_{kk} \end{bmatrix}$$

and note that $[\ e_{ij}\ ]_1^n = np_0$ for a one dimensional projection p_0

which is majorized by the projection q in the first term and is orthogonal to the third projection. Therefore,

$$q\,[\,\tau(\,e_{ij}\,)]\,q = (\,n-1\,)q - np_0 = (\,n-1\,)(\,q-p_0\,) - p_0$$

which is clearly not positive. On the other hand, we assert that the matrix $q[\,b_{ij}\,]q$ is of diagonal form, so that the right member of $q\,[\,\tau(\,e_{ij}\,)\,]\,q,\ q\,(\,[\,b_{ij}\,]\,)\,q$ becomes positive, a contradiction. In fact, as $q\,[\,b_{ij}\,]\,q = [\,e_{ii}b_{ij}e_{jj}\,],\ e_{ii}b_{ij}e_{jj} = 0$ for $i = 1,2,\ldots,n$ and $j = i+k+1,\ i+k+2,\ \ldots,\ i+n-1$ (where indexes are understood to be mod n), we have $e_{ii}b_{ij}e_{jj} = 0$ for $i \neq j$ because $b_{ij} = b_{ji}$. This completes the proof.

As an immediate consequence we see that the map τ_1 has the same degree of indecomposability. Furthermore, the trivial embedding of Choi's map Φ into $\mathcal{P}(M_n)$ shows that there exists at least one positive map in M_n ($n \geq 3$) which is not written as the sum of a 3-positive map and a 2-copositive map. However, as mentioned above, we can prove a stronger result, namely the existence of an atomic positive map in M_n ($n \geq 3$).

In [14], Woronowicz has proved that there exists an indecomposable positive map $\tau_w\colon M_2 \longrightarrow M_4$. Let $\sigma\colon M_4 \longrightarrow M_2$ be the norm 1 projection which $\sigma(\,[\,a_{ij}\,]_1^4\,) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Then $\tau_w \circ \sigma\colon M_4 \longrightarrow M_4$ is a positive map which cannot be decomposable into the sum of a 2-positive map and 2-copositive map. Therefore this embedding procedure shows the existence of an atomic map in M_n for $n \geq 4$. But unfortunately his proof is not constructive and rather complicated. On the other hand, as mentioned before there is a constructive example of an atomic map in M_4 by [8].

Theorem 5. Choi's map Φ is not decomposable into the sum of a 2-positive map and a 2-copositive map.

If we make use of the extremal property of Φ by [4] all what we have to show are Lemma 6 and the fact that Φ is not 2-copositive. But the following direct proof would be of own interest.

Lemma 6. Φ is not 2-positive.

Proof. If Φ were 2-positive, it would satisfy the Schwartz inequality,

$$\Phi(x)^*\Phi(x) \leq 2\Phi(x^* x), \quad x \in M_3.$$

But if we take the matrix

$$x = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix},$$

a straightforward calculation yields that

$$2\Phi(x^*x) - \Phi(x)^*\Phi(x) = \begin{bmatrix} 2 & -4 & 1 \\ -4 & 6 & 1 \\ 1 & 1 & 3 \end{bmatrix},$$

and it is easily seen that this matrix is not positive. This shows that the Schwartz inequality for Φ does not hold.

Proof of Theorem 5. Suppose that $\Phi = \rho_1 + \rho_2$ where ρ_1 is 2-positive and ρ_2 is 2-copositive. It is enough to prove that $\rho_2 = 0$, thus contradicting Lemma 6. We shall use the same notations as in the proof of Theorem 4 for the map Φ, such as

$$[\ \rho_1(\ e_{ij}\ )\ ]_1^3 = [\ a_{ij}\ ]_1^3\ ,\ q = \begin{bmatrix} e_{11} & 0 & 0 \\ 0 & e_{22} & 0 \\ 0 & 0 & e_{33} \end{bmatrix},\ \text{etc.}$$

