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On the initial boundary value problem for the Boltzmann equation

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1. Introduction

The motion of a gas contained in a vessel $\Omega \subset \mathbb{R}^n$ is described by the following initial boundary value problem for the Boltzmann equation;

(1.1) $$f_t + \xi\cdot\nabla_2 f + a(x\cdot\xi)\cdot\nabla_\xi f = Q[f] ,$$

(1.2) $$\gamma^- f = M\gamma^+ f ,$$

(1.3) $$f|_{t=0} = f_0 .$$

Here the unknown $f = f(t,x,\xi)$ is the density of gas particles at time $t \geq 0$, at the position $x = (x_1,x_2,\ldots,x_n) \in \Omega$ and velocity $\xi = (\xi_1,\xi_2,\ldots,\xi_n) \in \mathbb{R}^n$. (1.1) is the Boltzmann equation, in which $\cdot$ stands for the scalar product in $\mathbb{R}^n$, $\nabla_x = (\partial/\partial x_1,\ldots,\partial/\partial x_n)$ and similarly for ∇_ξ, while $a(x,\xi)$ is a given vector which is the external force acting on a gas particle and Q is a quadratic integral operator in ξ describing the collision of articles. In this paper we assume the cutoff interaction potential in the sense of Grad [3].

The equation (1.2) is the boundary condition which describes the reflection of gas particles by the wall $\partial\Omega$ of the vessel. Let $n(x)$ be the outward normal to $\partial\Omega$ and define the trace operators $\gamma^\pm$ by

(1.4) $$\gamma^\pm f = f|_{S_\pm}$$

where

(1.5) $$S^\pm = \{(x,\xi) \in \partial\Omega\times\mathbb{R}^n \mid n(x)\cdot\xi \gtrless 0\} \quad \text{(same signs)}.$$

Clearly γ^+f (resp. γ^-f) describes the density of particles incident to (resp. reflected by) the wall $\partial\Omega$. Thus (1.2) is the reflection law specified by the boundary operator M. Typical examples are

(1.6a) perfect absorption: $M = 0$,

(1.6b) specular reflection: $M\gamma^+f = f(t,x,\xi-2(\xi\cdot n(x))n(x))$,

(1.6c) diffuse reflection:

$$M\gamma^+f = \rho_w(\xi)\int_{n(x)\cdot\xi'>0} f(t,x,\xi')|n(x)\cdot\xi'|\, d\xi' ,$$

where $\rho_w(\xi) = (2\pi)^{-(n-1)/2}T_w^{-(n+1)/2}\exp(-|\xi|^2/2T_w)$, T_w being the temperature of the wall $\partial\Omega$.

The aim of this paper is to discuss the local (in t) solvability of the problem (1.1)-(1.3). To this end we shall first solve the linearized problem,

$$f_t + \xi\cdot\nabla_x f + a(x,\xi)\cdot\nabla_\xi f = 0 , \tag{1.7a}$$

$$\gamma^-f = M\gamma^+f , \tag{1.7b}$$

$$f|_{t=0} = f_0 . \tag{1.7c}$$

Denote the solution operator (semi-group) by U(t);

$$f = U(t)f_0 \tag{1.8}$$

is the solution to (1.7). Then if f is a solution of (1.1)-(1.3), it solves the integral equation

$$f(t) = U(t)f_0 + \int_0^t U(t-s)\, Q[f(s)]\, ds . \tag{1.9}$$

Since we assume Grad's cutoff potential, this can be solved locally by the successive approximations described in [3], once U(t) is shown to exist as a uniformly bounded operator in some time interval [0,T] in a suitable family of Banach spaces, see section 4.

