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Uniqueness of critical point of solutions

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§ 1. Introduction and results

Uniqueness of critical point of solutions to elliptic boundary value problems over convex domain in $\mathbb{R}^n$ ($n \geq 2$) was shown by many authors (see Kawohl [10] and its references). The typical examples are the following: Let Ω be a bounded convex domain in $\mathbb{R}^n$ with boundary $\partial\Omega$.

(1) (The fixed membrane eigenvalue problem) The positive first eigenfunction u_1 to the eigenvalue problem: $-\Delta u = \lambda u$ in Ω and $u = 0$ on $\partial\Omega$ is log-concave, that is, $\log u_1$ is a concave function (see [1], [4], [5], [11],and [15]).

(2) (The Saint Venant torsion problem) The root of the solution u (that is, $\sqrt{u}$) to the problem: $\Delta u = -1$ in Ω and $u = 0$ on $\partial\Omega$ is a concave function(see [12] and [13]).

(3) (The capillary free surfaces with zero contact angle against the wall over the cross section Ω) The solution u to the problem : $\operatorname{div} Tu = \kappa u$ (or nH) in Ω and $Tu\cdot\nu = 1$ on $\partial\Omega$ is a convex function, where $Tu = (1 + |\nabla u|^2)^{-\frac{1}{2}} \nabla u$, κ and H are positive constants, and ν denotes the unit outer normal vector

to $\partial\Omega$(see [6] and [14]).
In these three examples it is shown that the level sets of the solution u, $\Omega_s = \{ x \in \Omega ; u(x) \geq s \}$ or $\{ x \in \Omega ; u(x) \leq s \}$ for $s \in \mathbb{R}$ are convex, and therefore it is shown that the critical point of u (that is, the point p with $\nabla u(p) = 0$) is unique and the solution u achieves its maximum or minimum at this critical point.

On the other hand, in the two-dimensional case the uniqueness of critical point of solutions was shown by several authors for some broader classes of nonlinear elliptic boundary value problems, though the convexity of level sets are not shown. Concerning the Dirichlet problems, in [22] Sperb considered the semilinear elliptic problem : $\Delta u = f(u)$ in Ω and $u = 0$ on $\partial\Omega$ and he showed that any positive solution u to this has only one critical point for some f. His proof is based on an idea of Payne [18]. Payne's idea is, roughly speaking, to study the curve $\{ x \in \Omega ; \alpha\cdot\nabla u(x) = 0 \}$ for any direction α . Also, using the same idea, in [19] Philippin showed that the solution u to the Dirichlet problem for the prescribed constant mean curvature equation: $\mathrm{div}\, Tu = 2H$ in Ω and $u = 0$ on $\partial\Omega$ has only one critical point, where H is a positive constant. Concerning the problem which is not Dirichlet, we know the result of Chen. In [7], Chen showed that the capillary free surface over convex domain has only one critical point. Precisely, he considered the problem: $\mathrm{div}\, Tu = \kappa u$ (or $2H$) in Ω and $Tu\cdot\nu = \cos\gamma$ on $\partial\Omega$,where H, κ,and γ are positive constants with $0 < \gamma < \pi/2$. His proof is based on a nice comparison technique found in Chen & Huang [6] and the method of continuity with respect to the contact angle γ and the results of Chen & Huang [6] and Korevaar [14] (that is, the above example (3)).

In this paper we consider some two-dimensional semilinear elliptic boundary value problems, which are not Dirichlet, and prove the uniqueness of critical point of the solution. Our methods are based on an idea of Chen [7]. Precisely, let Ω be a bounded convex domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, and let f be a real valued C^∞- non-decreasing function on $\mathbb{R}$, which is positive somewhere.

Now, our results are the following:

Theorem 1. Let $u \in C^2(\bar{\Omega})$ be the solution to

$$(1.1)\qquad \begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = c & \text{on } \partial\Omega, \end{cases}$$

where c is a positive constant. Then u has one and only one critical point in Ω.

Theorem 2. Suppose that $f(0)$ is positive. Let $u \in C^2(\bar{\Omega})$ be the solution to

$$(1.2)\qquad \begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where β is a positive constant. Then u has one and only one critical point in Ω.

