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ON THE MULTIVALENT FUNCTIONS WITH BOUNDED ARGUMENT

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ABSTRACT

The object of the present paper is to give the starlike boundary and convex boundary of certain subclass of p-valent functions in the unit disk.

1. Introduction

Let F(p) be the subclass of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in N = \{1,2,3,\cdots\})$$

which are analytic in $E = \{ z \,|\, |z| < 1 \}$.

Also let $A(\alpha)$ denote the class of functions

$$g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n ,$$

analytic in E and satisfying

$$|\arg g(z)| \leqq \pi\alpha/2 \qquad (z \in E)$$

where $\alpha > 0$.

It is well known [3] that if $g(z) \in A(1)$, then

$$g(z) \prec \frac{1+z}{1-z}$$

where $\prec$ denotes " is subordinate to " and [1] that

$$(1) \qquad \left| \frac{g'(z)}{g(z)} \right| \leqq \frac{2}{1-r^2} , \qquad \text{for } |z| = r < 1.$$

In this paper, we need the following lemata.

LEMMA 1. Suppose $g(z) \in A(\alpha)$, where $\alpha > 0$. Then we have

$$\left| \frac{zg'(z)}{g(z)} \right| \leqq \frac{2\alpha r}{1-r^2} \qquad \text{for } |z| = r < 1.$$

PROOF. Let $h(z) = g(z)^{1/\alpha}$. Then we have $h(z) \in A(1)$ and from (1) we have

$$\left| \frac{h'(z)}{h(z)} \right| = \frac{1}{\alpha} \left| \frac{g'(z)}{g(z)} \right| \leqq \frac{2}{1 - r^2}$$

for $|z| = r < 1$. This completes our proof and a proof of this result can be found in [4, 7].

If a function

$$g(z) = 1 + \sum_{n=1}^{\infty} b_n z^n$$

is analytic in E and does not assume non-positive real values in E, then $g(z) \in A(2)$ and from LEMMA 1, it follows that

$$\left| \frac{zg'(z)}{g(z)} \right| \leqq \frac{4r}{1 - r^2} \qquad \text{for } |z| = r < 1.$$

REMARK. Throughout the paper, all powers are meant as principal values.

LEMMA 2. Suppose that $f(z) \in F(p)$ and

$$p + \mathrm{Re}\, \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} > 0 \qquad \text{in E.}$$

Then we have

$$1 + \mathrm{Re}\, \frac{zf''(z)}{f'(z)} > 0 \qquad \text{in E,}$$

and therefore $f(z)$ is p-valently convex in E.

A proof of this lemma can be found in [5, Theorem 1].

LEMMA 3. Suppose $f(z) \in F(p)$ and

$$\mathrm{Re}\, \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} > 0 \qquad \text{in E.}$$

Then we have

$$\mathrm{Re}\, \frac{zf'(z)}{f(z)} > 0 \qquad \text{in E,}$$

and therefore $f(z)$ is p-valently starlike in E.

A proof of this lemma can be found in [5, Theorem 5].

LEMMA 4. Let $h(z) \in A(2)$. Then we have

$$\left(\frac{1 - r}{1 + r} \right)^2 \leqq \operatorname{Re} h(z) \leqq \left(\frac{1 + r}{1 - r} \right)^2$$

for $|z| = r < 2 - \sqrt{3}$, and

$$\frac{1 - 6r^2 + r^4}{2(1 - r^2)^2} \leqq \operatorname{Re} h(z) \leqq \left(\frac{1 + r}{1 - r} \right)^2$$

for $2 - \sqrt{3} \leqq |z| = r < 1$. Therefore we have that

$$\operatorname{Re} h(z) > 0 \qquad \text{in } |z| < \sqrt{2} - 1.$$

This result is sharp. A proof can be found in [6].

2. Main theorems

THEOREM 1. Let $f(z) \in F(p)$ and

$$\frac{f(z)}{z^p} \in A(\alpha), \qquad \text{where } \alpha > 0.$$

Then $f(z)$ is p-valently starlike in $|z| < (\sqrt{\alpha^2 + p^2} - \alpha)/p$ and the result is sharp.

PROOF. Let $g(z) = f(z)/z^p$. Then $g(z) \in A(\alpha)$ and from LEMMA 1, we have

$$\left| \frac{zg'(z)}{g(z)} \right| = \left| \frac{zf'(z)}{f(z)} - p \right| \leqq \frac{2\alpha r}{1 - r^2}$$

for $|z| = r < 1$.

Then it follows that

$$\operatorname{Re} \frac{zf'(z)}{f(z)} \geqq p - \frac{2\alpha r}{1 - r^2} = \frac{p - 2\alpha r - pr^2}{1 - r^2}$$

for $|z| = r < 1$.

This shows that $f(z)$ is p-valently starlike in $|z| < (\sqrt{\alpha^2 + p^2} - \alpha)/p$.

The result is sharp as seen by letting

$$f(z) = z^p \left(\frac{1 + z}{1 - z} \right)^{\alpha}.$$

COROLLARY 1. Let $f(z) \in F(p)$ and let $f(z)/z^p$ do not assume non-positive real values in E. Then $f(z)$ is p-valently starlike in $|z| < (\sqrt{4 + p^2} - 2)/p$ and the result is sharp as seen by letting

$$f(z) = z^p\left(\frac{1+z}{1-z}\right)^2 .$$

Applying the same method as the proof of THEOREM 1, we have the following theorem.

THEOREM 2. Let $f(z) \in F(p)$ and $f^{(p)}(z)/(p!) \in A(\alpha)$. Then $f(z)$ is p-valently onvex in $|z| < (\sqrt{\alpha^2 + p^2} - \alpha)/p$.