Then the same argument as there shows that

$$e_{33}b_{11} = b_{11}e_{33} = e_{11}b_{22} = b_{22}e_{11} = e_{22}b_{33} = b_{33}e_{22} = 0,$$

and

$$q\ [\ \Phi(\ e_{ij}\ )\ ]_1^3\ q = 2q - [\ e_{ij}\ ]_1^3$$

$$= q\ [\ a_{ij}\ ]_1^3\ q \ + \begin{bmatrix} e_{11}b_{11}e_{11} & 0 & 0 \\ 0 & e_{22}b_{22}e_{22} & 0 \\ 0 & 0 & e_{33}b_{33}e_{33} \end{bmatrix}.$$

Multiply by the matrix $2p' = \begin{bmatrix} e_{11} & e_{12} & 0 \\ e_{21} & e_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ from both sides. Then, since the projection p' is majorized also by q, the first term vanishes, whence

$$p'\ [\ a_{ij}\ ]_1^3\ p' \ + p' \begin{bmatrix} e_{11}b_{11}e_{11} & 0 & 0 \\ 0 & e_{22}b_{22}e_{22} & 0 \\ 0 & 0 & e_{33}b_{33}e_{33} \end{bmatrix} p' = 0.$$

By the 2-positivity of ρ_1, the first term is positive, and hence it must be zero together with the second term. This implies that

$$e_{11}b_{11}e_{11} + e_{12}b_{22}e_{21} = 0,$$

and, as the b_{ii}'s are positive,

$$e_{11}b_{11} = b_{11}e_{11} = e_{22}b_{22} = b_{22}e_{22} = 0.$$

Similarly, we have that

$$e_{33}b_{33} = b_{33}e_{33} = 0.$$

Therefore,

$$b_{ii} = e_{i+1\ i+1}\ b_{ii}\ e_{i+1\ i+1} = \lambda_{i+1}\ e_{i+1\ i+1}\ ,$$

where the index i is understood to be mod 3, and $0 \le \lambda_i \le 1$ because

$$e_{i+1\ i+1} = e_{i+1\ i+1}\ a_{ii}\ e_{i+1\ i+1} \ + b_{ii} \ \ge b_{ii} \ge 0.$$

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It follows that

$$[\ a_{ij}\ ]_1^3 = \begin{bmatrix} e_{11} + (1-\lambda_2)e_{22} & -e_{12} & -e_{13} \\ -e_{21} & e_{22} + (1-\lambda_3)e_{33} & -e_{23} \\ -e_{31} & -e_{32} & e_{33} + (1-\lambda_1)e_{11} \end{bmatrix},$$

and ρ_1 may be written as

$$\rho_1 = 2\varepsilon(x) + \tau'(x) - x$$

where τ' is a 3-positive map, defined by the matrix

$$[\ \tau'(\ e_{ij}\ )]_1^3 = \begin{bmatrix} (1-\lambda_2)e_{22} & 0 & 0 \\ 0 & (1-\lambda_3)e_{33} & 0 \\ 0 & 0 & (1-\lambda_1)e_{11} \end{bmatrix}.$$

We now assert that the map ρ_1 can not be positive unless the λ_i's are all zero. In fact, as in the proof of Theorem 1, the positivity of ρ_1 is equivalent to the assertion that $2\varepsilon(p) + \tau'(p) \geq p$ for every one dimensional projection $p = (\ \bar{\alpha}_i\alpha_j\ )$, and this is further converted into the inequality $\sum_{i=1}^{3} \frac{1}{2 + \mu_i a_i} \leq 1$, by putting $a_i = \frac{|\ \alpha_{i-1}\ |^2}{|\ \alpha_i\ |^2}$ and $\mu_i = 1 - \lambda_i$. This is equivalent to saying that