As will be shown in section 3, (1.7) is easy to solve if $\|M\| < 1$, but the case $\|M\| = 1$, which involves the important classical examples (1.6b) and (1.6c), is delicate. So far, three different methods have been developed, using the duality [6], the principle of limiting absorption [2] and the monotonicity [4], respec-tively. Each method solves (1.7) in a generalized sense. However it is not known in general whether or not the solutions coincide with each other. Thus, the uniqueness of the solution is open. All the difficulty comes from the fact that the trace theorem associated with the operator

$$\Lambda = \frac{\partial}{\partial t} + \xi \cdot \nabla_x + a(x,\xi) \cdot \nabla_\xi \ , \tag{1.10}$$

is not compatible with the boundary condition (1.2) or (1.7b). This will be described in the next section, and the duality method in section 3. The result of this paper is reported in [8].

2. Trace theorem

We begin by studying the characteristic equation of (1.7a);

$$\begin{aligned} &\frac{d\tau}{ds} = 1 \ , \quad \frac{dX}{ds} = \Xi \ , \quad \frac{d\Xi}{ds} = a(x,\Xi) \ , \\ &\tau|_{s=0} = t \ , \quad X|_{s=0} = x \ , \quad \Xi|_{s=0} = \xi \ . \end{aligned} \tag{2.1}$$

Putting $y = (t,x,\xi)$, we write the solution as

$$Y(s,y) = (\tau(s,y), X(s,y), \Xi(s,y)) \ . \tag{2.2}$$

Assume that

(2.3) Ω is a domain in $\mathbb{R}^n$, bounded or unbounded, with piecewise C^2 boundary $\partial\Omega$;

(2.4) $a(x,\xi) = -\nabla_x b(x) + a_1(x,\xi)$ where

(i) $b(x) \in C^2(\bar{\Omega})$, with $b(x) \geq 1$ (a potential bounded below).

(ii) $a_1(x,\xi) \in C^1(\bar{\Omega} \times \mathbb{R}^n)$ with $\xi \cdot a_1 = \nabla_\xi \cdot a_1 = 0$.

Furthermore, for any fixed $T > 0$, we set

$$
\begin{aligned}
&D = \Omega \times \mathbb{R}^n , \\
&V = (0,T) \times D, \quad \Sigma = (0,T) \times \partial\Omega \times \mathbb{R}^n , \\
(2.5) \qquad &\Sigma^{\pm} = (0,T) \times S^{\pm} , \\
&D^+ = \{T\} \times D, \quad D^- = \{0\} \times D , \\
&\partial V^{\pm} = \Sigma^{\pm} \cup D^{\pm}
\end{aligned}
$$

Theorem 2.1. Assume (2.3) and (2.4). Then for each $y \in V \cup \partial V^+ \cup \partial V^-$, there exists a closed bounded interval $I_y = [-t^-(y), t^+(y)]$, $t^{\pm}(y) \geq 0$, such that

(i) (2.1) has a unique solution $Y(s,y)$ on I_y ,

(ii) $Y(s,y) \in V$ for all $s \in \mathring{I}_y = (-t^-(y), t^+(y))$,

(iii) $Y(\pm t^{\pm}(y), y) \in \overline{\partial V^{\pm}}$,

(iv) $t^{\pm}(y) = 0$ for $y \in \partial V^{\pm}$ (same signs).

Proof: Let $y \in V$. Since $a \in C^1$, Y exists at least for small $|s|$. Then the energy,

$$(2.6) \qquad H(y) = H(x,\xi) = b(x) + \frac{1}{2} |\xi|^2 ,$$

is conserved,

$$(2.7) \qquad H(Y(s,y)) = H(y) ,$$

due to (2.4). Since $b(x) \geq 1$, this implies an a priori estimate for Y, so Y can be continued as long as it remains in V. Since $\tau(s,y)$ is simply solved by $\tau(s,y) = t-s$, and since $\tau \in (0,T)$ is required, the maximal interval of extension is bounded. At some $s = \pm t^{\pm}(y)$, Y meets ∂V for the first time as $|s|$ increases, implying (ii). To prove (iii), we put $Y^+ = Y(t^+(y),y)$. Obviously $Y^+ \in \overline{D^+}$ or $Y^+ \in (0,T) \times \partial\Omega \times \mathbb{R}^n$. If the latter occurs, then $X^+ = X^+(t^+(y),y) \in \partial\Omega$. Suppose $\partial\Omega$ is smooth near X^+, i.e., it is expressed by the equation $\phi(y) = 0$ near X^+ where $\phi \in C^2$ with $|\nabla_x\phi| = 1$. Assume further $\phi(y) < 0$ if $x \in \Omega$ and $\phi(x) > 0$ if $x \in \overline{\Omega}^c$ and note thereof $n(x) = \nabla_x \phi$. Then we have,

