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NOTES ON P-VALENTLY BAZILEVIĆ FUNCTIONS

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ABSTRACT

The object of the present paper is to improve the former results for p-valently Bazilević functions which were recently proved by the author and others.

I. INTRODUCTION

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \qquad (p \in \mathbb{N} = \{1,2,3,\ldots\}) \tag{1.1}$$

which are analytic in the unit disk $E = \{z: |z| < 1\}$. A function $f(z)$ belonging to A_p is said to be p-valently starlike if and only if it satisfies the condition

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \qquad (z \in E). \tag{1.2}$$

We denote by S_p^* the subclass of A_p consisting of functions which are p-valently starlike in E.

A function $f(z)$ belonging to the class A_p is said to be p-valently Bazilević of type β and order γ if there exists a function $g(z)$ belonging to S_p^* such that

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)^{1-\beta} g(z)^{\beta}}\right\} > \gamma \qquad (z \in E) \tag{1.3}$$

for some β ($\beta > 0$) and γ ($0 \leqq \gamma < p$).

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Also we denote by $B_p(\beta,\gamma)$ the subclass of A_p consisting of all p-valently Bazilevič functions of type β and order γ in E. The concept of Bazilevič functions was first introduced by Bazilevič [6]. Thomas [8] has called a function in the class $B_1(\beta,0)$ a Bazilevič function of type β. Further, Nunokawa [3] has proved that a function f(z) in the class $B_p(\beta,0)$ is p-valent in the unit disk E.

In particular, the class $B_p(\beta,\gamma)$ for g(z) = f(z) is called is the class of p-valently starlike functions of order γ. Further, we note that the class $B_p(\beta,0)$ for g(z) = f(z) is equivalent to S_p^*.

Let $A_p(\alpha,\beta)$ be the subclass of A_p consisting of functions which satisfy the condition

$$(1.4) \qquad \mathrm{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left[1 + \frac{zf''(z)}{f'(z)}\right]\right\} > \beta \qquad (z \in E)$$

for some real α and β.

The class $A_1(\alpha,0)$ when p = 1 and $\beta = 0$ was introduced by Mocanu [2], and was studied by Miller, Mocanu and Reade [1], and Sakaguchi and Fukui [7].

Let $C_p(\alpha,\beta)$ be the subclass of A_p consisting of functions satisfying the condition

$$(1.5) \qquad \mathrm{Re}\left\{(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left[1 + \frac{zf''(z)}{f'(z)}\right]\right\} < \beta \qquad (z \in E)$$

for some real α and β $(\beta > p)$.

The class $C_p(\alpha,\beta)$ was recently introduced by Nunokawa and Owa [4].

2. SOME PROPERTIES

In order to derive our results, we need the following lemmas.

LEMMA I ([5]). If a function f(z) belongs to the class $A_p(\alpha,\beta)$ with $\alpha > 0$ and $0 \leq -\beta/\alpha \leq 1/2$, then $f(z) \in B_p(1/\alpha, 2^{2\beta/\alpha})$, and therefore f(z) is p-valent in the unit disk E.

LEMMA 2 ([4]). Let a function $f(z)$ belong to the class $\mathcal{C}_p(\alpha,\beta)$ with $\alpha \neq 0$, $\beta > p$, and $|\beta/\alpha| \leqq 1/2$. Then $f(z)$ is p-valent in the unit disk E. Moreover, if $0 \leqq -\beta/\alpha \leqq 1/2$, then $f(z) \in \mathcal{B}_p(1/\alpha, 2^{2\beta/\alpha})$.

Applying the above lemmas, we prove

THEOREM I. If a function $f(z)$ belongs to the class $\mathcal{A}_p(\alpha,\beta)$ with $\alpha > 0$ and $0 \leqq -\beta/\alpha \leqq 1/2$, then $f(z) \in \mathcal{B}_p(1/\alpha, p2^{2\beta/\alpha})$.

