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Notes on Imperfect Repair by Fumio OHI Osaka University

0. Introduction

A component with failure time distribution $F(t) = 1-e^{-\Lambda(t)}$ is repaired at failure. Each repaire results in minimal repair or perfect repair. Let N be a positive integer valued random variable denoting the number of repaires performed till the perfect repair, i.e., $N = k$ means the event that $k-1$ minimal repairs were performed and the repair for the k-th failure was perfect and the component returned to the good-as-new-state. After the perfect repair the process is renewed. Time for repair is assumed to be negligible.

The dynamic process of the component is governed by a non-homogeneous Poisson process $\{N(t), t\geq 0\}$ with mean value function $\Lambda(t)$. Then T_N means the time that the component returns to the good-as-new-state, where $T_k = \inf \{t | N(t) = k\}$ $k\geq 1$, the time that the k-th failure occurs supposed that the repairs for the previous $k-1$ failures were minimal.

In this paper we study monotonic properties of T_N and other stochastic quantities, e.g., steady-state-distributions and so on. Our results may be of interest in renewal theory as well as in relibility theory.

1. Preliminaries

$\{N(t),t\geq 0\}$: a non-homogeneous Poisson Process with differentiable mean value function $\Lambda(t)$,

$$\Pr\{N(t) = k\} = e^{-\Lambda(t)}\frac{[\Lambda(\tau)]^k}{k!} ,$$

$$\lambda(t) = \frac{d}{dt} \Lambda(t),$$

$$T_k = \inf \{t \mid N(t) = k\}, \quad k = 1,2,\ldots,$$

$$T_0 \equiv 0 .$$

The following theorem is easily proved.

Theorem 1. For $k \geq 0$, $\ell \geq 1$,

(1) $\Pr[T_{k+\ell}-T_k>x \mid T_k=y] = \Pr[N(x+y) - N(y)<y]$,

(2) $\Pr[T_{k+\ell}-T_k>x] = \int_0^\infty \Pr[N(x+y)-N(y)<\ell]d\Pr[T_k\leq y]$,

(3) $E[T_{k+\ell}-T_k \mid T_k=y] = \int_0^\infty \Pr[N(x+y)-N(y)<\ell]dx$,

(4) $E[T_{k+\ell}-T_k] = \int_0^\infty\int_0^\infty \Pr[N(x+y)-N(y)<\ell]dxd\Pr[T_k\leq y]$. □

Corollary 2. Letting $\ell=1$ in the previous theorem, for $k\geq1$,

(1) $\Pr[T_{k+1}-T_k>x \mid T_k=y] = e^{-[\Lambda(x+y)-\Lambda(y)]} = \dfrac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}}$,

(2) $\Pr[T_{k+1}-T_k>x] = \int_0^\infty \dfrac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} d\Pr[T_k\leq y]$,

(3) $E[T_{k+1}-T_k \mid T_k=y] = \int_0^\infty \dfrac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} dx$,

(4) $E[T_{k+1}-T_k] = \int_0^\infty\int_0^\infty \dfrac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} dxd\Pr[T_k\leq y]$. □

In the sequel we use the following lemmas, of which proof is easy and then omitted.

Lemma 3.

Let $g(x)\downarrow$, $g(x)\geq0$ for $x\geq0$, $f(x)\uparrow$, $f(x)\geq0$ for $x\geq0$. If two distribution functions F_1 and F_2 satisfy $F_1(0^-) = F_2(0^-) = 0$ and $F_1(x)\leq F_2(x)$ for $x\geq0$, then

$$\int_0^\infty g(x)dF_1(x) \leq \int_0^\infty g(x)dF_2(x) \quad \text{and} \quad \int_0^\infty f(x)dF_1(x) \geq \int_0^\infty f(x)dF_2(x),$$

supposing that the integrations finitely exist. □

Lemma 4. For $\lambda\geq\mu$, $\sum_{k=0}^n e^{-\lambda} \dfrac{\lambda^k}{k!} \leq \sum_{k=0}^n e^{-\mu} \dfrac{\mu^k}{k!}$ holds for $\forall n\geq0$. □

