数理解析研究所講究録
第664巻 1988年 5-11

A Note on Closed-to-Convex Functions

(An Application of Schwarz-Christoffel Formula)

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Let z_k : $k = 1, 2, \cdots, n$ be points on the unit circle $z = 1$ such that

$$z_k = e^{i\theta_k} \ (0 \leqq \theta 1 < \theta 2 < \cdots < \theta n < 2\pi)$$

and $w(z)$ defined in the unit disc $z < 1$ be the function which satisfies the following equation

$$(1) \quad \frac{dw}{dz} = C(z - z_1)^{\alpha_1 - 1}(z - z_2)^{\alpha_2 - 1} \cdots (z - z_n)^{\alpha_n - 1}$$

where C is a constant complex nummber, and α_k ($k = 1, 2, \cdots, n$) satisfy

$$0 \leqq \alpha_k \leqq 2 \ , \ \sum_{k=1}^{n} \alpha_k = n - 2$$

and (*) are assumed to take values of the branch $1^{\alpha_k} = 1$.

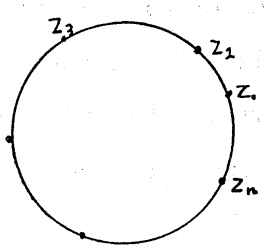

Fig. 1

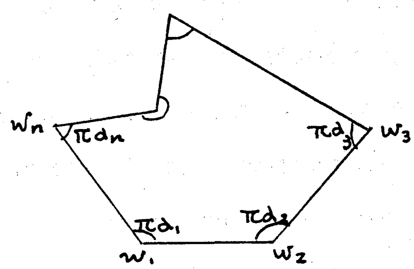

Fig. 2

If points z_k on the unit circle $z = 1$ are suitably chosen, by the function $w(z)$, the interior of unit disc in z-plane is transformed into the interior of a polygon with n sides in the w-plane, and each vertex w_k of the polygon corresponds to z_k on the unit circle $z = 1$. And each interior angle at w_k is equal to $\pi\alpha_k$.

But if points z_k on the unit circle are arbitrary chosen, the polygon defined by (1) may be not in general a bounded polygon in common sense. This formula (1) is well-known as Schwarz-Christoffel's transformation.

The function $w(z)$ defined in (1) can be normalised to $w(0) = 0$ and $w'(0) = 1$, as follows

$$(2)\quad \frac{dw}{dz} = (1-\varepsilon_1 z)^{\delta_1}(1-\varepsilon_2 z)^{\delta_2}\cdots(1-\varepsilon_n z)^{\delta_n}$$

where $\varepsilon_k = z_k^{-1}$, $\delta_k = \alpha_k - 1$, $|\delta_k| \leqq 1$ and $\sum_{k=1}^{n} \delta_k = -2$.

At first, we consider the case when a polygon is convex. In this case, interior angles $\pi\alpha_k$ at all vertices are smaller than π, and all δ_k satisfy

$$-1 < \delta_k < 0 , \quad \sum_{k=1}^{n} \delta_k = -2$$

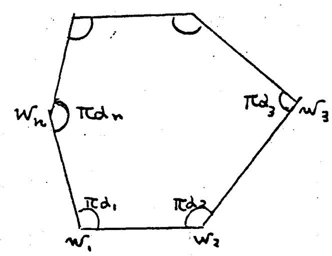

Fig. 3

And we have a following relation from (2)

$$z\,\frac{w''(z)}{w'(z)} = -\sum_{k=1}^{n} \frac{\delta_k \varepsilon_k z}{1-\varepsilon_k z} = -\frac{1}{2}\sum_{k=1}^{n} \delta_k \frac{1+\varepsilon_k z}{1-\varepsilon_k z} - 1$$

and from $\delta k < 0$, we have an equality

$$\text{Re.}\ z\frac{w''(z)}{w'(z)} > -1 : \ |z| < 1$$

Accordingly, in the case when all δ_k are negative, for arbitrary points z_k on the unit circle, the function $w(z)$ defined by (2) is a convex function.

Next, we consider the estimation of coefficients by the Taylor expansion of a convex function $w(z)$ defined by (2). Generally, when γ is a positive number, in the power series

$$(1-z)^{-\gamma} = 1 + \frac{\gamma}{1!}z + \frac{\gamma(\gamma+1)}{2!}z^2 + \frac{\gamma(\gamma+1)(\gamma+2)}{3!}z^3 + \cdots$$

all coefficients are positive.