$$n^+(x^+) \cdot \Xi^+(x^+) = \nabla \phi(x^+) \cdot \left.\frac{dx^+}{dt}\right|_{s=t^+(y)}$$
$$= \left.\frac{d\phi(X)}{ds}\right|_{s=t^+(y)} \geq 0 ,$$

whence $(X^+,\Xi^+) \in \overline{S^+}$ or $Y^+ \in \overline{\Sigma^+}$. This proves the (+) part of (iii), and similarly for the (-) part. The above argument is also true of $y \in \partial V^{\pm}$, with $t^{\pm}(y) = 0$, and the proof of the theorem is complete.

Recall Λ of (1.10) and let $f \in L^1_{loc}(V)$. If there exists a $g \in L^1_{loc}(V)$ such that

$$(f,\phi) = - (g,\Lambda\phi)$$

holds for all $\phi \in C^1_0(V)$, we write $g = \Lambda f$. Here (,) means

$$(f,h) = \int_V f(y)\,\overline{h(y)}\,dy \quad (dy = dt\,dx\,d\xi) .$$

Note that the formal adjoint of Λ is $-\Lambda$ by the assumption $\nabla_\xi \cdot a_1 = 0$. For $f = f(y)$, put $\tilde{f}(s,y) = f(Y(s,y))$. Then we can show

Lemma 2.2. $\Lambda f = \partial \tilde{f}/\partial s$ a.a. $y \in V$ and $s \in \mathring{I}_y$.

Therefore if $\Lambda f \in L^1_{loc}(\bar{V})$, then f is absolutely continuous along the characteristic curve $C_y = \{Y(s,y) \mid s \in I_y\}$, so that we can define the traces $\gamma_\pm f$ of f n $\partial V^\pm$ by the limits

(2.8) $$(\gamma_\pm f)(Y^\pm) = \lim_{\epsilon \to \pm 0} \tilde{f}(\pm t^\pm(y)+\epsilon, y) ,$$

for almost every $y \in V$ and subsequently for almost all $Y^\pm \in \partial V^\pm$. We can show that these $\gamma_\pm$ are bounded operators. More precisely, define, for $p \in [1,\infty]$,

(2.9) $$W_p = \{f \in L^p(V) \mid \Lambda f \in L^p(V)\}, \quad \|f\|_{W_p} = \|f\|_{L^p(V)} + \|\Lambda f\|_{L^p(V)}$$
$$Y_p^\pm = L^p(\partial V^\pm; \rho\, d\sigma_y) ,$$

where

(2.10) $$\rho(y) = t^+(y) + t^-(y) ,$$
$$d\sigma_y = \begin{cases} |n(x),\xi|\ dt\ d\sigma_x\ d\xi , & y \in \Sigma^\pm \\ ds\ d\xi , & y \in D^\pm \end{cases},$$

$d\sigma_x$ denoting the measure on $\partial\Omega$. Note that $\rho(y) \le T$. We can prove the

Theorem 2.3. For any $p \in [1,\infty]$, $\gamma^\pm$ are bounded from W_p to $Y_p^\pm$ (same signs).

In the above the weight function $\rho(y)$ which is not bounded away from 0 cannot be removed, so some authors have discussed traces only belonging to $L^p_{loc}(\Sigma^\pm; d\sigma_y)$ $(\supset Y_p^\pm)$, [4], [8]. The above theorem was given in [6] for the case $a(x,\xi) = 0$.