Theorem 5. For $k\geq0$, $\ell\geq1$,

(1) $1-e^{(t)}$: IFR $\Rightarrow$ $\Pr[T_{k+\ell}-T_k>x]\downarrow_k$ for $\forall x\geq0$,

(2) $1-e^{-\Lambda(t)}$: DMRL $\Rightarrow$ $E[T_{k+\ell}-T_k]\downarrow_k$,

(3) $1-e^{-\Lambda(t)}$: NBU $\Rightarrow$ $Pr[T_{k+\ell}-T_k>x] \le Pr[T_\ell>x]$ for $\forall x\ge 0$,

(4) $1-e^{-\Lambda(t)}$: NBUE $\Rightarrow$ $E[T_{k+\ell}-T_k] \le \ell E[T_1]$.

Proof. We notice that $T_k\uparrow_k$ a.s.

(1) $1-e^{-\Lambda(t)}$:IFR $\iff$ $\Lambda(x+y)-\Lambda(y)\uparrow_y$ $\Rightarrow$ $Pr[N(x+y)-N(y)<\ell]\downarrow_y$ (by Lemma 4)

$\Rightarrow$ $Pr[T_{k+\ell}-T_k>x]\downarrow_k$ (by Theorem 1 (2) and Lemma 3).

(2) $E[T_{k+\ell}-T_k]=\sum_{i=1}^{\ell-1}E[T_{k+i+1}-T_{k+i}]$ is decreasing in k, since each term of the right hand side is decreasing in k by Corollary 2 and Lemma 3.

(3) By Lemma 4, $Pr[N(x+y)-N(y)<\ell] \le Pr[N(x)<\ell] = Pr[T_\ell>x]$. Then (3) is obvious.

(4) $E[T_{k+\ell}-T_k]=\sum_{i=0}^{\ell-1}E[T_{k+i+1}-T_{k+i}]$ and $E[T_{k+i+1}-T_{k+i}]\le\int_0^\infty e^{-\Lambda(x)}dx=E[T_1]$ by the assumption and Corollary 2 (4). Then we have the inequality.

□

2. Monotonic Properties of T_k

Let N be a positive integer valued r.v. independent with $\{N(t),t\ge 0\}$. In this section we study monotonic properties of T_N.

Theorem 6. (1) Suppose that Z_i $(i\ge 1)$ are i.i.d. r.v.'s with common distribution function same to the one of T_1, and are independent with N.

(1) $1-e^{-\Lambda(t)}$: NBU $\Rightarrow$ $Pr[T_N>t] \le Pr[\sum_{i=1}^{N}Z_i>t]$.

(2) $1-e^{-\Lambda(t)}$: NBUE $\Rightarrow$ $E[T_N] \le E[N]E[T_1]$.

Proof. (2) is obvious from Theorem 5 (4).

(1) It is sufficient to prove $Pr[T_k>t]\le Pr[Z_1+...+Z_k>t]$ for $k\ge 1$. We prove the inequality by the mathematical induction on k.

$$\begin{aligned}Pr[T_{k+1}>t] &= \int Pr[T_{k+1}>t\,|\,T_k=x]dPr[T_k\le x]\\ &= \int Pr[T_{k+1}-T_k>t-x\,|\,T_k=x]dPr[T_k\le x]\\ &\le \int Pr[Z_{k+1}>t-x]dPr[T_k\le x]\end{aligned}$$ (by Corollary 2 (1) and th

assumtion)

$$\le \int \Pr[Z_{k+1}>t-x]d\Pr[Z_1+\dots+Z_k\le x] \quad \text{(by the inductive assumption and Lemma 3)}$$

$$= \Pr[Z_1+\dots+Z_{k+1}>t]. \qquad \square$$

3. Stochastic Comparisons of T_N and $T_{N'}$

Let
$$\bar{F}_N(t) = \Pr[T_N>t] = \textstyle\sum_{k=0}^{\infty}\Pr[N(t)=k]\Pr[N>k],$$
$$f_N(t) = \frac{d}{dt}\Pr[T_N\le t] = \textstyle\sum_{k=0}^{\infty}\Pr[N(t)=k]\lambda(t)\Pr[N=k+1],$$
$$\lambda_N(t) = \frac{f_N(t)}{\bar{F}_N(t)} .$$