Accordingly, all coefficients of power series of $(1-\varepsilon_k z)^{\delta_k}$ in (2) are majorated by coefficients of power series of $(1-z)^{\delta_k}$, and coefficients of the power series of dw/dz in (2) are majorated by coefficients of

$$(1-z)^{\delta_1}(1-z)^{\delta_2}\cdots(1-z)^{\delta_n} = (1-z)^{-2}$$
$$= 1 + 2z + 3z^2 + 4z^3 + \cdots$$

and all coefficients of Taylor exapansion of $w(z)$ are majorated by coefficients of

$$(1-z)^{-1} = 1+z+z^2+z^3+\cdots\cdots$$

Next, we consider the case when a function defined by (2) is closed-to-convex. At first, we show a lemma as follows

<u>Lemma.</u> Let z_k ($k=1, 2, \cdots, 2n$) be points on the unit circle $|z|=1$ such that

$$z_k = e^{i\theta_k} \;:\; 0 \leqq \theta_1 \leqq \theta_2 \leqq \cdots \leqq \theta_{2n} \leqq 2\pi$$

and $\phi(z)$ be a function reresented by

$$\phi(z) = \frac{1-\varepsilon_2 z}{1-\varepsilon_1 z}\,\frac{1-\varepsilon_4 z}{1-\varepsilon_3 z}\cdots\frac{1-\varepsilon_{2n} z}{1-\varepsilon_{2n-1} z}$$

where $\varepsilon_k = z_k^{-1}$.

Then the function $\phi(z)$ takes values on a half plane which contains the unit in its interior, and is borderd by a line which passes the origin.

<u>Proof.</u> In Fig.4 , when $|z|=1$, we have

$$\arg\frac{z_2-z}{z_1-z} = \begin{cases} \frac{1}{2}(\theta_2-\theta_1) & : z \in \widehat{z_1 z_2} \\ \frac{1}{2}(\theta_2-\theta_1)+\pi & : z \in \widehat{z_1 z_2} \end{cases}$$

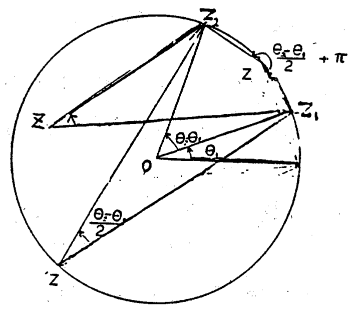

Fig. 4

and when $|z|<1$, we have

$$\frac{1}{2}(\theta_2-\theta_1) < \arg\frac{z-z_2}{z-z_1} < \frac{1}{2}(\theta_2-\theta_1)+\pi$$

Accordingly, when $|z|=1$, we have

$$\arg\frac{1-\varepsilon_2 z}{1-\varepsilon_1 z} = \begin{cases} -\frac{1}{2}(\theta_2-\theta_1) & : z \in \widehat{z_1 z_2} \\ -\frac{1}{2}(\theta_2-\theta_1)+\pi & : z \in \widehat{z_1 z_2} \end{cases}$$

When z varies on the unit circle, if z is not on any one of arcs $\widehat{z_1 z_2}, \widehat{z_3 z_4}, \cdots, \widehat{z_{2n-1} z_{2n}}$

$$\arg\phi(z) = \Theta = \frac{-1}{2}(\theta_2-\theta_1+\theta_4-\theta_3+\cdots+\theta_{2n}-\theta_{2n-1})$$

and if z is on any one of these arcs, $\arg\phi(z)$ is equal to $\Theta+\pi$.

And when z is an interior point on the unit disc, we have

$$\Theta < \arg\phi(z) < \Theta + \pi$$

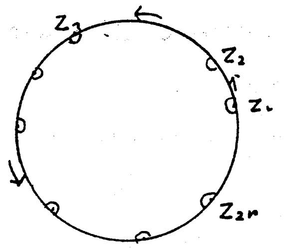

Fig. 5

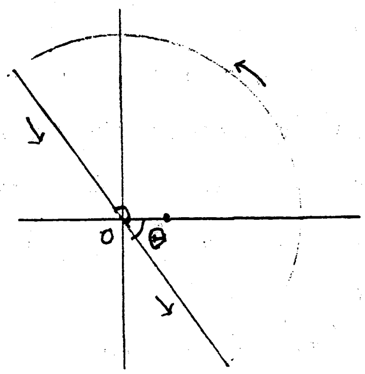

Fig. 6

Now the lemma has been proved.