The space W_p is a nice space to solve (1.7a), but $Y_p^\pm$-traces which are natural traces for Λ are not adequate for the boundary condition (1.7b); $\gamma_\pm f$ are required to belong to the space

(2.11)
$$\tilde{Y}_p^{\pm} = L^p(\Sigma^{\pm};\ d\sigma_y)\ ,$$
$$|\gamma_{\pm} f|_{\pm,p} = \|\gamma_{\pm} f\|_{L^p(\Sigma^{\pm};d\sigma_y)}\ .$$

Note that $Y_\infty^{\pm} = \tilde{Y}_\infty^{\pm}$ but $\tilde{Y}_p^{\pm} \subsetneq Y_p^{\pm}$ if $p \in [1,\infty)$. Set

(2.12)
$$\tilde{W}_p = \{f \in w_p \mid \gamma_{\pm} f \in \tilde{Y}_p^{\pm}\}\ .$$

The following Green's formulas are useful in the next section.

Theorem 2.4. (i) For any $f \in \tilde{W}_p$, $p \in [1,\infty)$, we have

(2.13)
$$|\gamma_+ f|_{+,p}^p - |\gamma_- f|_{-,p}^{+} = -p\ \mathrm{Re}(\Lambda f,\ |f|^{p-1}\ \mathrm{sgn}(f))\ ,$$

where $\mathrm{sgn}(f) = f/|f|$ $(f(x) \neq 0)$, $= 0$ $(f(x) = 0)$.

(ii) For $f \in \tilde{W}_p$ and $g \in \tilde{W}_q$ with $p^{-1} + q^{-1} = 1$, it holds that

(2.14)
$$(\Lambda f, g) + (f, \Lambda g) = \langle\gamma_+ f, \gamma_+ y\rangle_+ - \langle\gamma_- f, \gamma_- g\rangle_-\ ,$$

where

$$\langle\phi,\psi\rangle_{\pm} = \int_{\partial V^{\pm}} \phi(y)\ \overline{\psi(y)}\ d\sigma_y\ .$$

Corollary 2.5. Suppose $f \in W_p$, $p \in [1,\infty)$. If $\gamma_+ f \in \tilde{Y}_p^+$, then $\gamma_- f \in \tilde{Y}_p^-$, and vice versa.

3. Linear problem

Let us solve (1.7). Evidently $\gamma^{\pm}$ in (1.7b) and $f|_{t=0}$ in (1.7c) should be understood as

$$(3.1) \qquad \gamma^{\pm} f = \gamma_{\pm} f|_{\Sigma^{\pm}}, \quad f|_{t=0} = f(0) = \gamma_{-} f|_{D^{-}},$$

where $\gamma_{\pm}$ are trace operators defined by (2.8). We also write $f(T) = \gamma_{+} f|_{D^{+}}$. Note that $\tilde{Y}_{p}^{\pm} = Y^{p,\pm} \oplus L^{p}(D)$ where

$$(3.2) \qquad Y^{p,\pm} = L^{p}(\Sigma^{\pm}; |n(x)\cdot\xi| \, dt \, d\sigma_{x} d\xi) .$$

We set

$$(3.3) \qquad \begin{aligned} \|\cdot\|_{p,\pm} &= \|\cdot\|_{Y^{p,\pm}}, \\ \|\cdot\|_{p} &= \|\cdot\|_{Y^{p}(D)}, \end{aligned}$$

The boundary operator M in (1.7b) is assumed to satisfy

$$(3.4) \qquad M; \ Y^{p,+} \rightarrow Y^{p,-} \ \text{ is bounded with } \|M\| \le 1 .$$

Let M^{*} be the adjoint of M. If $p \in [1,\infty)$, (3.4) implies $(p^{-1}+q^{-1}=1)$

$$(3.5) \qquad M^{*}; \ Y^{q,-} \rightarrow Y^{q,+} \ \text{ is bounded with } \|M^{*}\| \le 1 .$$

For $p = \infty$, this is taken as an additional assumption.