Since our theorems concern only qualitative properties of the solutions, so only under the hypothesis of the existence of solutions we show the uniqueness of critical point of the solutions. For the existence of solutions, for example , see Lieberman [16] or Lieberman & Trudinger [17].

In the following sections we prove these theorems. Section 2 provides some preliminary results for the problems (1.1), (1.2). In section 3 we introduce two families of problems indexed by a bounded closed interval [0,1] in order to use the method of continuity. In section 4 we prove several basic lemmas with the help of one modification of Chen & Huang 's comparison technique ,and complete the proofs. For the details see [20].

§ 2. Preliminaries

First of all, using the strong maximum principle, we get

Proposition 2.1. Let $u \in C^2(\bar{\Omega})$ be the solution to (1.1) or (1.2). Then $f(u)$ is positive in $\bar{\Omega}$ and $\frac{\partial u}{\partial \nu} > 0$ on $\partial\Omega$. Furthermore, $u < 0$ in $\bar{\Omega}$ in the case of (1.2).

Concerning the uniqueness and the regularity of the solution, we have

Proposition 2.2. (1) The solution to (1.1) is unique up to an additive constant. (2) The solution to (1.2) is unique. (3) The solution to (1.1) or (1.2) belongs to $C^\infty(\bar{\Omega})$.

Proof. Since f is non-decreasing, (1) and (2) follow from the strong maximum principle. Since f is smooth, the regularity theory of elliptic partial differential equations implies (3).

§ 3. Families of problems for the method of continuity

For t ($0 \leq t \leq 1$), we introduce the following problems:

$$(3.1.t) \quad \begin{cases} \Delta u = t f(u) + (1-t) k & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = c & \text{on } \partial\Omega, \end{cases}$$

where $k = \dfrac{|\partial\Omega|}{|\Omega|} c$ and c is the positive constant in (1.1),

$$(3.2.t) \quad \begin{cases} \Delta u = t f(u) + (1-t) f(m) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where m is the minimum value of the unique solution to (1.2) and β is the positive constant in (1.2).

Remark 3.1. (3.1.1) = (1.1) and (3.2.1) = (1.2).

Remark 3.2. Concerning the uniqueness of the solution to these problems, we obtain the same results as in Proposition 2.2, since the right hand sides of these equations are non-decreasing functions with respect to u. Also we have the same results as in Proposition 2.1 to these problems, if we replace $f(u)$ by the right hand sides of these equations.

Concerning the existence, by the method of sub and supersolutions we obtain

Proposition 3.3. *Under the hypothesis of the existence of the solution to the problem* (1.1) *we see that there exists a solution* $u_t \in C^\infty(\bar{\Omega})$ *to the problem* (3.1.t) *satisfying*

(3.3) $\quad \| u_t \|_{C^{2+\alpha}(\bar{\Omega})} \leq C$ *for all* $t \in [0,1]$,

where C *and* α *are positive constants independent of* $t \in [0,1]$, *and the solution to* (3.1.t) *is unique up to an additive constant*.

Proposition 3.4. *Under the hypothesis of the existence of the solution to* (1.2) *we see that there exists a unique solution* $u_t \in C^\infty(\bar{\Omega})$ *to the problem* (3.2.t) *satisfying the same inequality as* (3.3) *for all* $t \in [0,1]$.

Proof. Using the solutions u_1 and u_0, we can construct the sub and supersolutions to (3.1.t) and (3.2.t). Therefore, by the method of sub and supersolutions (see Sattinger [21, Theorem 2.1 , p. 980]) we can prove these theorems. The estimates follow from the results of Agmon, Douglis & Nirenberg [2].