Next I show a theorem as follows.

Theorem . Let $z_k = e^{i\theta_k}$: $k = 1,2,\cdots,n$ $(0 \leqq \theta_1 \leqq \theta_2 \leqq \cdots \leqq \theta_n)$ and $z_{j,k} = e^{i\theta_{j,k}}$: $k = 1,2,\cdots,2n_j$ $(0 \leqq \theta_{j,1} \leqq \theta_{j,2} \leqq \cdots \leqq \theta_{j,2n_j})$; $j=1,2,\cdots,m$ be points on the unit cicle $z = 1$, and $w(z)$ defined in the unit disc $|z| < 1$ be a function which satisfies

$$(3)\qquad \frac{dw}{dz} = \prod_{k=1}^{n}(1-\varepsilon_k z)^{\delta_k} \prod_{j=1}^{m}\left[\prod_{k=1}^{n_j}\frac{1-\varepsilon_{j,2k}\, z}{1-\varepsilon_{j,2k-1}\, z}\right]^{\lambda_j}$$

where δ_k are negative numbers satisfying $\sum_{k=1}^{n}\delta_k = -2$, and λ_j are real numbers satisfying $\sum_{j=1}^{m}|\lambda_j| = 1$.

Then, in the Taylor expansion of the function $w(z)$,i.e.

$$w(z) = z + A_2 z^2 + \cdots + A_k z^k + \cdots$$

$$|A_n| \leqq n \quad : \quad n = 1, 2, 3, \cdots$$

and $w(z)$ is a closed-to-convex function.

Proof of Theorem. In the equation (3), $\phi(z) = \prod_{k=1}^{n}(1-\varepsilon_k z)^{\delta_k}$ is the derivative of a convex function, and in the Taylor expansion of $\phi(z)$,i.e.

$$\phi(z) = 1 + a_1 z + a_2 z^2 + \cdots + a_k z^k + \cdots$$

all coefficient a_k : $k = 1,2,\cdots$ are majorated by coefficients of

$$(1-z)^{-2} = 1 + 2z + 3z^2 + \cdots$$

And in (3), $\phi_j(z)=\prod_{k=1}^{n_j}\dfrac{1-\varepsilon_{j,2k}\ z}{1-\varepsilon_{j,2k-1}\ z}$ takes values on a half plane which contains the unit in its interior , and is bordered by a line which passes the origin.

And $\psi_j(z)=\left[\phi_j(z)\right]^{\lambda_j}=\left[\prod_{k=1}^{n_j}\dfrac{1-\varepsilon_{j,2k}\ z}{1-\varepsilon_{j,2k-1}\ z}\right]^{\lambda_j}$ takes values on a domain, which contains the unit in its interior, and is bordered by two lines meet at the angle $\pi\lambda_j$ in the origin.

Accordingly, the function

$$\psi(z)=\prod_{j=1}^{m}\psi_j(z)=\prod_{j=1}^{m}\left[\prod_{k=1}^{n_j}\frac{1-\varepsilon_{j,2k}\ z}{1-\varepsilon_{j,2k-1}\ z}\right]^{\lambda_j}$$

takes values on a half plane, which contains the unit in its interior and is bordered by a line passes the origin.

In the Taylor expansion of $\psi(z)$ i.e.

$$\psi(z)=1+b_1 z+b_2 z^2+b_3 z^3+\cdots$$

it is well-known that all cosfficient b_k are majorated by coefficients of the expansion

$$\frac{1+z}{1-z}=1+2z+2z^2+2z^3+\cdots$$

Accordigly, in the expansion of the function dw/dz defined in (3), i.e.

$$\frac{dw}{dz}=\phi(z)\,\psi(z)=1+a_1 z+a_2 z^2+\cdots$$

every coefficient a_k is majorated by the function

$$\frac{1+z}{(1-z)^3}=1+2^2 z+3^2 z^2+\cdots+n^2 z^{n-1}+\cdots$$

and in the Taylor expansion

$$w(z) = z + A_2 z^2 + \cdots + A_k z^k + \cdots$$

all coefficients A_k are majorated by coefficients of the function

$$\frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \cdots + nz^n + \cdots$$

Thus, $|A_k| \leqq n$ has been proved.

