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Multi-objective Optimization Using Interval Analysis

Yasuo Fujii
Educational center for
information processing,
Kyoto university.

1. Introduction

We described an interval method to compute the global maximum value of the multimodal multivariable function. By using "interval analysis", we can obtain an exact estimate of the global maximum value or the minimum value of unitary objective functions, including the rigorous error bounds [1].

In the engineering and scientific fields, as well as in the field of social science, a number of objective functions exist, and in many cases they come into conflict with each other [2],[3].

In this paper, an interval analysis method is applied for finding the minima (or maxima) of multi-objective optimization.

2. Multi-objective optimization

Multi-objective optimization problems or multi-objective programming problems can be formulated under the condition of an inequality constraint and/or equality constraints, as a problem to obtain a decision variable which optimizes more than one objective function at the same time.

Here, objective functions are considerd to be the minima, but we can treat them in the same way when a part or all of the objective functions are the maxima.

Consider the following problem:

Minimize

$$\dot{f}(\dot{x}) = (f_1(\dot{x}), f_2(\dot{x}), ..., f_k(\dot{x}))^T \tag{1}$$

subject to

$$\dot{x} \in X = \{\dot{x} \in E^n | \dot{g}(\dot{x}) \leq \dot{0}, \dot{h}(\dot{x}) = \dot{0}\} \tag{2}$$

where $\dot{x} = (x_1, x_2, ..., x_n)^T$ is an n-dimensional decision variable, $\dot{f}(\dot{x}) = (f_1(\dot{x}), f_2(\dot{x}), ..., f_k(\dot{x}))^T$ is a k-dimensional vector function, and $\dot{g}(\dot{x}) = (g_1(\dot{x}), g_2(\dot{x}), ..., g_r(\dot{x}))^T$ and $\dot{h}(\dot{x}) = (h_1(\dot{x}), h_2(\dot{x}), ..., h_q(\dot{x}))^T$ are r-dimensional and q-dimensional vector constraint functions respectively.

If we apply the concept of the problem of an unitary objective case to the multi-objective optimization problem, we can define the concept of the following complete optimal solution.

Definition 2.1 A vector $\dot{x}^*$ is called a *complete optimal solution* of the Eqs.(1) and (2) if there is $\dot{x}^* \in X$ with $\dot{f}(\dot{x}^*) \leq \dot{f}(\dot{x})$.

A complete optimal solution which minimizes more than one objective function at the same time can not exist when the objective functions conflict with each other. It is impossible to discuss the multi-objective optimization problem with the same way of the case of the unitary optimization problem, because the objective functions are vectors.

Instead, as a noninferior solution, we are forced to obtain the Pareto optimal solution.

Definition 2.2 A vector $\dot{x}^*$ is called a *Pareto optimal solution* of the Eqs. (1) and (2) if there is no $\dot{x} \in X$ with $\dot{f}(x) \leq \dot{f}(x^*)$.

As a method to obtain a Pareto optimal solution of the multi- objective optimization problem, the so called scalar method is well known in which we can change multicriterion optimization problem to a scalar optimization problem.

Here, as with the typical scalar methods, we performed the experiments of weighting method, global criterion method and min-max method to obtain the experimental values. In weighting method, the total sum of weighted objective functions is minimized as an unitary objective function. That is, by obtaining the minimum value of $f(\dot{x})$ in the following equation

$$f(\dot{x}) = \sum_{i=1}^{k} w_i f_i(\dot{x}) \tag{3}$$

where $w_i > 0$ are the weighting coefficients representing the relative importance of the criteria and it is usually assumed that

$$\sum_{i=1}^{k} w_i = 1.$$

In this method an optimal solution is a vector of decision variables which minimizes some global criterion. A function which describes this global criterion is a measure of 'how close the decision maker can get to the ideal vector $\dot{f}^0$'.

This can be formulated as follows,

$$\min_{x \in X} f(\dot{x}) = \min \sum_{i=1}^{k} \left| \frac{f_i(\dot{x}) - f_i^0}{f_i^0} \right|^p, \qquad 1 \leq p \leq \infty. \tag{4}$$

In the min-max method, the maximum value of the difference from the minimum value of each objective function can be minimized.

When

$$z_i(\dot{x}) = \left| \frac{f_i(\dot{x}) - f_i^0}{f_i^0} \right|, \tag{5}$$

$\dot{x}$ which satisfies

$$\min_{\dot{x} \in X} \max_{i \in K} \{z_i(\dot{x})\}, \tag{6}$$

is the solution of min-max $(K = \{1, 2, ..., k\})$. But when $f_i^0 = 0$, the right side of Eq. (5) is $|f_i(\dot{x})|$.