Let $\lambda > 0$ fixed. Instead of (1.7), we solve

$$(3.6) \qquad \begin{cases} (\Lambda + \lambda)f = 0 , \\ \gamma^{-}f = M\gamma^{+}f , \\ f(0) = f_0 . \end{cases}$$

f is said to be a strong solution if $f \in \tilde{W}_p$ and solves (3.6) in the L^p-sense. Define for $p \in (1,\infty]$,

$$\tilde{W}_p^* = \{g \in \tilde{W}_q \mid \gamma^{+}g = M^*\gamma^{-}g,\ g(T) = 0\} , \quad p^{-1} + q^{-1} = 1.$$

If $f \in \tilde{W}_p$ is a strong solution of (3.6), we see from (2.14) that

$$(3.7) \qquad (f,(\Lambda-\lambda)g) = - \langle f_0, g(0)\rangle ,$$

holds for all $g \in \tilde{W}_p^*$, where

$$\langle \phi,\psi\rangle = \int_D \phi\, \bar{\psi}\, dx\, d\xi .$$

Definition 3.1. Let $p \in (1,\infty]$ and $f_0 \in L^p(D)$. $f \in L^p(V)$ is a weak solution of (3.6) if (3.7) holds for all $g \in \tilde{W}_p^*$.

Theorem 3.2. Let $\lambda > 0$ and $p \in (1,\infty]$. Under the assumptions (2.3), (2.4), (3.4) and (3.5), a weak solution exists for any $f_0 \in L^p(D)$.

Proof: Since $p \in (1,\infty]$, we have $q \in [1,\infty)$. Apply (2.13) to $g \in \tilde{W}_p^*$ to deduce

$$(3.8) \qquad \|g\|_{L^q(V)} ,\ \|g(0)\|_q \le C\|(\Lambda-\lambda)g\|_{L^q(V)} .$$

This indicates that the mapping $g \to h = (\Lambda-\lambda)g$ is one to one so that

$$\ell(h) = -\langle f_0, g(0)\rangle$$

is a well-defined functional on the space

$$Z_p = \{h = (\Lambda-\lambda)g \mid g \in \tilde{W}_p^*\} .$$

Clearly, Z_p is a linear subset of $L^q(V)$, and (3.8) also shows that ℓ is bounded;

$$|\ell(h)| \le \|f_0\|_p \|g(0)\|_q \le C\|h\|_{L^p(V)} .$$

By the Hahn-Banach theorem, therefore, ℓ has an extension $\tilde{\ell}$ to $L^q(V)$, and then, by Riesz' representation theorem, there exists an $f \in L^p(V)$ such that $\tilde{\ell}(h) = (f,h)$ holds for any $h \in L^q(V)$. Restricting h in Z_p, this f is seen to be a desired weak solution.

Note that this proof does not work in $L^1(V)$ but it does in $L^\infty(V)^* =$ ba(V) if $\|M\| < 1$.

Theorem 3.3. If $\|M\| < 1$, the weak solution of Theorem 3.2 is a unique strong solution. And it satisfies

$$(3.8)_\infty \qquad \|f\|_{L^\infty(V)},\ \|f(T)\|_\infty,\ \|\gamma^+ f\|_{\infty,+} \le \|f_0\|_\infty ,$$

or

$$(3.8)_p \qquad \lambda p\|f\|^p_{L^p(V)} + \|f(T)\|^p_p + (1-\|M\|^p)\ \|\gamma^+ f\|^p_{p,+} \le \|f_0\|^p_p ,$$

according to $p = \infty$ or $p \in (1,\infty)$.

This is proved using the following characterization of the weak solution.

Theorem 3.4. Let $f \in L^p(V)$ be a weak solution. Then,

(i) $f \in W_p$ and satisfies $(\Lambda+\lambda)f = 0$.

(ii) $f(0) = f_0 \in L^p(D)$.

(iii) Let $\chi_\epsilon(y)$ be such that $\chi_\epsilon = 1$ $(\rho(y) > \epsilon)$, $= 0$ $(\rho(y) < \epsilon)$, and set $f_\epsilon = \chi_\epsilon f$. Then as $\epsilon \to 0$,

$$\gamma^- f_\epsilon - M\gamma^+ f_\epsilon \to 0 \quad \text{in } Y^{p,-}$$

weakly $(p < \infty)$ or weakly* $(p = \infty)$.