In this section N and N' are positive integer valued r.v.'s independent with $\{N(t),\ t\ge 0\}$

Theorem 7. (1) $\dfrac{\Pr[N=\ell]}{\Pr[N=k]} \le \dfrac{\Pr[N'=\ell]}{\Pr[N'=k]}$ for $k<\ell \Rightarrow \dfrac{f_N(x+\Delta)}{f_N(x)} \le \dfrac{f_{N'}(x+\Delta)}{f_{N'}(x)}$ for $x\ge 0$ and $\Delta\ge 0$.

(2) $\dfrac{\Pr[N=k+1]}{\Pr[N>k]} \le \dfrac{\Pr[N'=k+1]}{\Pr[N'>k]}$ for $k\ge 1 \iff \dfrac{\Pr[N>\ell]}{\Pr[N>k]} \ge \dfrac{\Pr[N'>\ell]}{\Pr[N'>k]}$ for $k<\ell$

$$\Rightarrow \frac{\bar{F}_N(t+\Delta)}{\bar{F}_N(t)} \ge \frac{\bar{F}_{N'}(t+\Delta)}{\bar{F}_{N'}(t)} \text{ for } t\ge 0,\ \Delta\ge 0 \iff \lambda_N(t) \le \lambda_{N'}(t) \text{ for } t\ge 0.$$

(3) $\Pr[N>\ell] \ge \Pr[N'>\ell]$ for $\ell\ge 1 \Rightarrow \bar{F}_N(t) \ge \bar{F}_{N'}(t)$ for $t\ge 0$.

Proof. (1)

$$\begin{vmatrix} \sum_{k=0}^{\infty}\Pr[N(x+\Delta)=k]\lambda(x+\Delta)\Pr[N=k+1] & \sum_{k=0}^{\infty}\Pr[N(x+\Delta)=k]\lambda(x+\Delta)\Pr[N'=k+1] \\ \sum_{k=0}^{\infty}\Pr[N(x)=k]\lambda(x)\Pr[N=k+1] & \sum_{k=0}^{\infty}\Pr[N(x)=k]\lambda(x)\Pr[N'=k+1] \end{vmatrix}$$

$$=\sum_{k<\ell}\sum \begin{vmatrix} \Pr[N(x+\Delta)=k]\lambda(x+\Delta) & \Pr[N(x+\Delta)=\ell]\lambda(x+\Delta) \\ \Pr[N(x)=k]\lambda(x) & \Pr[N(x)=\ell]\lambda(x) \end{vmatrix} \begin{vmatrix} \Pr[N=k+1] & \Pr[N'=k+1] \\ \Pr[N=\ell+1] & \Pr[N'=\ell+1] \end{vmatrix}$$

≤ 0.

(2) The equivalent relations of (2) is obvious.

$$\begin{vmatrix} \bar{F}_N(t+\Delta) & \bar{F}_{N'}(t+\Delta) \\ \bar{F}_N(t) & \bar{F}_{N'}(t) \end{vmatrix} = \sum_{k<\ell}\sum \begin{vmatrix} \Pr[N(t+\Delta)=k] & \Pr[N(t+\Delta)=\ell] \\ \Pr[N(t)=k] & \Pr[N(t)=\ell] \end{vmatrix} \begin{vmatrix} \Pr[N>k] & \Pr[N'>k] \\ \Pr[N>\ell] & \Pr[N'>\ell] \end{vmatrix}$$

$\geq 0.$

The relation (3) is easily proved by usin Lemma 3. □

We present simple bounds for the distribution and the expectation of T_N.