3. Interval analysis applied to min-max algorithm

In interval analysis, we calculate the values which are considered to have an interval [4].

In case of the unitary objective function, if it is a multimodal multivariable functions, we can always obtain the global optimal solution (We can obtain all sets of the decision variable, the value of at which gives the global optimal solution). In this method, we can obtain an optimal solution, extending objective functions and decision variables to the intervals, estimating the upper and lower limits of function value in each region by dividing the variable regions in order, and eliminating the region which has no probability of having the optimal solution. Further the convergence can be made fast by the interval analysis version of Newton's method is very efficient for reducing the interval width [5].

In case of the multi-objective optimization, the weighting method and global criterion method are the method used to transform into the unitary objective optimization problems. Thus, we can apply this interval analysis optimization which we have been developing.

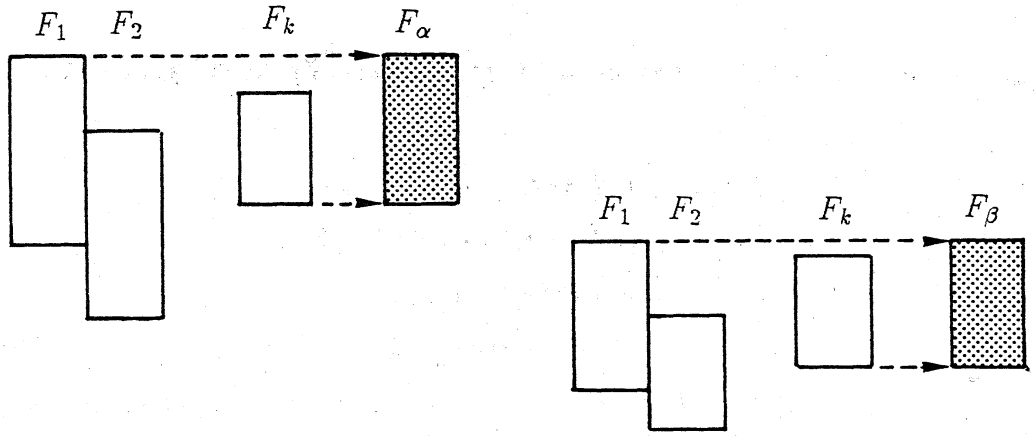
The optimal solution in the min-max method is obtained by applying the interval analysis, as follows;

When interval functions of each objective function in two subregions S_α and S_β are $F_i \quad (i = 1, ..., k)$.

In the Figure 1,

$$F_\alpha = \left[\max_{\substack{i \in k \\ x \in s_\alpha}} \underline{F_i}, \max_{\substack{i \in k \\ x \in s_\alpha}} \overline{F_i} \right]$$

is an interval whose upper and lower limits are the maximum values of each objective functions upper and lower limits at S_α respectively. F_β is described similaly. $\underline{F_i}$ and $\overline{F_i}$ show the upper and lower limits of all the F_i.



Interval function value in subregion S_α.

Interval function value in subregion S_β.

Fig.1 Division and garning and eliminating the partial region.

In Fig. 1 if $\overline{F_\beta} < \underline{F_\alpha}$, it is obvious that there is no optimal solution, therefore we can eliminate S_α. Thus, dividing the region in order, we can obtain the optimal solution by estimating the function values on a subregion in that region. But this division method takes much time, and it is not accurate enough. So we use the Lagrange-multiplier technique and Newton's method. Now, we assume that the following problem is to be solved by the min-max method:

$$min f_1(\dot{x}), \quad min f_2(\dot{x}) \tag{7}$$

subject to the two inequality constraints

$$g_1(\dot{x}) \leq 0, \quad g_2(\dot{x}) \leq 0. \tag{8}$$

It follow that Eqs. (7) and (8) can be rewritten

$$min x_0 \tag{9}$$

subject to

$$\left\{ \begin{array}{r} z_1(\dot{x}) - x_0 \leq 0 \\ z_2(\dot{x}) - x_0 \leq 0 \\ g_1(\dot{x}) \leq 0 \\ g_2(\dot{x}) \leq 0 \end{array} \right\} \tag{10}$$

The Lagrange function obtained by introducing the Lagrange-Multiplier technique is as follows,

$$\begin{aligned} L = x_0 &+ p_1\{z_1(\dot{x}) - x_0 + x_{n+1}^2\} \\ &+ p_2\{z_2(\dot{x}) - x_0 + x_{n+2}^2\} \\ &+ p_3\{g_1(\dot{x}) + x_{n+3}^2\} \\ &+ p_4\{g_2(\dot{x}) + x_{n+4}^2\}. \end{aligned} \tag{11}$$

If partial differentiation of L with respect to $x_0, x_1, ..., x_n, x_{n+1}, ..., x_{n+4}, p_1, p_2, p_3, p_4$ is equal 0, i.e.

$$\frac{\partial L}{\partial x_j} = 0, \quad j = 0, 1, ..., n+4, \tag{12}$$

and

$$\frac{\partial L}{\partial p_i} = 0, \quad i = 1, ..., 4 \tag{13}$$

the stationary points can be computed by solving these nonlinear simulataneous equations with by the interval Newton's method.