§ 4. One modification of Chen & Huang's comparison technique

We begin with

Lemma 4.1. *For any* h *with* $f(h) > 0$, *there exists a number* L

($0 < L \leq \infty$) <u>such that the initial value problem</u>

$$(4.1)\quad \begin{cases} v''(s) = f(v(s)) & (-L < s < L), \\ v(0) = h \ \text{and} \ v'(0) = 0, \end{cases}$$

<u>has a unique</u> C^∞ - <u>solution</u> v ,<u>which satisfies the following</u>:

$$(4.2)\qquad v(s) = v(-s) \quad (-L < s < L),$$

$$(4.3)\qquad v(s) \geq h \ \text{and} \ v''(s) \geq f(h) > 0 \quad (-L < s < L),$$

$$(4.4)\qquad v'(s) \geq 0 \quad (0 \leq s < L),$$

$$(4.5)\qquad \lim_{s \to L} v(s) = +\infty \ \text{and} \ \lim_{s \to L} v'(s) = +\infty.$$

Using one modification of Chen & Huang's comparison technique, we obtain

Lemma 4.2. <u>Let</u> $u \in C^\infty(\bar{\Omega})$ <u>be the solution to</u> (3.1.t) <u>or</u> (3.2.t) <u>for</u> $t \in [0,1]$. <u>Suppose that</u> $\nabla u(p) = 0$ <u>at some point</u> $p \in \Omega$. <u>Then the Gaussian curvature</u> $K(p)$ <u>of the graph</u> $(x, u(x))$ <u>at</u> p <u>does not vanish</u>, <u>where</u> $K(p) = D_{11}u(p) \cdot D_{22}u(p) - \{D_{12}u(p)\}^2$ <u>and</u> $D_{ij} = \dfrac{\partial^2}{\partial x_i \partial x_j}$.

<u>Proof</u>. As in Remark 3.2, it suffices to show this lemma when $t = 1$. Therefore, let u be the solution to (1.1) or (1.2). Let $p \in \Omega$ be a point with $\nabla u(p) = 0$. Suppose that $K(p) = 0$. Then, by using a parallel translation and a rotation of coordinates, we may assume that

$$(4.6)\qquad p = 0 \ \text{and} \ [D_{ij}u(0)] = \text{diag}[f(u(0)), 0].$$

It follows from Proposition 2.1 that $f(u(0))$ is positive. Using Lemma 4.1 for $h = u(0)$, we get a unique solution to (4.1), say v. Put $w(x)$ $(= w(x_1,x_2))$ $= v(x_1)$. Then w satisfies

$$
(4.7)\quad \begin{cases} \Delta w = f(w) & \text{in } (-L,L) \times \mathbb{R}, \\ [\, D_{ij} w(0) \,] = \operatorname{diag} [\, f(u(0)), 0 \,], \\ w(0) = u(0) \text{ and } \nabla w(0) = \nabla u(0) = 0. \end{cases}
$$

Hence $u - w$ satisfies

$$
(4.8)\quad \Delta (u - w) = a(x)(u - w) \text{ and } a(x) \geq 0 \quad \text{in } (-L,L) \times \mathbb{R} \cap \Omega,
$$

where $a(x) = \int_0^1 f'(w + \theta(u - w)) \, d\theta$, and $u - w$ vanishes up to second order derivatives at 0. Furthermore, since $u - w$ is not identically zero, it follows from (4.8) and a unique continuation theorem for solutions to elliptic partial differential equations (see Aronszajn [3]) that $u - w$ never vanishes up to infinite order at 0. Therefore, by Taylor's formula we get for some integer $n \geq 3$

$$
(4.9)\quad (u - w)(x) = P_n(x) + o(|x|^n) \text{ as } |x| \longrightarrow 0,
$$

where $P_n(x)$ is a homogeneous polynomial of degree n and $P_n(x)$ is not identically zero. Furthermore, since $u - w$ is a C^∞-function, using (4.8), we see that $P_n(x)$ is a harmonic polynomial. On the other hand, it follows from the result of Hartman & Wintner [9, Corollary 1, p. 450] that every interior critical point of $u - w$ is isolated. Therefore, as in [6] we see that the zero set of $u - w$ in some neighborhood U of origin consists of n smooth arcs, all intersecting at origin and dividing U into $2n$ sectors ($n \geq 3$). Put

$$
A = \{ x \in \Omega \cap (-L,L) \times \mathbb{R} \;;\; u(x)-w(x) > 0 \},
$$

$$
B = \{ x \in \Omega \cap (-L,L) \times \mathbb{R} \;;\; u(x)-w(x) < 0 \}.
$$

Then, it follows from the maximum principle that

(4.10) <u>Both</u> A <u>and</u> B <u>have at least three components each of which meets the boundary</u> $\partial(\Omega \cap (-L,L) \times \mathbb{R})$.

