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# Factorized Cumulant Expansion for Isotropic Turbulence

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## 1. Introduction

The factorized cumulant expansion of the characteristic functional was introduced to homomeneous turbulence by Tatsumi, Yamada and Takei {1}. In this expansion, the n-th cumulant $C^{(n)}$ is factorized into products of the (n-1)-th cumulants $C^{(n-1)}$ and a recursion factor, $\Lambda$ say. As a closure approximation of the order n, it gives an explicit expression for $C^{(n)}$, which was either neglected or taken into account only partially in previous approximations related with the cumulant expansion {2}, {3}. As an untruncated expansion of the characteristic functional, it amounts to assuming that the multiple interactions among n wave-number components associated with $C^{(n)}$ can be expressed as a multiple superposition of the triad interaction represented by $\Lambda$. This assumption is compatible with the notion of the dynamical equilibrium of turbulence which is essentially governed by the turbulent energy and the energy dissipation.

## 2. Factorization approximation

A substantial difficulty lies in the mathematical formulation of the assumption since a straightforward expression of the factorized cumulants leads to a rather complicated system of equations. In the previous work {1}, the factorization was applied to the degenerate cumulant,

$$\overline{C}^{(4)} = \int C^{(4)}(k,k',k''-h,h)dh, \tag{1}$$

and the ratio of the cumulants,

$$\overline{C}^{(3)}/C^{(2)} \propto \psi(k)/\phi(k), \tag{2}$$

was employed as the recursion factor, where $\phi(k)$ and $\psi(k)$ represent the energy spectrum density and the energy transfer density respectively.

The energy spectrum,

$$E(k) = 4\pi k^2 \phi(k), \tag{3}$$

and the energy dissipation rate spectrum,

$$\gamma(k) = \nu k^2 - \psi(k)/2\phi(k), \tag{4}$$

which was calculated numerically under this approximation are depicted in Fig.1 and Fig.2 respectively. The behaviour of these and other statistical functions are generally similar to those due to the modified zero fourth cumulant approximation {3}. There exists, however, an essential difference in the relative strength of the inertial to the viscous effects, which is much more enhanced in the former approximation than in the latter. It is also confirmed analytically that the former approximation gives the energy dissipation wavenumber $k_d$ of the order $\nu^{-1}$, while Kolmogorov's similarity requires that $k_d \propto \nu^{-3/4}$. Such features of overestimation of the inertial effect seems to be attributed to the choice of the recursion factor (2), which is dimensionally correct but does not satisfy the proper functional form of the cumulant expansion

## 3. New factorization scheme

In principle, the recursion factor $\Lambda$ is not a function to be assumed in any specific form but an unknown function which should be determined as the coefficient of the recursion operator,

$$\int\int \Lambda_{i_p i_q i_r}(k_p,k_q|k_p+k_q) z_{i_p}(k_p) z_{i_q}(k_q) \frac{\partial}{\partial z_{i_r}(k_p+k_q)} dk_p dk_q, \quad (5)$$

which connects two consecutive terms in the cumulant expansion.

The recursion operator (5) applied to a cumulant $C^{(n)}$ gives the following recursion formula for $C^{(n)}$:

$$C^{(n)}_{i_1 \ldots i_n}(k_1,\ldots,k_n) = \sum_{{}_nC_2} \Lambda_{i_p i_q i_r}(k_p,k_q|k_p+k_q) \times$$
$$\times\, C^{(n-1)}_{i'_1 \ldots i'_n i_r}(k'_1,\ldots,k'_n,k_p+k_q), \quad (6)$$

where $(i'_1 \ldots i'_n)$ and $(k'_1,\ldots,k'_n)$ denote the sequences $(i_1 \ldots i_n)$ and $(k_1,\ldots,k_n)$ devoid of the members $(i_p i_q)$ and $(k_p,k_q)$ respectively, and the condition $k_1+\ldots+k_n = 0$ is assumed.

For the two lowest orders, eq.(6) is written as

$$C^{(3)}_{ijk}(k,k',k'') = \Lambda_{ijl}(k,k'|-k'')C^{(2)}_{kl}(k'') + \Lambda_{ikm}(k,k''|-k')C^{(2)}_{jm}(k')$$
$$+ \Lambda_{jkn}(k',k''|-k)C^{(2)}_{in}(k), \quad (7)$$

with $k+k'+k'' = 0$, and

$$C^{(4)}_{ijkl}(k,k',k'',k''') = \Lambda_{ijm}(k,k'|k+k')C^{(3)}_{klm}(k'',k''',k+k')$$
$$+ \Lambda_{ikn}(k,k''|k+k'')C^{(3)}_{jln}(k',k''',k+k'')$$
$$+ \Lambda_{ilo}(k,k'''|k+k''')C^{(3)}_{jko}(k',k'',k+k''')$$
$$+ \Lambda_{jkp}(k',k''|k'+k'')C^{(3)}_{ilp}(k,k''',k'+k'')$$
$$+ \Lambda_{jlq}(k',k'''|k'+k''')C^{(3)}_{ikq}(k,k'',k'+k''')$$

$$+ \Lambda_{klr}(k'',k'''|k''+k''')C^{(3)}_{ijr}(k,k',k''+k'''), \quad (8)$$

with $k+k'+k''+k''' = 0$.