4. Numerical Examples

Several numerical examples have been computed by the method described above. The calculations were done with HITAC M-680H of the Educational Center for Information Processing of Kyoto University.

Find the minimum of the function,

$$f_1(\dot{x}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 + 1,$$

$$f_2(\dot{x}) = 100(x_2 - x_1^2)^2 + (2 - x_1)^2 + 1 \tag{14}$$

subject to additional constraints of the form

$$0 \le x_1 \le 3, \qquad 0 \le x_2 \le 5.$$

Example 1:

Weighting method and Newton's method are applied to Eq. (14), in $w_1 = 0.5$, $w_2 = 0.5$. The computed result is:

$X_1 =$ [1.50000 00000 00000, 1.50000 00000 00218],
$X_2 =$ [2.24999 99999 99506, 2.25000 00000 00655],
$F_1 =$ [1.25000 00000 00000, 1.25000 00000 00218],

$F_2 = [\,1.24999\ 99999\ 99781, \quad 1.25000\ 00000\ 00000\,].$

Example 2:

Global criterion method is applied to Eq. (14), in p=2. The computed result is:

$X_1 = [\,1.49047\ 85156\ 25000, \quad 1.50952\ 14843\ 75000\,],$
$X_2 = [\,2.21954\ 34570\ 31250, \quad 2.28073\ 12011\ 71875\,],$
$F_1 = [\,1.24861\ 71823\ 83895, \quad 1.25832\ 45457\ 37095\,],$
$F_2 = [\,1.24861\ 71823\ 83895, \quad 1.25832\ 45457\ 37094\,].$

Example 3:

Min-max algorithm is applied to Eq. (14). The computed result is:

$X_1 = [\,1.49999\ 99998\ 25377, \quad 1.50000\ 00001\ 74623\,],$
$X_2 = [\,2.24999\ 90652\ 43173, \quad 2.25000\ 09347\ 27722\,],$
$F_1 = [\,1.24999\ 99998\ 25377, \quad 1.25000\ 00002\ 62100\,],$
$F_2 = [\,1.24999\ 99998\ 25376, \quad 1.25000\ 00002\ 62099\,].$

Example 4:

Lagrange-multipler tequnique and Newton's method are applied to Eq. (14). The computed result is:

$X_1 = [\,1.50000\ 00000\ 00000, \quad 1.50000\ 00000\ 00000\,],$
$X_2 = [\,2.25000\ 00000\ 00000, \quad 2.25000\ 00000\ 00000\,],$
$F_1 = [\,1.25000\ 00000\ 00000, \quad 1.25000\ 00000\ 00001\,],$
$F_2 = [\,1.24999\ 99999\ 99999, \quad 1.25000\ 00000\ 00000\,].$

The values of Lagrange-multiplier p_1, p_2, slack variable x_3, x_4 and additional variable x_0 are:

$P_1 = [\,4.99999\ 99999\ 99999, \quad 5.00000\ 00000\ 00000\,],$
$P_2 = [\,4.99999\ 99999\ 99999, \quad 5.00000\ 00000\ 00000\,],$
$X_3 = [\,0.0, \quad 0.22539\ 92852\ 19884 \times 10^{-19}\,],$
$X_4 = [\,0.0, \quad 0.0\,],$
$X_0 = [\,2.49999\ 99999\ 99999, \quad 2.50000\ 00000\ 00000\,].$

The computed time is each case is shown in Table 1.

Table 1. Computation time

	Ex. 1	Ex. 2	Ex. 3	Ex. 4
CPU Time(sec.)	690.59	325.46	3189.54	32.08
Ratio	21.5	10.1	99.4	1.0

5. Conclusion

We described an algorithm for minimizing multi-objective functions by using interval analysis. It enables us to obtain the minimum in the domain or on the boundary. Both constrained and unconstrained minimum can be computed. So far we have calculated minima of two objective functions. If effective devices for reducing interval width of functions are developed, this methods can be applied to many objective function problems.

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