Corollary 8. Let $q_m=\inf_k \Pr[N>k+1|N>k]$, $q_M=\sup_k \Pr[N>k+1|N>k]$, and N_m and N_M be positive integer valued r.v.'s independent with $\{N(t),\ t\geq 0\}$ such that $\Pr[N_m=k]=q_m^{k-1}(1-q_m)$, $\Pr[N_M=k]=q_M^{k-1}(1-q_M)$. Since $\Pr[N_m>k]\leq\Pr[N>k]\leq\Pr[N_M>k]$ for $k\geq 1$, by Theorem 7 we have $\Pr[T_{N_m}>t]\leq\Pr[T_N>t]\leq\Pr[T_{N_M}>t]$ for $t\geq 0$ and $E[T_{N_m}]\leq E[T_N]\leq E[T_{N_M}]$. □

Remark 9. Theorem 7 (1) (2) (3) show that stochastically-lager-relations between N and N' are preserved to the same stochastic relations between T_N and $T_{N'}$ without any assumption on $1-e^{-\Lambda(t)}$. □

Theorem 10.

$$1-e^{-\Lambda(t)}:\text{DMRL},\quad \frac{\sum_{k=j}^{\infty}\Pr[N\geq k]}{E[N]} \geq \frac{\sum_{k=j}^{\infty}\Pr[N'\geq k]}{E[N]} \text{ for } j\geq 1$$
$$\Rightarrow \frac{E[T_N]}{E[N]} \leq \frac{E[T_{N'}]}{E[N']} .$$

Proof. Since $1-e^{-\Lambda(t)}$ is DMRL, $E[T_j-T_{j-1}]$ is decreasing in j by Theorem 5 (2). Then using Lemma 3,

$$\frac{E[T_N]}{E[N]} = \sum_{j=1}^{\infty}E[T_j-T_{j-1}]\frac{\Pr[N\geq j]}{E[N]} \leq \sum_{j=1}^{\infty}E[T_j-T_{j-1}]\frac{\Pr[N'\geq j]}{E[N']} = \frac{E[T_{N'}]}{E[N']} . \square$$

Remark 11. The following relation holds.

$$\frac{\Pr[N=\ell]}{\Pr[N=k]} \geq \frac{\Pr[N'=\ell]}{\Pr[N'=k]} \text{ for } k\leq\ell \Rightarrow \frac{\Pr[N\geq\ell]}{\Pr[N\geq k]} \geq \frac{\Pr[N'\geq\ell]}{\Pr[N'\geq k]} \text{ for } k\leq\ell$$
$$\Rightarrow \frac{\sum_{k=j}^{\infty}\Pr[N\geq k]}{E[N]} \geq \frac{\sum_{k=j}^{\infty}\Pr[N'\geq k]}{E[N']} \text{ for } j\geq 1 . \quad (1)$$
$$\Rightarrow \Pr[N\geq k] \geq \Pr[N'\geq k] \quad \text{for } k\geq 1 . \quad (2)$$

There is generally no relation between (1) and (2). □

Remark 12. It is easily verifed that if $\Pr[N\leq 2]=\Pr[N'\leq 2]=1$, $\Pr[N=2]\geq\Pr[N'=2]$ and $1-e^{-\Lambda(t)}$ is NBU, then $\frac{E[T_N]}{E[N]} \leq \frac{E[T_{N'}]}{E[N']}$. □

Lemma 13. For $a_1 \geq a_2 \geq \ldots$, $\frac{a_1+\ldots+a_n}{n} \geq \frac{a_1+\ldots+a_{n+1}}{n+1}$ holds for $n \geq 1$. The proof is easy and omitted. □

Theorem 14. $1-e^{-\Lambda(t)}$:DMRL and $Pr[N>k] \geq Pr[N'>k]$ for $k \geq 1$

$$\Rightarrow \quad E[\frac{T_N}{N}] \leq E[\frac{T_{N'}}{N'}] \ .$$

Proof. Since $1-e^{-\Lambda(t)}$ is DMRL, $E[T_j-T_{j-1}]$ is decreasing in j by Theorem 5 (2). Then by Lemma 13, $E[\frac{T_k}{k}] = \frac{\sum_{j=1}^{k} E[T_j-T_{j-1}]}{k}$ is decreasing in k. Then Theorem 14 is obvious by Lemma 3. □