<u>Corollary 3.5</u>. A weak solution f is a strong solution if $f \in \tilde{W}_p$.

<u>Proof of Theorem 3.3</u>: Let f be a weak solution. For the case $p = \infty$, since $W_\infty = \tilde{W}_\infty$, f is a strong solution. By Lemma 2.2, we have $\partial(e^{\lambda s}|\tilde{f}|)/\partial s = e^{\lambda s}(\Lambda+\lambda)\tilde{f} = 0$, and hence

$$e^{\lambda t^+(y)}|\gamma_+ f(Y^+)| = e^{-\lambda t^-(y)}|\gamma_- f(Y^-)| .$$

Taking the supremum over $y \in V$,

$$\|\gamma^+ f\|_{\infty,+}, \ \|f(T)\|_\infty \le \max(\|f(0)\|_\infty, \ \|\gamma^- f\|_{\infty,-}) .$$

Since $\gamma^- f = M\gamma^+ f$ and $\|M\| < 1$, we get $(3.8)_\infty$.

For the case $p < +\infty$, put $g_\epsilon = \gamma^- f_\epsilon - M\gamma^+ f_\epsilon$, which are uniformly bounded in $Y^{p,-}$ for $\epsilon > 0$, due to Theorem 3.4 (iii). Since $\|M\| < 1$, we have

$$\begin{aligned}\|\gamma^- f_\epsilon\|^p_{p,-} &\le (\|M\| \ \|\gamma^+ f_\epsilon\|_{p,+} + \|g_\epsilon\|_{p,-})^p \\ &\le (1-\delta) \ \|\gamma^+ f_\epsilon\|^p_{p,+} + C ,\end{aligned}$$

with some $\delta, C > 0$ independent of ϵ. This and (2.13) then give

$$(3.9) \quad \|f_\epsilon(T)\|^p_p + \delta\|\gamma^+ f_\epsilon\|^p_{p,+} \le \|f_0\|^p_p + C + p\|f_\epsilon\|^{p-1}_{L^p(V)} \|(\Lambda+\lambda)f_\epsilon\|_{L^p(V)} .$$

But $(\Lambda+\lambda)f_\epsilon = \chi_\epsilon(\Lambda+\lambda)f = 0$ since $\Lambda\chi_\epsilon = 0$, so that passing to the limit $\epsilon \to 0$, and by Theorem 2.1, (3.9) shows that $\gamma f_{+} \in \tilde{Y}^{+}_{p}$. Hence f is a strong solution, thanks to Corollaries 2.4 and 3.5. Now $(3.8)_p$ follows from (2.13).

The case $\|M\| = 1$ is hard to prove the existence of a strong solution. Instead, we will show the

Theorem 3.6. Even if $\|M\| = 1$, there exists a weak solution $f \in L^p(V)$, $p \in (1,\infty]$, satisfying

$$(3.10) \qquad \|f(T)\|_p \le \|f_0\|_p .$$

Proof: Owing to Theorem 3.3, the problem (3.6) with M replaced by κM, $\kappa \in (0,1)$, has a unique strong solution f_κ. By (3.8) (with $\|M\| = \kappa$), there exists a subsequence such that as $\kappa \to 1$,

$$f_\kappa \to f \ \text{ in } L^p(V), \quad f_\kappa(T) \to \psi \ \text{ in } L^p(D) ,$$

weakly $(p < +\infty)$ or weakly* $(p = \infty)$. It is not hard to see that this limit f is a weak solution and $f(T) = \psi$. Then (3.10) follows from (3.8), going to the limit.

However, the uniqueness of the weak solution is not known without further conditions on Ω and M, [2].

Note that if f is a (weak) solution of (3.6), then $e^{\lambda t}f$ is a solution of (1.7), i.e., (3.7) for $\lambda = 0$, and (3.10) holds also for $\lambda = 0$. Since T may be arbitrary (even negative), we can define the operator U(T), by

$$U(T)f_0 = f(T) , \quad -\infty < T < \infty . \tag{3.11}$$

This is the operator desired in (1.8).