PROOF. For a function $f(z)$ in the class $\mathcal{A}_p$, we define the function $g(z)$ by

$$g(z) = f(z)^{1/p} = z + g_2 z^2 + g_3 z^3 + \cdots \quad . \tag{2.1}$$

Then $g(z)$ is in the class $\mathcal{A}_1$, and satisfies

$$\frac{zf'(z)}{f(z)} = p\,\frac{zg'(z)}{g(z)} \tag{2.2}$$

and

$$1 + \frac{zf''(z)}{f'(z)} = 1 + \frac{zg''(z)}{g'(z)} + (p-1)\frac{zg'(z)}{g(z)} \ . \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = (p-\alpha)\frac{zg'(z)}{g(z)} + \alpha\left(1 + \frac{zg''(z)}{g'(z)}\right). \tag{2.4}$$

Therefore, we have

$$f(z) \in \mathcal{A}_p(\alpha,\beta) \Longleftrightarrow \mathrm{Re}\left\{\left(1 - \frac{\alpha}{p}\right)\frac{zg'(z)}{g(z)} + \frac{\alpha}{p}\left(1 + \frac{zg''(z)}{g'(z)}\right)\right\} > \frac{\beta}{p} \tag{2.5}$$

$$\Longleftrightarrow g(z) \in A_1(\alpha/p, \beta/p).$$

Applying Lemma 1 for p = 1, we see that

$$g(z) \in A_1(\alpha/p, \beta/p) \implies g(z) \in B_1(p/\alpha, 2^{2\beta/\alpha}).$$

It follows that

$$f(z) \in A_p(\alpha, \beta) \implies g(z) \in B_1(p/\alpha, 2^{2\beta/\alpha})$$

$$\Longleftrightarrow \mathrm{Re}\left\{ \frac{zg'(z)}{g(z)^{1-p/\alpha}h(z)^{p/\alpha}} \right\} > 2^{2\beta/\alpha} \qquad (h(z) \in S_1^*)$$

$$\Longleftrightarrow \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)^{1-1/\alpha}(h(z)^p)^{1/\alpha}} \right\} > p2^{2\beta/\alpha} \qquad (h(z)^p \in S_p^*)$$

$$\Longleftrightarrow f(z) \in B_p(1/\alpha, p2^{2\beta/\alpha}).$$

This completes the assertion of Theorem 1.

REMARK 1. Noting $p2^{2\beta/\alpha} \geqq 2^{2\beta/\alpha}$, we see that

$$B_p(1/\alpha, p2^{2\beta/\alpha}) \subseteq B_p(1/\alpha, 2^{2\beta/\alpha}).$$

Thus Theorem 1 is the improvement of Lemma 1 by Nunokawa, Owa, Saitoh, Yaguchi and Lee [5].

Taking p = 1 in Theorem 1, we have

COROLLARY 1. If $f(z) \in A_1(\alpha, \beta)$ with $\alpha > 0$ and $0 \leqq -\beta/\alpha \leqq 1/2$, then $f(z) \in B_1(1/\alpha, 2^{2\beta/\alpha})$.

Using the same manner as in the proof of Theorem 1, we have

THEOREM 2. If a function f(z) belongs to the class $C_p(\alpha, \beta)$ with

$\alpha \neq 0$, $\beta > p$, and $0 \leqq -\beta/\alpha \leqq 1/2$, then $f(z) \in B_p(1/\alpha, p2^{2\beta/\alpha})$.

Finally, letting $p = 1$ in Theorem 2, we have

COROLLARY 2. If $f(z) \in C_1(\alpha,\beta)$ with $\alpha \neq 0$, $\beta > 1$, and $0 \leqq -\beta/\alpha \leqq 1/2$, then $f(z) \in B_1(1/\alpha, 2^{2\beta/\alpha})$.

REMARK 2. We note that Theorem 2 is the improvement of Lemma 2 due to Nunukawa and Owa [4].

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