4. Stochastic Comparisons of Steady-State-Distributions

$\{N^j(t), t \geq 0\}$ $(j \geq 1)$: independent non-homogeneous Poisson processes with common mean value function $\Lambda(t)$,

$T_k^j = \inf \{t | N^j(t) = k\}$

N_j $(j \geq 1)$: independent positive inyeger valued r.v.'s with common distribution same to the one of N,

i.e., $\{N^j(t), t \geq 0\}$ $(j \geq 1)$ are replicas of $\{N(t), t \geq 0\}$ and N_j $(j \geq 1)$ are replicas of N. We assume that $\{N^j(t), t \geq 0\}$ $(j \geq 1)$ are independent with N_j $(j \geq 1)$.

We define a counting process $\{M(T), t \geq 0\}$ as

$$M(t) = \sum_{j=1}^{n-1} N_j + N^n(t - \sum_{j=1}^{n-1} T^j_{N_j}) \quad \text{if} \quad \sum_{j=1}^{n-1} T^j_{N_j} \leq t \leq \sum_{j=1}^{n} T^j_{N_j} \ .$$

We notice that $T^j_{N_j}$ $(j \geq 1)$ are i.i.d. randome variables with common distribution function $F_N(t)$, $1-F_N(t) = \sum_{k=0}^{\infty} Pr[N(t)=k] Pr[N>k]$. In this section we consider stochastic quantities with erespect to $\{M(t), t \geq 0\}$, which means the number of repairs performed in $[0,t]$.

Let's define

$$Z(t) = T^n_{N^n(t - \sum_{j=1}^{n-1} T^j_{N_j})+1} - (t - \sum_{j=1}^{n-1} T^j_{N_j}) \quad \text{if} \quad \sum_{j=1}^{n-1} T^j_{N_j} \leq t \leq \sum_{j=1}^{n} T^j_{N_j},$$

which means the time to the next failure from the time epock t.

Theorem 15.

$$\lim_{t\to\infty} \Pr[Z(t)>x] = \int_0^\infty \frac{\bar{F}_N(y)}{E[T_N]} \cdot \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} dy .$$

Proof. Simple caluculation verifies the above equality. Since $\Pr[Z(t)>x]=\int_0^t \Pr[T_{N(x-y)+1}>t-y+x, t-y<T_N] d\{\sum_{n=1}^\infty (F_N)^{(n-1)}(y)\}$, where $(F_N)^{(n-1)}$ is the (n-1)-hold convolution of $_NF$, then by the basic renewal theory we have

$$\lim_{t\to\infty} \Pr[Z(t)>x] = \frac{1}{E[T_N]} \int_0^\infty \Pr[T_{N(y)+1}>y+x, y<T_N] dy .$$

Noticing that $\Pr[T_{N(y)+1}>y+x, y<T_N] = \bar{F}_N(y)\Pr[N(y+x)-N(y)=0]$, the theorem is proved. □

We write the steady-state-distribution of Z(t) as H_N, i.e.,

$$\bar{H}_N(x) = 1-H_N(x) = \int_0^\infty \frac{\bar{F}_N(y)}{E[T_N]} \cdot \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} dy .$$

Theorem 16. (1) The density function $h_N(x)$ and the failure rate function $r_N(x)$ of $H_N(x)$ are

$$h_N(x) = \int_0^\infty \frac{\bar{F}_N(y)}{E[T_N]} \cdot \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} \cdot \lambda(x+y) dy ,$$

$$r_N(x) = \frac{\int_0^\infty \bar{F}_N(y) \cdot \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} \cdot \lambda(x+y) dy}{\int_0^\infty \bar{F}_N(y) \cdot \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} dy} .$$

(2) $\frac{\Pr[N>\ell]}{\Pr[N>k]} \le \frac{\Pr[N'>\ell]}{\Pr[N'>k]}$ for $k<\ell$, $1-e^{-\Lambda(t)}$ has PF_2-density

$$\Rightarrow \frac{h_N(t+\Delta)}{h_N(t)} \ge \frac{h_{N'}(t+\Delta)}{h_{N'}(t)} \quad \text{for } t>0, \Delta>0 .$$

