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SOLUTIONS OF SOME SEMILINEAR ELLIPTIC PROBLEMS

BY

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Abstract

In this article we study some semilinear elliptic problems on an infinite strip, and prove their existences of various classical solutions, which are spherically symmetric and decreasing in the $|x|$-direction and decay exponentially at infinite.

0. INTRODUCTION

In the part III of his lecture notes [5], Ni gave systematic studies of semilinear elliptic equaitons on unbounded domains in the Euclidean space $\mathbb{R}^n$, and gave extensive references. A typical equation in [5] is as follows:

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$$\begin{cases} -\Delta u = \lambda f(u) & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where Ω is an unbounded domain in $\mathbb{R}^n$. This type of equaitons in the case $\Omega = \mathbb{R}^N$ have been studied in great detail in[3,5,7]. The treatments in which use variational arguments to solve the problems. Those techniques, espectially from [3] involving the radial and the compactness theorems of strauss, form one of our basic methods. This type of equations in the case $\Omega = \mathbb{R} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ have been studied in [1,2,4,7]. In [2,4], they use the finite domain approximations to treat the existence results. The bifurcation and asymptotic bifurcation of these equations have been studied in great detail in [2]. In [7] the double Steiner symmetrizations have been used, and in [1], finite domain approximaitons have been used to study the bifurcation problem of some more general equations. We treat here in the case $\Omega = \mathbb{R}^N \times (0,1)$, $N = 2, 3$, and develop some new techniques of uniform analysis to obtain our results. Throughout this article we use the same notation C for different constants in various inequalities.

1. EXISTENCES

Let $\Omega = \mathbb{R}^2 \times (0,1)$ or $\Omega = \mathbb{R}^3 \times (0,1)$. Denote by a point (x,y) in Ω with $x \in \mathbb{R}^N$, $N = 2$ or 3, $y \in (0,1)$. Consider the semilinear elliptic eigenvalue equation

$$\text{(A)} \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $f: \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, odd, $f(0) = 0$, and satisfies the following conditions:

(1.1) $\quad -\infty < \underline{\lim}_{s \to 0+} \frac{f(s)}{s} \le \overline{\lim}_{s \to 0+} \frac{f(s)}{s} = -m \le 0$

$(1.2)_2$ $\quad -\infty < \overline{\lim}_{s \to \infty} \frac{f(s)}{s^\ell} \le 0$ for any $\ell > 1$,

if $\Omega = \mathbb{R}^2 \times (0,1)$

$(1.2)_3$ $\quad -\infty < \overline{\lim}_{s \to \infty} \frac{f(s)}{s^3} \le 0$ if $\Omega = \mathbb{R}^3 \times (0,1)$

(1.3) There is $\alpha > 0$ with $F(\alpha) = \int_0^\alpha f(s)ds > 0$.

Define a new function $\tilde{f}: \mathbb{R} \longrightarrow \mathbb{R}$ as follows:

(i) if $f(s) \ge 0$ for all $s \ge \alpha$, put $\tilde{f} = f$

(i i) if there is $s_0 \ge \alpha$ with $f(s_0) = 0$ put

$$\tilde{f}(s) = \begin{cases} f(s) & \text{on } [0, s_0] \\ 0 & \text{for } s \geq s_0 \end{cases}$$

(iii) for $s \leq 0$, $\tilde{f}(s) = -\tilde{f}(-s)$.

Observe that $\tilde{f}$ satisfies the same condition as f. Furthermore, by the maximum principle, solutions of problem (A) with $\tilde{f}$ are also solutions of (A) with f. We henceforth adopt that f has been replaced by $\tilde{f}$. In this case, $(1.2)_2$ and $(1.2)_3$ can be replaced by the followings respectively

$$\lim_{|s|\to\infty} \frac{|f(s)|}{|s|^{\ell}} = 0 \quad \text{for any } \ell > 1, \text{ in case } \Omega = \mathbb{R}^2 \times (0,1)$$

and

$$\lim_{|s|\to\infty} \frac{|f(s)|}{|s|^{3}} = 0 \quad \text{in case } \Omega = \mathbb{R}^3 \times (0,1).$$

There are some typical examples of the equation (A)

1.4. <u>EXAMPLE</u>. Consider the equation

$$\begin{cases} -\Delta u + mu = \beta|u|^{p-1}u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where m, β are positive constants, and $p > 1$.

1.5. **EXAMPLE**. Consider the equation

$$\begin{cases} -\Delta u + mu = \beta|u|^{p-1}u - \gamma|u|^{q-1}u & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where m, β, γ are positive constants, and $1 < q < p < \infty$ for the case $\Omega = \mathbb{R}^2 \times (0,1)$ and $1 < q < p < 3$ for the case $\Omega = \mathbb{R}^3 \times (0,1)$.