If we take the cumulant equations for $C^{(2)}$ and $C^{(3)}$ which involve three unknowns $C^{(2)}$, $C^{(3)}$ and $C^{(4)}$, and employ the factorization formulae (7) and (8) as the closure relations, we obtain a closed set of equations for the four unknowns $C^{(2)}$, $C^{(3)}$, $C^{(4)}$ and $\Lambda$. This constitutes a new scheme of the factorized cumulant approximation of the fourth order.

## 4. Energy spectrum equation

For isotropic turbulence, the cumulants $C^{(2)}$ and $C^{(4)}$ are expressed as

$$C^{(2)}_{ij}(k,-k) = \phi(k)\Delta_{ij}(k), \quad (9)$$

$$C^{(3)}_{ijk}(k,k',k'') = -i\Delta_{in}(k)\Delta_{jm}(k')\Delta_{kl}(k'')\times$$
$$\times\{\psi(k,k'|k'')(k''_n\delta_{lm} + k''_m\delta_{ln})$$
$$+ \psi(k,k''|k')(k'_l\delta_{mn} + k'_n\delta_{ml})$$
$$+ \psi(k',k''|k)(k_l\delta_{nm} + k_m\delta_{nl})\}, \quad (10)$$

with

$$\Delta_{ij}(k) = \delta_{ij} - k_i k_j/k^2, \quad (11)$$

where $\psi(k,k'|k'')$ is a scalar function which depends upon the magnitudes of the wavenumber vectors $k=|k|$, $k'=|k'|$ and $k''=|k''|$. In general, the isotropic form of $C^{(3)}$ must involve a term proportional to $A(k,k'\ k'')k_l k_m k'_n$, A being a scalar function antisymmetric with respect to k and k', in the curly brackets on the right-hand side, but this term is omitted in eq.(10) assuming that $C^{(3)} = 0$ as an initial condition.

On substitution from (9), (10) and (11) into (7), we obtain the isotropic form of $\Lambda$ as follows:

$$\Lambda_{ijl}(k,k'|-k") = -i\lambda(k,k'|k")\Delta_{in}(k)\Delta_{jm}(k')(k"_n\delta_{lm} + k"_m\delta_{ln}), \quad (12)$$

where

$$\lambda(k,k'|k") = \psi(k,k'|k")/\phi(k") \quad (13)$$

is the defining scalar function of $\Lambda$.

The isotropic form of the degenerate cumulant $\overline{C}^{(3)}$ is expressed as

$$\overline{C}^{(3)}_{ijk}(k) = \int C^{(3)}_{ijk}(k,-k-h,h)dh$$

$$= -\frac{i}{2k^2}\psi(k)\{k_j\Delta_{ik}(k) + k_k\Delta_{ij}(k)\}, \quad (14)$$

where $\psi(k)$ denotes the energy transfer density introduced in (2). Thus,

$$k_k\overline{C}^{(3)}_{iik} = -i\psi(k). \quad (15)$$

Substituting the isotropic form (10) into the left-hand side of eq.(15), we find that

$$\psi(k) = \int\Big[\psi(k,k'|k")\mu(k^2 - k'^2) + \psi(k,k"|k')\{kk' + \mu(k"^2 - k^2)\}$$
$$-\psi(k',k"|k)\{kk' + \mu(k"^2 - k'^2)\}\Big](\mu - \mu'\mu")dk', \quad (16)$$

where $\mu' = (k'.k")/k'k"$ and $\mu" = (k".k)/k"k$.