Theorem 3.7. If $p \in (1,\infty)$, U(t) is a C_0-group on $L^p(D)$.

This is not the case for $p = \infty$. For other methods of constructing U(t), see [2], [4], [8]. Note that in order to solve the nonlinear problem (1.9), the L^∞-solution operator U(t) is required, in contrast to the transport theory of [4], [8], in which the L^1-solution is discussed.

4. Nonlinear problem.

Now (1.9) is solved by the contraction mapping principle in the space

$$X_{\alpha,\beta} = \{f = f(x,\xi) \mid \rho_{\alpha,\beta} f \in L^\infty(D)\} , \tag{4.1}$$

where D is as in (2.5) and $\rho_{\alpha,\beta}$ is the weight function defined by

$$\rho_{\alpha,\beta}(x,\xi) = \exp(\alpha H(x,\xi))\, H(x,\xi)^{\beta/2} ,$$

with $H(x,\xi)$ given by (2.6). Formula (4.1) is a Banach space with the norm

$$\|f\|_{\alpha,\beta} = \|\rho_{\alpha,\beta}f\|_{L^\infty(D)} .$$

First, we shall discuss $U(t)$ of (3.11) (for $\lambda = 0$) in $X_{\alpha,\beta}$. Putting $g = \rho_{\alpha,\beta}U(t)f_0$, we see that since $\Lambda\rho_{\alpha,\beta} = 0$ due to (2.7), g solves

$$\text{(4.2)}\qquad \begin{cases} \Lambda g = 0 & \text{in } V , \\ \gamma^- g = M_{\alpha,\beta}\gamma^+ g & \text{on } \Sigma^+ , \\ g(0) = g_0 & \text{on } D^- , \end{cases}$$

where

$$\text{(4.3)}\qquad M_{\alpha,\beta} = \rho_{\alpha,\beta}M\rho^{-1}_{\alpha,\beta} ,$$

and $g_0 = \rho_{\alpha,\beta}f_0$. Therefore, if $M_{\alpha,\beta}$ satisfies (3.4) and (3.5) for $p = \infty$, (3.10) gives $\|g(T)\|_\infty \le \|g_0\|_\infty$, or equivalently,

$$\text{(4.4)}\qquad \|U(t)f_0\|_{\alpha,\beta} \le \|f_0\|_{\alpha,\beta} .$$

Next, since we assume Grad's angular cutoff potential it holds [3] that

$$\text{(4.5)}\qquad \|Q[f]\|_{\alpha,\beta-\gamma} \le C_{\alpha,\beta}\|fR^2_{\alpha,\beta} ,$$

for any $\alpha > 0$, $\beta \ge 0$. Here $\gamma \in [0,1]$ is a constant specific to the potential, while $C_{\alpha,\beta} > 0$ depends only on α,β decreasing in α, and $C_{\alpha,\beta} \to 0$ $(\alpha \to \infty)$, $\to \infty$ $(\alpha \to 0)$. Note that if $\gamma > 0$, Q is an unbounded operator on $X_{\alpha,\beta}$ and (1.9) cannot be solved by successive approximations, due to the loss of weight.

In order to control this unboundedness, we vary α with t. Then the integration in t in the last term of (1.9) plays a role of the smoothing operator. To see this, define the operator N_0 by

$$N_0[f] = \int_0^t U(t-s)\, Q[f(s)]\, ds \ ,$$

and put $T = \alpha/2\kappa$ and

$$|||f|||_{\alpha,\kappa,\beta} = \sup_{0\le t\le T} \|f(t)\|_{\alpha-\kappa t,\beta} \ .$$

Lemma 4.1. Let $\alpha > 0$ and $\beta \ge 0$. There is a constant $C_0 \ge 0$ such that for any $\kappa > 0$,

$$|||N_0[f]|||_{\alpha,\kappa,\beta} \le \frac{C_0}{\kappa}\ |||f|||^2_{\alpha,\kappa,\beta} \tag{4.6}$$

holds, if (4.4) with α replaced by $\alpha-\kappa t$ holds for any $t \in [0,T]$.