Moreover, in $dw/dz = \phi(z)\ \psi(z)$, $\phi(z)$ is the derivative of a convex function, and $\psi(z)$ takes values on a half plane which contains the unit in its interior, and is bordered by a line which passes the origin. Accordingly, the function $w(z)$ is a closed-to convex function. Thus the theorem has been proved.

At last, I show a simple example of closed-to-convex functions. Let us consider a simply connected polygon Ω which has 2n sides parallel to the real axis or imaginary axis in the w-plane. If we call its vertices w_1, $w_2, \cdots w_{2n}$ and denote its interior angles $\pi\alpha_1, \pi\alpha_2, \cdots, \pi\alpha_{2n}$ respectively, α_k take the value 1/2 or 3/2, and satisfy $\sum_{k=1}^{n} \alpha_k = 2n - 2$.

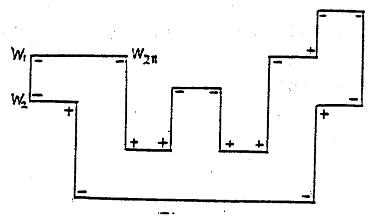

Fig. П

We can construct the function $w = f(z)$ which maps the interior of unit circle $|z| < 1$ onto the interior of such a polygon by

$$\frac{dw}{dz} = C(z - z_1)^{\alpha_1 - 1}(z - z_2)^{\alpha_2 - 1} \cdots (z - z_{2n})^{\alpha_{2n} - 1}$$

where C is a complex number, and z_k are points on the unit circle, such that

$$z_k = e^{i\theta_k}: \ k = 1, 2, \cdots, 2n \ ; \quad 0 < \theta_1 < \theta_2 < \cdots < \theta_{2n} < 2\pi$$

Now ,if we put $z_k^{-1} = \varepsilon_k$, the function $w = f(z)$ is normalized by

$$\frac{dw}{dz} = (1 - \varepsilon_1 z)^{\delta_1}(1 - \varepsilon_2 z)^{\delta_2} \cdots (1 - \varepsilon_{2n} z)^{\delta_{2n}}$$

where $\delta_k (= \alpha_k - 1)$ take the value $-1/2$ or $1/2$, and $\sum_{k=1}^{2n} \delta_k = -2$.

We consider the polygon shown in Fig. 7. In this case, we can write signs of δ_k in order and if we take apart suitable four minus signs, we can arrange a sequence of couples (−+) or (+−) as follows

$$\ominus\ominus\ (+-)(-+)\ \ominus\ominus\ (-+)(+-)(-+)(+-)$$

In such a case, as the function w(z) satisfies

$$\frac{dw}{dz} = \prod_{k=1}^{4}(1-\varepsilon_{1,k}\ z)\left[\prod_{k=1}^{3}\frac{1-\varepsilon_{2,2k}\ \ z}{1-\varepsilon_{2,2k-1}\,z}\right]^{1/2}\left[\prod_{k=1}^{4}\frac{1-\varepsilon_{3,2k}\ \ z}{1-\varepsilon_{3,2k-1}\,z}\right]^{-1/2}$$

and satisfies the codition of the theorem, such a mapping fucntion w(z) is closed-to-convex, and in the Taylor expansion $w(z) = z+A_2z^2+A_3z^3+\cdots$, all coefficients A_k satify $|A_k| \leqq k$.

Remark. Equalities $|A_k| = k$ ($k = 1,2,\cdots$) can be satisfied only when

$$z_1 = z_2 = z_3 = z_6 = z_7 = z_8 = \varepsilon$$

$$z_4 = z_5 = -\varepsilon \quad (|\varepsilon| = 1)$$

as the limit case of such a polygon in Fig. 8.

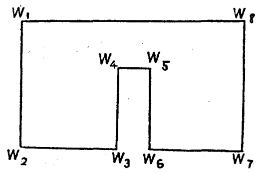

Fig. 8