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On a decay property of weak solution of the M.H.D. equations in a 3-dimensional exterior domain

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Introduction

The purpose of this report is to present the abstract theorems on the existence and a certain decay property of weak solution for a semilinear evolution equation of parabolic type. Then we shall show that these theorems are applicable to the M.H.D. equations <u>in an exterior domain</u>. In the case of the Navier-Stokes equations, the energy decay for a weak solution was shown by Masuda [3] and Sohr [6].

Let X be a separable Hilbert space. The abstract equation of evolution we consider has the form

$$du(t)/dt + Au(t) + Nu(t) = f(t) \quad t > 0,$$
$$u(0) = a. \tag{E}$$

Here A is a non-negative self-adjoint operator in X, while N is a non-linear operator in X specified later.

Our problem reads as follows.

Problem. <u>Construct a weak solution</u> $u = u(t)$ <u>of</u> (E) <u>on</u> $(0, \infty)$ <u>such that</u> $\|u(t)\| \to 0$ as $t \to \infty$. <u>Here</u> $\| \ \|$ <u>denotes the norm on</u> X.

Concernig our problem, we follow Madsuda [3] and Sohr [6]. At the first step, we show that there exists a positive number α such that any weak solution u of (E) with $\int_0^\infty \|A^{1/2}u(\tau)\|^2 d\tau < \infty$ satisfies

$$\|(1 + A)^{-\alpha}u(t)\| \to 0, \text{ as } t \to \infty. \qquad \text{(W.D.)}$$

An immediate consequence of (W.D.) is that

$$\lim_{t\to\infty} \int_t^{t+1} \|u(\tau)\|^2 d\tau = 0. \qquad (0.1)$$

See Masuda [3, Corollary 1].
Hence in order to solve our problem, it is sufficient to construct a weak solution u of (E) satisfying the energy inequality of strong form:

$$\|u(t)\|^2 + 2\int_s^t \|A^{1/2}u(\tau)\|^2 d\tau \le \|u(s)\|^2 + 2\int_s^t (f(\tau), u(\tau))\, d\tau \qquad \text{(E.I.S.)}$$

for almost all $s \ge 0$, including $s = 0$, and all $t > 0$ such that $s < t$. ((,); the scalar product on X)

In fact, if f is a summable function on $(0, \infty)$ with values in X, we see by (E.I.S.) that the inequality

$$\|u(t)\|^2 + 2\int_s^t \|A^{1/2}u(\tau)\|^2 d\tau \leq \|u(s)\|^2 + M_0\int_s^\infty \|f(\tau)\| \, d\tau \qquad (0.2)$$

holds for almost all $s \geq 0$ and all $t > s$. ($M_0 := \sup_{\tau > 0} \|u(\tau)\|$)

It follows from (0.1) and (0.2) that a weak solution u of (E) with (E.I.S.) satisfies $\lim_{t\to\infty} \|u(t)\|^2 = 0$.

In section 1, we state the general results of existence and decay property for weak solutions of (E) under some assumptions on A and N. Section 2 is devoted to apply the theorems obtained in section 1 to the magnetohydrodynamic(M.H.D.) equations in a three-dimensional exterior domain.

1 Existence and decay property (W.D.) of weak solutions of (E)

Let X be a separable Hilbert space. We denote by (,) and $\| \|$ the scalar product and the norm on X. Suppose that A is a non-negative self-adjoint operator in X with the core D. we set

$$V = D(A^{1/2}). \quad (D(S); \text{ domain of } S).$$

Equipped with the scalar product

$$((u, v)) = (u, v) + (A^{1/2}u, A^{1/2}v),$$

V is a Hilbert space. We denote by $|||\ \ |||$ the norm on V defined by $|||u||| = ((u, u))^{1/2}$.

Let X^* and V^* denote the dual spaces X and V, respectively. Identifying X with X^*, we have the usual inclusions

$$V \subset X \equiv X^* \subset V^*,$$

where each space is dense in the following one and the injections are continuous.