Proof: For $0 \le s \le t$, we have

$$\rho^{-1}_{\alpha-\kappa s,\beta-\gamma} = \rho^{-1}_{\alpha-\kappa t,\beta}\ e^{-\kappa(t-s)H}\ H^{\gamma/2} \ .$$

Therefore by (4.4) and (4.5), both with α replaced by $\alpha-\kappa s$, we get

$$\rho_{\alpha-\kappa t,\beta}|N_0[f](t)| \le \int_0^t e^{-\kappa(t-s)H}\ H^{1/2}\|Q[f(s)]\|_{\alpha-\kappa s,\beta-\gamma}\ ds$$

$$\le \int_0^t e^{-\kappa(t-s)H}\ H^{\gamma/2}C_{\alpha-\kappa s,\beta}\|f(s)\|^2_{\alpha-\kappa s,\beta}\ ds$$

$$\leq C_{\alpha/2,\beta}\left(\int_0^t e^{-\kappa(t-s)H} H^{\gamma/2} ds \right) |||f|||^2_{\alpha,\kappa,\beta} .$$

The last integral is majorized by $\kappa^{-1} H^{-1+\gamma/2} \leq \kappa^{-1}$ for $\gamma \in [0,2]$, whence (4.6) follows with $C_0 = C_{\alpha/2,\beta}$.

Define the operator N by

$$N[f] = U(t)f_0 + N_0[f] .$$

Then to solve (1.9) is to find a fixed point of N. Let B_a be a ball $\{f \mid |||f|||_{\alpha,\kappa,\beta} \leq a\}$. For $f \in B_a$, (4.4) and (4.6) give

$$|||N[f]|||_{\alpha,\kappa,\beta} \leq \|f_0\|_{\alpha,\beta} + \frac{C_0}{\kappa} a^2 . \tag{4.7}$$

Choose $\kappa > 0$ so large that $d = 1-4C_0\|f_0\|_{\alpha,\beta}/\kappa \geq 0$. Then set $a = \kappa(1-\sqrt{d})/2C_0$ which is the smaller root of the quadratic equation $(C_0/\kappa)a^2-a+\|f_0\|_{\alpha,\beta} = 0$. With this choice of a, (4.7) indicates that N maps B_a into itself. Further, since Q is quadratic, it follows from (4.6) that

$$|||Q[f] - Q[g]|||_{\alpha,\kappa,\beta} \leq \frac{C_0}{\kappa} |||f+g|||_{\alpha,\kappa,\beta} |||f-g|||_{\alpha,\kappa,\beta} ,$$

which is majorized, if $f,g \in B_a$, by $\mu|||f-g|||_{\alpha,\kappa,\beta}$ with $\mu = 2C_0a/\kappa = 1-\sqrt{d} < 1$. Hence N is a contraction mapping on B_a, having a unique fixed point $f \in B_a$. Note that $a \leq 2\|f_0\|_{\alpha,\beta}$. To summarize, we proved

Theorem 4.1. Let $\alpha > 0$ and $\beta \geq 0$. Assume (2.3) for Ω and (2.4) for $a(x,\xi)$. Further, suppose M is such that for any $\alpha' \in [\alpha/2,\alpha]$, $M_{\alpha',\beta}$ of (4.3) satisfies (3.4) and (3.5) for $p = \infty$. Then for any $f_0 \in X_{\alpha,\beta}$, there is a $\kappa > 0$ and (1.9) has a unique solution for $0 \le t \le T - \alpha/2\kappa$ satisfying

$$|||f|||_{\alpha,\kappa,\beta} \le 2\|f_0\|_{\alpha,\beta} .$$

Note that if M is given by (1.6a) or (1.6b), then $M_{\alpha,\beta} = M$ and satisfies (3.4), (3.5) for any α. In the case of (1.6c), we see that $M_{\alpha,\beta}$ then satisfies then only for $\alpha = T_w$, so the above theorem does not apply. However, solutions can be constructed by the method of [1].

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