Now, we first consider the case of Neumann boundary condition (1.1). Furthermore we devide the proof into two cases. One is the case that L is finite, and the other is that L is infinite. Consider the former. Choose a number $L^{\sim}$ with $L^{\sim} <$ L ,which is sufficiently near to L. Put $\Omega^{\sim} = (-L^{\sim},L^{\sim}) \times \mathbb{R}$.
Look at the boundary $\partial(\Omega \cap \Omega^{\sim})$. Since Ω is convex, observing the boundary condition of u and the shape of the graph of w (see Lemma 4.1), we see that $\partial(\Omega \cap \Omega^{\sim})$ consists of at most four connected arcs, in which $\frac{\partial}{\partial\nu}(u-w)$ changes sign alternatively. Put

$$\Gamma_+ = \{ x \in \partial(\Omega \cap \Omega^{\sim}) \ ; \ \frac{\partial}{\partial\nu}(u-w)(x) > 0 \},$$
$$\Gamma_- = \{ x \in \partial(\Omega \cap \Omega^{\sim}) \ ; \ \frac{\partial}{\partial\nu}(u-w)(x) < 0 \}.$$

(At a corner, we choose ν to be the unit outer normal vector to $\partial\Omega^{\sim}$.) Then, it never occurs that a component of $A \cap \Omega^{\sim}$ meets $\partial(\Omega \cap \Omega^{\sim})$ exclusively in Γ_-. Indeed, let ω be a component of $A \cap \Omega^{\sim}$ which meets $\partial(\Omega \cap \Omega^{\sim})$ exclusively in Γ_-. Hence the strong maximum principle implies that a positive maximum of u-w in $\overline{\omega}$ is attained at $p \in \Gamma_-$ and $\frac{\partial}{\partial\nu}(u-w)(p) \geq 0$. This contradicts the definition of Γ_-. Also, by the same argument as this, we see that it never occurs that a component of $B \cap \Omega^{\sim}$ meets $\partial(\Omega \cap \Omega^{\sim})$ exclusively in Γ_+. However, these facts contradict (4.10).

Next consider the latter when L is infinite. Only replacing $\Omega \cap \Omega^{\sim}$ by Ω, we can use the same argument as above.

In the case of the third kind boundary condition (1.2), replacing $\frac{\partial}{\partial \nu}(u-w)$ by $\frac{\partial}{\partial \nu}(u-w) + \beta(u-w)$, we can use the same argument as in the case of Neumann boundary condition (1.1). This completes the proof of Lemma 4.2.

Lemma 4.3. Let $u \in C^{\infty}(\overline{\Omega})$ be the solution to (3.1.0) or (3.2.0). Suppose that $\nabla u(p) = 0$ at some point $p \in \Omega$. Then the Gaussian curvature $K(p)$ of the graph $(x, u(x))$ at p is positive.

Proof. Let p be a point with $\nabla u(p) = 0$. Suppose that $K(p) \leqq 0$. For simplicity, by a parallel translation and a rotation of coordinates we may assume that

(4.11) $p = 0$ and $[\, D_{ij}u(0)\,] = \mathrm{diag}\,[\, \lambda_1, \lambda_2]$,

where $\lambda_1 > 0$ and $\lambda_2 \leqq 0$ and $\lambda_1 + \lambda_2 = k$ in the case of (3.1.0), $= f(m)$ in the case of (3.2.0). Then $u(x) = w(x) + P(x)$, where $w(x) = u(0) + \frac{1}{2}\lambda_1(x_1)^2 + \frac{1}{2}\lambda_2(x_2)^2$ and $P(x)$ is a harmonic function in Ω. Therefore, by the same argument as in the proof of Lemma 4.2 we can prove this.

Now, we complete the proofs. Let u_t be the solution to (3.1.t) or (3.2.t). Roughly speaking, Lemma 4.3 implies that u_0 has only one critical point, and Lemma 4.2 implies that the uniqueness of critical point of u_t is preserved as t varies with the help of Proposition 3.3 and Proposition 3.4. Therefore, u_1, that is the solution to (1.1) or (1.2), has only one critical point by the method of continuity.

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