For $T > 0$, we set $\mathcal{Y}_T = L^\infty(0, T; X) \cap L^2(0, T; V)$. $\mathcal{Y}_T$ is the Banach space with norm $|\ \ |_T$:

$$|u|_T = \sup_{0<t<T} \|u(t)\| + \{\int_0^T \|A^{1/2}u(\tau)\|^2 d\tau\}^{1/2}.$$

We can now introduce the assumptions on N.

N is a continuous mapping from V into V^* satisfying the following conditions.

<u>Assumption</u> 1. There exists a monotone increasing function $L_1 = L_1(\lambda)$ such that the inequality

$$|\langle Nu, \phi\rangle| \leq L_1(\|u\|)(1 + \|A^{1/2}u\|)\|A^{1/2}u\|\|A^{1/2}\phi\|$$

holds for all u and Φ in V. Here and hereafter $\langle\ ,\ \rangle$ denotes the duality between V^* and V.

Assumption 2. There exist a constant $0 \leqq p < 2$ and a monotone increasing function $L_2 = L_{2,\Phi}(\lambda)$ depending only on Φ in D such that the inequality

$$|\langle Nu, \Phi\rangle| \leqq L_2(\|u\|)(1 + \|A^{1/2}u\|)^p$$

holds for all u in V.

Assumption 3. For each $T > 0$, there exist a monotone increasing function $L_3 = L_{3,v,\Phi}(\lambda)$ depending only on $v \in \mathcal{Y}_T$ and $\Phi = \Phi(t) = h(t)\phi$ $(h \in C_0^1([0, T)),\ \phi \in D)$, an interger $K = K_{\varepsilon,v,\Phi}$ depending only on $\varepsilon > 0$, v and Φ as above, positive numbers $\alpha_i (i=1,\dots,K)$ and functions $\Psi_i (i=1,\dots,K)$ in $C^0([0, T]; X)$ such that the inequality

$$\left|\int_0^T \langle Nu(\tau) - Nv(\tau), \Phi(\tau)\rangle\, d\tau\right|$$

$$\leqq L_3(|u|_T)\{\varepsilon + \int_0^T (\textstyle\sum_{i=1}^K |(u(\tau) - v(\tau), \Psi_i(\tau))|)^{\alpha_i}\, d\tau\}$$

holds for all $u \in \mathcal{Y}_T$.

For a weak solution of (E), we give the following definition.

Definition. Let the initial data a be in X and let f be in $L^1(0, \infty; X)$. Suppose that the assumptions 1, 2 and 3 hold. u is called a weak solution of (E), if (i) and (ii) hold:

(i) $u \in L^\infty(0, \infty; X) \cap L^2_{loc}(0, \infty; V)$.

(ii) For all $\Phi \in C^1_0([0, \infty); V)$, the equality

$$\int_0^\infty \{-(u(t), \partial_t\Phi(t)) + (A^{1/2}u(t), A^{1/2}\Phi(t)) + \langle Nu(t), \Phi(t)\rangle\} \, dt$$

$$= (a, \Phi(0)) + \int_0^\infty (f(t), \Phi(t)) \, dt$$

is satisfied.

Our result on existence of weak solutions reads:

Theorem 1. _Let the initial data_ a _be in_ X _and let_ f _be in_ $L^1(0, \infty; X)$. _Suppose that the assumptions_ 1, 2 _and_ 3 _hold_. _If_ $\langle Nu, u\rangle \geq 0$ _for all_ $u \in V$, _then there exists a weak solution_ u _of_ (E) _such that the inequality_

$$\|u(t)\|^2 + 2\int_0^t \|A^{1/2}u(\tau)\|^2 \, d\tau \leq \|a\|^2 + 2\int_0^t (f(\tau), u(\tau)) \, d\tau$$

holds for all $t > 0$.

Our result on the decay property of weak solutions reads:

Theorem 2. *Let a be in X and let f be in $L^1(0, \infty; X)$. Suppose that the assumptions 1, 2 and 3 hold. If zero is not an eigenvalue of A, then any weak solution of (E) with $\int_0^\infty \|A^{1/2}u(\tau)\|^2 d\tau < \infty$ satisfies*

$$\lim_{t\to\infty} \|(1 + A)^{-1/4}u(t)\| = 0.$$

For the proof of Theorems 1 and 2, see Kozono [2].