1.6. **THEOREM**. Suppose f satisfies the conditions (1.1) - (1.3). There is a solution (λ,u) of the equation (A), where u is of $C^2(\Omega)$, and is spherically symmetric and decreasing in the $|x|$-direction.

1.7. **REMARK**. In Theorem 1.6, we obtained a solution (λ,u) of equation (A)

$$\text{(A)} \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In general, the Lagrange multiplier λ can not be absorbed.

Note that λ can be absorbed implies that u is a solution of the equation

$$(A1) \quad \begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

However, the equation (A1) has a solution in the following particular cases.

(1) In Theorem 1.8 below we modify our proof of Theorem 1.6 to obtain a solution of the equation

$$(B) \quad \begin{cases} -\Delta u + mu = u^p & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $m > 0$ a constant.

(2) In Theorem 1.13 below, we use Nehari's method to construct a solution of the equation

$$(C) \quad \begin{cases} -\Delta u = g(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

1.8. **THEOREM**. For either $\Omega = \mathbb{R}^2 \times (0,1)$, $2 < p < \infty$ or $\Omega = \mathbb{R}^3 \times (0,1)$, $2 < p < 3$, there is a C^2 solution $u(x,y)$ of the equation

$$\text{(B)} \qquad \begin{cases} -\Delta u + mu = u^p & \text{in} \quad \Omega \\ u > 0 & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

where m is a positive constant. Moreover $u(x,y)$ is spherically symmetric and decreasing in the $|x|$-direction for each y in $(0,1)$.

Let the function $g: \mathbb{R} \longrightarrow \mathbb{R}$ be continuous, odd, $g(0) = 0$, satisfies (1.1)-(1.3), and

(1. 9) g is increasing on $[0,\infty)$

(1.10) $tg(t) - 2G(t) \geq \theta G(t)$ for large t, where θ a positive constant, and $G(t) = \int_0^t g(s)ds$

(1.11) Consider the equation $g \in C^1(0,\infty)$ with $g'(t) > \frac{g(t)}{t}$ for all $t > 0$

1.12. **EXAMPLE**. $g(u) = u^p$, $2 < p < \infty$ in case $\Omega = \mathbb{R}^2 \times (0,1)$ or $2 < p < 3$ in case $\Omega = \mathbb{R}^3 \times (0,1)$.

Consider the equation

$$(C)\quad \begin{cases} -\Delta u = g(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

1.13. **THEOREM**. There is a C^2 solution $u(x,y)$ of the equation (C). The solution $u(x,y)$ is spherically symmetric and decreasing in the $|x|$-direciton for each y in $(0,1)$.

Follow from the proof of Theorem 1.6, 1.8 and Berestycki-Lions [3], we obtain

1.14. **THEOREM**. Let w be the solution of the equaiton (B) obtained as Theroem 1.8, and u any other solution of (B), then

$$0 < s(w) \leq s(u)$$

where $s(v) = A(v) - B(v)$, $A(v) = \frac{1}{2}\int_\Omega [|Dv|^2 + m|v|^2]$, $B(v) = \frac{1}{p+1}\int_\Omega |v|^{p+1}$.

Such a solution w is called a ground state for the equaiton (B). Any solutions u of (B) with

$$s(w) < s(u) < \infty$$

are called bound states. We'll prove that the equation (B) possesses infinite many solutions of buond states, through a dual variational method: For $n = 1,2,\cdots$

$$\text{maximize } \{B(u) \mid u \in H_0^1(\Omega),\ A(u) = n^2\}.$$

1.15. **THEOREM**. For either $\Omega = \mathbb{R}^2 \times (0,1)$, $2 < p < \infty$, or $\Omega = \mathbb{R}^3 \times (0,1)$, $2 < p < 3$, and for $n = 1,2,\cdots$, there is a C^2 solution $w_n(x,y)$ of the equation (B), which is spherically symmetric and decreasing in the $|x|$-direction with $A(w_n) = n^2$.

We study the decay property of the solutions of the equaiton (B).

1.16. **THEOREM**. If $u(x,y)$ is a C^2 solution of the equation (B) which is spherically symmetric and decreasing in the $|x|$-direciton, then

$$|D^\alpha u(x,y)| \le ce^{-\delta|x|} \qquad \text{for large } x$$

where $C, \delta > 0$ are constants independent of y in $(0,1)$ and $|\alpha| \le 1$.

1.17. **REMARK.** In an article in preparation, Nirenberg-Berestycki asserts that if $\Omega = \mathbb{R}^N \times (0,1)$, and u is a solution of the equation

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \partial\Omega \\ \lim_{|x|\to\infty} u(x) = 0 & \end{cases}$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is locally Lipschtz continuous function, $f'(0) < \pi^2$, then u is symmetric in x about some x_0 and $u_{|x|} > 0$ for $|x| < |x_0|$. After shifting, u can be considered symmetric in $x = 0$. If we apply this result, in the assumptions of Theorem 2.1, we may only assume that u is a C^2 solution of the equation (B).

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