It may easily be seen that (16) satisfies the consistency condition for the energy transfer function,

$$\int_0^\infty \psi(k)dk = 0. \quad (17)$$

Further, substitution from (13) into (16) gives that

$$\psi(k) = \phi(k)\lambda(k)$$
$$+\int\Big[\phi(k")\lambda(k,k'|k")\mu(k^2 - k'^2) \quad (18)$$
$$+ \phi(k')\lambda(k,k"|k')\{kk' + \mu(k"^2 - k^2)\}\Big](\mu - \mu'\mu")dk',$$

$$\lambda(k) = -\int \lambda(k',k''|k)\{kk' + \mu(k''^2 - k'^2)\}(\mu - \mu'\mu'')dk'. \quad (19)$$

Then, the energy spectrum equation of isotropic turbulence is written in the following scalar form:

$$\left(\frac{\partial}{\partial t} + 2\Gamma(k)\right)\phi(k) =$$
$$= \int\Big[\phi(k'')\lambda(k,k'|k'')\mu(k^2 - k'^2) + \phi(k')\lambda(k,k''|k')\{kk' + \mu(k''^2 - k^2)\}\Big](\mu - \mu'\mu'')dk', \quad (20)$$

$$\Gamma(k) = \nu k^2 - \frac{1}{2}\lambda(k). \quad (21)$$

Equations (20) and (21) provide us with the energy spectrum equation under the present factorization approximation. These equations should be compared with another exact form of the energy spectrum equation,

$$\left(\frac{\partial}{\partial t} + 2\gamma(k)\right)\phi(k) = 0, \quad (22)$$

which were used in the previous factorization scheme. The factorization of $\bar{C}^{(4)}$ in the equation for $C^{(3)}$ in terms of the factor $\psi(k)/\phi(k)$ led to the scaling of the energy-dissipation wavenumber $k_d \propto \nu^{-1}$ in the previous work {1}. In the present work, the factorization of the same term will be made in terms of $\lambda(k)$, and this can lead to the usual Kolmogorov's scaling $k_d \propto \nu^{-3/4}$.

The energy spectrum equation (20) together with the energy transfer equation, that is the isotropic form of the equation for $C^{(3)}$, constitute the closed set of dynamical equations under the present factorization scheme. Analytical properties and numerical solutions of these equations will be worked out and discussed in a separate paper.

References

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{2} Tatsumi, T. (1957) The theory of decay process of incompressible isotropic turbulence. Proc. Roy. Soc. London, A 239, 16-45.

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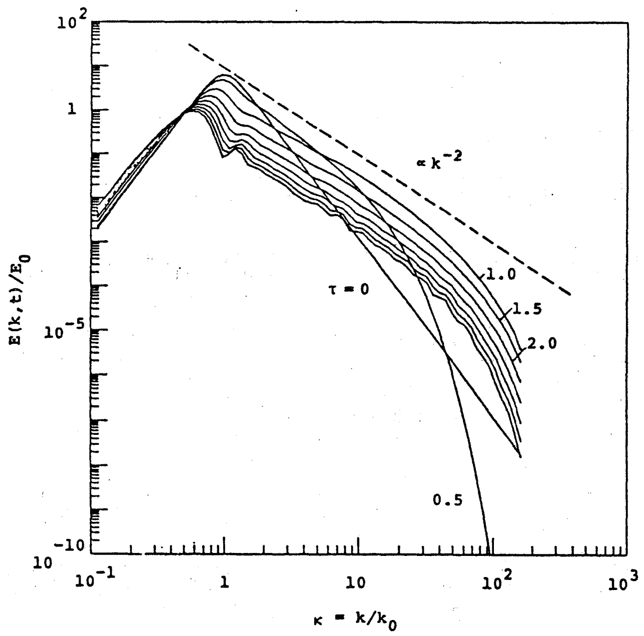


Fig. 1

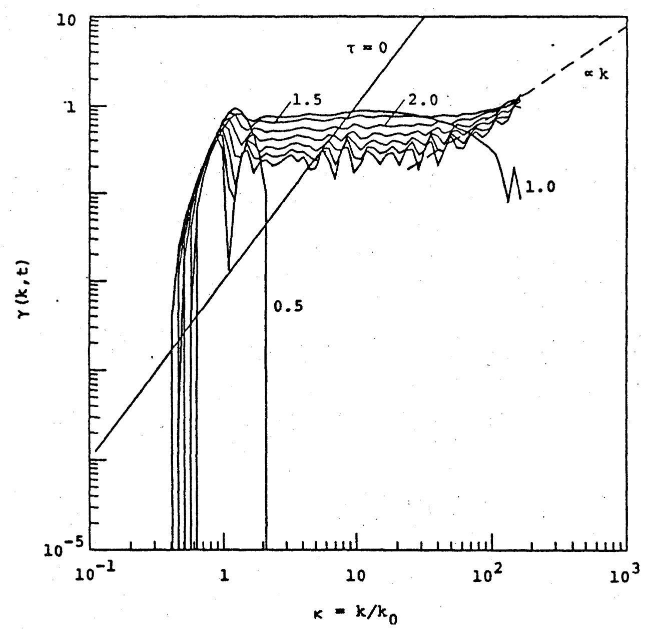


Fig. 2