2 Application

In this section, we apply the theorems obtained in the preceding section to the M.H.D. equations in a three-dimensional exterior domain. In the case of the Navier-Stokes equations, such a result was obtained by Masuda [3] and Sohr [6]. The main work is to check whether or not the assumptions 1, 2 and 3 hold.

Let O be a bounded domain in R^3 with smooth boundary ∂O. We set $\Omega = \mathbf{R}^3 - O$. For simplicity, we assume that Ω is simply connected. In $Q := \Omega\times(0, \infty)$, we consider the following magnetohydrodynamic(M.H.D.) equations:

$$
\begin{aligned}
&\partial_t v - \Delta v + (v, \nabla)v + B\times \mathrm{rot}B + \nabla\pi = b && \text{in } Q,\\
&\partial_t B - \Delta B + (v, \nabla)B - (B, \nabla)v = 0 && \text{in } Q,\\
&\mathrm{div}\, v = 0, \quad \mathrm{div}\, B = 0 && \text{in } Q,\\
&v = 0, \quad B\cdot\nu = 0, \ \mathrm{rot}B\times\nu = 0, \quad \text{on} && \partial\Omega\times(0, \infty),\\
&v\big|_{t=0} = v_0, \quad B\big|_{t=0} = B_0 .
\end{aligned}
$$

Here $v = v(x,t) = (v^1(x,t), v^2(x,t), v^3(x,t))$, $B = B(x,t) = (B^1(x,t), B^2(x,t), B^3(x,t))$ and $\pi = \pi(x,t)$ denote respectively the unknown velocity field of the fluid, magnetic field and pressure of the fluid, $b = b(x,t) = (b^1(x,t), b^2(x,t), b^3(x,t))$ denotes the given external force, $v_0 = v_0(x) = (v_0^1(x), v_0^2(x), v_0^3(x))$ and $B_0 = B_0(x) = (B_0^1(x), B_0^2(x), B_0^3(x))$ denote the given initial data and ν denotes the unit outward normal on $\partial\Omega$.

We introduce some function spaces.

Let $C_{0,\sigma}^{\infty}(\Omega)$ be the set of all C^{∞} vector functions Φ with compact support in Ω, such that $\mathrm{div}\,\Phi = 0$ $(x\in\Omega)$. $L_{\sigma}^2(\Omega)$ denotes the closure of $C_{0,\sigma}^{\infty}(\Omega)$ in $L^2(\Omega) := L^2(\Omega)^3$.

We denote by P the orthogonal projetion from $L^2(\Omega)$ onto $L_{\sigma}^2(\Omega)$.

We define the operators A_D and A_N on $L_{\sigma}^2(\Omega)$ as follows:

$$D(A_D) = H^2(\Omega)^3 \cap \{v \in H_0^1(\Omega)^3;\ \mathrm{div}\ v = 0\},$$

$$A_D v = -P\Delta v \quad \text{for} \quad v \in D(A_D),$$

$$D(A_N) = \{B \in H^2(\Omega)^3;\ B\cdot\nu = 0,\ \mathrm{rot}\ B\times\nu = 0 \quad \text{on} \quad \partial\Omega\} \cap L^2_\sigma(\Omega),$$

$$A_N B = -\Delta B \quad \text{for} \quad B \in D(A_N).$$

Note that $-\Delta B \in L^2_\sigma(\Omega)$ if and only if $B \in D(A_N)$. With the aid of Miyakawa [5, Theorem 1.8] and [4, Theorem 3.8], we see that A_D and A_N are non-negative self-adjoint operators in $L^2_\sigma(\Omega)$. Moreover, we have

$$\|A_D^{1/2} v\|_{L^2} = \|\nabla v\|_{L^2} \quad \text{for} \quad v \in D(A_D^{1/2}), \tag{2.1}$$

$$\|A_N^{1/2} B\|_{L^2} = \|\mathrm{rot} B\|_{L^2} \quad \text{for} \quad B \in D(A_N^{1/2}). \tag{2.2}$$

In the context of section 1, we set

$$X = L^2_\sigma(\Omega)\times L^2_\sigma(\Omega), \quad D = D(A_D)\times D(A_N)$$

and

$$A = \begin{pmatrix} A_D & 0 \\ 0 & A_N \end{pmatrix} \quad \text{with the domain} \quad D(A) = D(A_D)\times D(A_N).$$

Then we have $V = D(A_D^{1/2})\times D(A_N^{1/2})$.