$$\begin{aligned} T = &(\ 2 + \mu_1 a_1\ )(\ 2 + \mu_2 a_2\ )(\ 2 + \mu_3 a_3\ ) \\ &- (\ 2 + \mu_2 a_2\ )(\ 2 + \mu_3 a_3\ ) - (\ 2 + \mu_1 a_1\ )(\ 2 + \mu_3 a_3\ ) \\ &- (\ 2 + \mu_1 a_1\ )(\ 2 + \mu_2 a_2\ ) \\ = &-4 + \mu_1\mu_2 a_1 a_2 + \mu_2\mu_3 a_2 a_3 + \mu_3\mu_1 a_3 a_1 + \mu_1\mu_2\mu_3 \geq 0. \end{aligned}$$

However, if one of the λ_i's is not zero, the corresponding μ_i is strictly less than 1 and the value of T can not be positive when

$$a_1 = a_2 = a_3 = 1.$$

As this implies that ρ_1 is not positive, we conclude that all the λ_i's are zero. This shows that $\rho_2 = 0$, as desired.

3. Concluding remarks.

We have to mention that the situation surrounding the decomposability of positive maps is not so simple. In [12] the second author has analyzed those positive maps sitting on the lines ,onnecting the identity map σ, the transpose map θ, and the completely positive map $\tau(x) = \frac{1}{n} Tr(x)1_n$ from the point of view of the rank of positivity of those maps. Here, σ is clearly completely positive and θ is known to be plain positive but completely copositive by definition. The analysis of these maps from the point of view of copositivity, however, shows that there exists a completely positive map on the segment between θ and τ, which is therefore also the sum of a completely positive map and a (nonzero) positive scalar multiple of θ. Similarly, on the ray through the identity map and τ, we can find a segment on which there appears the highest class of positive maps which are both completely positive and completely copositive. Here we emphasize that the ranges of those maps are not commutative, because there are certainly trivial examples of such maps defined by positive functionals on M_n (cf.[13]). We also point out the fact (cf.[12]) that a completely positive map as well as a completely copositive map may be decomposable into the sum of positive maps with lower positivity and copositivity.

References

1. T. Ando, Positivity of certain maps, seminar note 1985.

2. M. D. Choi, Positive linear maps on C^*-algebras, Canad. J. Math., 24 (1972), 520-529.

3. ———— , Positive semidefinite biquadratic forms, Linear Algebra and Appl., 12 (1975), 95-100.

4. M. D. Choi and T. Y. Lam, Extremal positive semidefinite forms, Math. Ann., 231 (1977), 1-18.

5. ———— , Some assorted inequalities for positive linear maps on C^*-algebras, J. Operator Theory 4 (1980), 271-285.

6. E. C. Lance, On nuclear C^*-algebras, J. Funct. Analysis, 12 (1973), 157-176.

7. A. G. Robertson, Schwartz inequalities and decomposition of positive maps on C^*-algebras, Math. Proc. Camb. Phil. Soc., 94 (1983), 291-296.

8. ———— , Positive projections on C^*-algebras and an extremal positive map, J. London Math. Soc., 32 (1985), 133-140.

9. E. Størmer, Positive linear maps of operator algebras, Acta Math., 110 (1963), 233-278.

10. ———— , Decomposable positive maps on C^*-algebras, Proc. Amer. Math. Soc., 86 (1982), 402-404.

11. J. Tomiyama, On the transpose map of matrix algebras, Proc. Amer. Math. Soc., 88 (1983), 635-638.

12. ———— , On the geometry of positive maps in matrix algebras II, Linear Algebra and Appl., 69 (1985), 169-177.

13. ———— , On the geometry of positive maps in matrix algebras, to appear in the Proceedings of the Conference on Operator Algebras and Mathematical Physics, Iowa, 1985.

14. S. L. Woronowicz, Positive maps of low dimensional matrix

algebras, Rep. Math. Phys., 10 (1976), 165-183.

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