PROOF. Let $g(z) = f^{(p)}(z)/(p!)$. From LEMMA 1, we have

$$\left|\frac{zg'(z)}{g(z)}\right| = \left|\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right| \leq \frac{2\alpha r}{1-r^2}$$

for $|z| = r < 1$.

Then, it follows that

$$p + \mathrm{Re}\,\frac{zf^{(p+1)}(z)}{f^{(p)}(z)} \geqq p - \frac{2\alpha r}{1-r^2} > 0$$

in $|z| = r < (\sqrt{\alpha^2 + p^2} - \alpha)/p$.

Therefore, from LEMMA 2, we have

$$1 + \mathrm{Re}\,\frac{zf''(z)}{f'(z)} > 0 \qquad \text{in } |z| < (\sqrt{\alpha^2 + p^2} - \alpha)/p.$$

This shows that $f(z)$ is p-valently convex in $|z| < (\sqrt{\alpha^2 + p^2} - \alpha)/p$..

This completes our proof.

REMARK. For the case $p = 1$ and $\alpha = 1$, this result is sharp [1, 4].

COROLLARY 2. Let $f(z) \in F(p)$ and let $f^{(p)}(z)/(p!)$ do not assume non-positive real values in E ($f^{(p)}(z)/(p!) \in A(2)$).

Then $f(z)$ is p-valently convex in $|z| < (\sqrt{4 + p^2} - 2)/p$.

THEOREM 3. Let $f(z) \in F(p)$ and $f^{(p-1)}(z)/(p!z) \in A(\alpha)$.

Then $f(z)$ is p-valently starlike in $|z| < \sqrt{\alpha^2 + 1} - \alpha$.

PROOF. Let $g(z) = f^{(p-1)}(z)/(p!z)$. Then from LEMMA 1, we easily have

$$\left| \frac{zg'(z)}{g(z)} \right| = \left| \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - 1 \right| \leqq \frac{2\alpha r}{1-r^2}$$

for $|z| = r < 1$.

It follows that

$$\operatorname{Re} \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} > 0 \qquad \text{in } |z| < \sqrt{\alpha^2 + 1} - \alpha .$$

Then, from LEMMA 3, we have

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \qquad \text{in } |z| < \sqrt{\alpha^2 + 1} - \alpha .$$

This shows that $f(z)$ is p-valently starlike in $|z| < \sqrt{\alpha^2 + 1} - \alpha$.

This completes our proof.

REMARK. For the case $p = 1$ and $\alpha = 1$, this result is sharp [1, 2, 4].

COROLLARY 3. Let $f(z) \in F(p)$ and let $f^{(p-1)}(z)/z$ do not assume non-positive real values in E. Then $f(z)$ is p-valently starlike in $|z| < \sqrt{5} - 2$.

THEOREM 4. Let $f(z) \in F(p)$ and let

$$\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \in A(\alpha), \qquad 0 < \alpha \leqq 1.$$

Then $f(z)$ is p-valently convex in $|z| < \rho$ where ρ is the root of the equation

$$\left(\frac{1-r}{1+r} \right)^{\alpha} - \frac{2\alpha r}{1-r^2} = 0.$$

PROOF. Let $g(z) = zf^{(p)}(z)/f^{(p-1)}(z)$. Then from LEMMA 1, we have

$$\left| \frac{zg'(z)}{g(z)} \right| = \left| 1 + \frac{zf^{(p+1)}(z)}{f^{(p)}(z)} - \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right|$$

$$\leqq \frac{2\alpha r}{1-r^2} \qquad \text{for } |z| = r < 1.$$

Because of the assumption that $0 < \alpha \leqq 1$, we easily have

$$\operatorname{Re} \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \geqq \left(\frac{1-r}{1+r} \right)^{\alpha}$$

for $|z| = r < 1$. Then we have

$$1 + \operatorname{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)} \geqq \operatorname{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} - \frac{2\alpha r}{1-r^2}$$

$$= \left(\frac{1-r}{1+r}\right)^{\alpha} - \frac{2\alpha r}{1-r^2} \qquad \text{for } |z| = r < 1.$$

Hence we have

$$1 + \operatorname{Re}\frac{zf^{(p+1)}(z)}{f^{(p)}(z)} > 0 \qquad \text{for } |z| < \rho$$

where ρ is as stated in the hypothesis of the THEOREM 4. Clearly $0 < \rho < 1$.

THEOREM 5. Let $f(z) \in F(p)$ and let

$$\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \in A(\alpha), \qquad \alpha > 1.$$

Then $f(z)$ is p-valently starlike for $|z| < (1 - \cos(\pi/2\alpha))/\sin(\pi/2\alpha)$.

PROOF. From the hypothesis of the Theorem, we easily have

$$\left| \arg \frac{zf^{(p)}(z)}{f^{(p-1)}(z)} \right| \leqq \alpha \operatorname{Sin}^{-1}\left(\frac{2r}{1+r^2}\right)$$

for $|z| = r < 1$.

This shows that

$$\operatorname{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} > 0 \qquad \text{for } |z| < \frac{1-\cos(\pi/2\alpha)}{\sin(\pi/2\alpha)}.$$

Therefore, from LEMMA 3 we have

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > 0 \qquad \text{for } |z| < \frac{1-\cos(\pi/2\alpha)}{\sin(\pi/2\alpha)}.$$

This completes our proof.

COROLLARY 4. Let $f(z) \in F(p)$ and let $zf^{(p)}(z)/f^{(p-1)}(z) \in A(2)$.

Then $f(z)$ is p-valently starlike in $|z| < \sqrt{2} - 1$.

PROOF. From LEMMA 4, we have

$$\operatorname{Re}\frac{zf^{(p)}(z)}{f^{(p-1)}(z)} > 0 \qquad \text{in } |z| < \sqrt{2} - 1.$$

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