We define $Nu \in V^*$ for $u = {}^t[v, B] \in V$ by

$$\langle Nu, \Phi\rangle := \int_\Omega ((v, \nabla)v + B\times \mathrm{rot}B)\cdot\phi\, dx + \int_\Omega ((v, \nabla)B - (B, \nabla)v)\cdot\psi\, dx$$

for $\Phi = {}^t[\phi, \psi] \in V$.

By integration by parts, we have $\langle Nu, u\rangle = 0$ for all $u \in V$.

By the Holder inequality, the Gagliardo-Nirenberg inequality and Duvaut-Lions [4, Chapter 7 Theorem 6.1], the inequalities

$$\left|\int_\Omega ((v, \nabla)v + B\times \mathrm{rot}B)\cdot\phi\, dx\right|$$

$$\le (\|v\|_{L^3}\|\nabla v\|_{L^2} + \|B\|_{L^3}\|\mathrm{rot}B\|_{L^2})\|\phi\|_{L^6}$$

$$\le C(\|v\|_{L^2}^{1/2}\|\nabla v\|_{L^2}^{3/2} + \|B\|_{L^3}^{1/2}\|B\|_{H^1}^{1/2}\|\mathrm{rot}B\|_{L^2})\|\nabla\phi\|_{L^2}$$

$$\le C\{(1+\|v\|_{L^2})(1+\|A_D^{1/2}v\|_{L^2})\|A_D^{1/2}v\|_{L^2}+(1+\|B\|_{L^2})(1+\|A_N^{1/2}B\|_{L^2})\times$$

$$\times\|A_N^{1/2}B\|_{L^2}\}\|A_D^{1/2}\phi\|_{L^2},$$

and

$$\left|\int_\Omega ((v, \nabla)B - (B, \nabla)v)\cdot\psi\, dx\right|$$

$$= \left|\int_\Omega \mathrm{rot}(B\times v)\ \psi\, dx\right| = \left|\int_\Omega B\times v\ \mathrm{rot}\psi\, dx\right| \text{ (by integration by patrs)}$$

$$\le \|B\|_{L^3}\|v\|_{L^6}\|\mathrm{rot}\psi\|_{L^2}$$

$$\le C\|B\|_{L^2}^{1/2}\|B\|_{H^1}^{1/2}\|\nabla v\|_{L^2}\|\mathrm{rot}\psi\|_{L^2}$$

$$\le C(\|B\|_{L^2} + \|\mathrm{rot}B\|_{L^2})\|\nabla v\|_{L^2}\|\mathrm{rot}\psi\|_{L^2}$$

$$\leq C(1 + \|B\|_{L^2})(1 + \|A_N^{1/2}B\|_{L^2})\|A_D^{1/2}v\|_{L^2}\|A_N^{1/2}\psi\|_{L^2}$$

hold for all $u = {}^t[v, B] \in V$ and all $\Phi = {}^t[\phi, \psi] \in V$.
Hence the assumption 1 holds.
Moreover, for each $\phi \in D(A_D)$ and each $\psi \in D(A_N)$ the inequalities

$$\left|\int_\Omega ((v, \nabla)v + B\times \mathrm{rot}B)\cdot\phi \, dx\right|$$

$$\leq \operatorname*{ess\,sup}_{x\in\Omega} |\phi(x)|(\|v\|_{L^2}\|\nabla v\|_{L^2} + \|B\|_{L^2}\|\mathrm{rot}B\|_{L^2})$$

$$= \operatorname*{ess\,sup}_{x\in\Omega} |\phi(x)|(\|v\|_{L^2}\|A_D^{1/2}v\|_{L^2} + \|B\|_{L^2}\|A_N^{1/2}B\|_{L^2})$$

and

$$\left|\int_\Omega ((v, \nabla)B - (B, \nabla)v)\cdot\psi \, dx\right|$$

$$\leq \operatorname*{ess\,sup}_{x\in\Omega} |\psi(x)|\{\|v\|_{L^2}(\|B\|_{L^2} + \|A_N^{1/2}B\|_{L^2}) + \|B\|_{L^2}\|A_D^{1/2}v\|_{L^2}\}$$

hold for all ${}^t[v, B] \in V$.

Taking $p = 1$ and

$$L_{2,\Phi}(\lambda) := (\operatorname*{ess\,sup}_{x\in\Omega}|\phi(x)| + \operatorname*{ess\,sup}_{x\in\Omega}|\psi(x)|)(1 + \lambda^2)$$

for each $\Phi = {}^t[\phi, \psi] \in D \subset L^\infty(\Omega)^3\times L^\infty(\Omega)^3$, we see that the assumption 2 holds.

We obtain the assumption 3 by using Masuda's beautiful technique. See Masuda [3, Lemma 2.5].

Finally, it remains to show that zero is not an eigenvalue of A. In fact, by (2.1) it is easy to see that zero is not an eigenvalue of A_D. Suppose that $A_N B = 0$ for $B \in D(A_N)$. Then by (2.2), $\mathrm{rot}\, B = 0$ in Ω. Since $\mathrm{div}\, B = 0$ in Ω and since $B \cdot \nu = 0$ on $\partial\Omega$, it follows from the classical potential theory that there is a scalar function p with $p \in L^2_{loc}(\Omega)$, $\nabla p \in L^2(\Omega)$ and

$$\Delta p = 0 \quad \text{in} \quad \Omega, \quad \partial p/\partial \nu = 0 \quad \text{on} \quad \partial\Omega$$

such that $B = \nabla p$.
According to Miyakawa [5, Lemma 1.4], such p must satisfy that $\nabla p = 0$ and hence $B = 0$. Thus zero is not an eigenvalue of A_N.

Remark. <u>Theorems 1 and 2 are appplicable to the following system of semilinear parabolic equations</u>:

$$\partial_t u^i - \sum_{j,k=1}^{L} \partial_{x^j}(a_{jk}(x)\partial_{x^k} u^i) + |u|^\rho \sum_{j=1}^{L} b^i_j(x) u^j = f^i \quad \text{in} \quad Q,$$

$$(i = 1, \ldots, L)$$

$u^i = 0$ on $\partial\Omega \times (0, \infty)$,

$u^i|_{t=0} = a^i$,

where $Q = \Omega\times(0, \infty)$ (Ω; bounded or unbounded domain in R^n) and $|u| = \{ \Sigma_{j=1}^{L}(u^i)^2\}^{1/2}$.

Suppose that $3 \leq n \leq 6$ and $1 \leq \rho \leq 6/n$.

We assume that a_{ij} and b_j^i are in $\mathcal{B}^\infty(\bar{\Omega})$ and $a_{ij} = a_{ji}$ for all $i,j = 1,\ldots,L$. Moreover, we assume that there is a positive constant δ such that

$$\sum_{j,k=1}^{L} a_{jk}(x)\xi^j\xi^k \geq \delta|\xi|^2, \quad \sum_{i,j=1}^{L} b_j^i(x)\xi^i\xi^j \geq \delta|\xi|^2$$

hold for all $x \in \bar{\Omega}$ and all $\xi = (\xi^1,\ldots,\xi^L) \in R^L$.

References

1.Duvaut, G., Lions, J.L.: Inequalities in Mechanics and Physics. Berlin - Heidelberg - New York : Springer 1976

2.Kozono, H.: On a decay property of weak solution for semilinear evolution equation of parabolic type and its applications. to appear

3.Masuda, K.: Weak Solutions of Navier-Stokes Equations. Tôhoku Math. J. 36, 623-646 (1984)

4.Miyakawa, T.: The L^p approch to the Navier-Stokes equations with the Neumann boundary condition. Hiroshima Math. J. 10, 517-537 (1980)

5.Miyakawa, T.: On nonstationary solutions of the Navier-Stokes equations in an exterior domain. Hiroshima Math. J. 12, 115-140 (1982)

6.Sohr, H.: On the Decay of Weak Solutions of the Navier-Stokes Equations. preprint