(3) $\frac{\Pr[N>\ell]}{\Pr[N>k]} \le \frac{\Pr[N'>\ell]}{\Pr[N'>k]}$ for $k<\ell$, $1-e^{-\Lambda(t)}$: IFR

$$\Rightarrow r_N(x) \le r_{N'}(x) \text{ for } \forall x>0 \Rightarrow H_N(x) \le H_{N'}(x) \text{ for } \forall x>0 .$$

Proof. Differentiating $H_N(x)$, (1) is easily obtained.

$$(2)\ \begin{vmatrix} \int_0^\infty \bar{F}_N(y)\cdot\frac{e^{-\Lambda(x+\Delta+y)}}{e^{-\Lambda(y)}}\cdot\lambda(x+\Delta+y)dy & \int_0^\infty \bar{F}_{N'}(y)\cdot\frac{e^{-\Lambda(x+\Delta+y)}}{e^{-\Lambda(y)}}\cdot\lambda(x+\Delta+y)dy \\ \int_0^\infty \bar{F}_N(y)\cdot\frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}}\cdot\lambda(x+y)dy & \int_0^\infty \bar{F}_{N'}(y)\cdot\frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}}\cdot\lambda(x+y)dy \end{vmatrix}$$

$$= \iint_{y_1<y_2} \begin{vmatrix} \bar{F}_N(y_1) & \bar{F}_N(y_2) \\ \bar{F}_{N'}(y_1) & \bar{F}_{N'}(y_2) \end{vmatrix} \begin{vmatrix} \frac{e^{-\Lambda(x+\Delta+y_1)}}{e^{-\Lambda(y_1)}}\cdot\lambda(x+\Delta+y_1) & \frac{e^{-\Lambda(x+\ +y_2)}}{e^{-\Lambda(y_2)}}\cdot\lambda(x+\Delta+y_2) \\ \frac{e^{-\Lambda(x+y_1)}}{e^{-\Lambda(y_1)}}\cdot\lambda(x+y_1) & \frac{e^{-\Lambda(x+y_2)}}{e^{-\Lambda(y_2)}}\cdot\lambda(x+y_2) \end{vmatrix}$$

≥ 0.

Using Basic Composition Theorem, (3) is proved similarly to the proof of (2). □

We define

$$Z^*(t) = \sum_{j=1}^{n} T_{N_j}^{j} - t \quad \text{if} \quad \sum_{j=1}^{n-1} T_{N_j}^{j} \leq t \leq \sum_{j=1}^{n} T_{N_j}^{j} ,$$

which means the time to the next perfect repair from the time epock t.

It is well known that

$$\lim_{t\to\infty} \Pr[Z^*(t)\leq x] = \frac{1}{E[T_N]} \int_0^x \bar{F}_N(u)du .$$

We write the right hand side of the above equality as $H_N^*(x)$.

Theorem 17. $\frac{\Pr[N>\ell]}{\Pr[N>k]} \leq \frac{\Pr[N'>\ell]}{\Pr[N'>k]}$ for $k<\ell$

$$\Rightarrow \frac{\bar{H}_N^*(t+\Delta)}{\bar{H}_N^*(t)} \leq \frac{\bar{H}_{N'}^*(t+\Delta)}{\bar{H}_{N'}^*(t)} \quad \text{for } t>0,\ \Delta>0 \quad \Rightarrow \quad H_N^*(t) \geq H_{N'}^*(t) \quad \text{for } t>0 .$$

Proof.

$$\begin{vmatrix} \int_{t+\Delta}^\infty \bar{F}_N(u)du & \int_{t+\Delta}^\infty F_{N'}(u)du \\ \int_t^\infty \bar{F}_N(u)du & \int_t^\infty \bar{F}_{N'}(u)du \end{vmatrix} = \begin{vmatrix} \int_{t+\Delta}^\infty \bar{F}_N(u)du & \int_{t+\Delta}^\infty \bar{F}_{N'}(u)du \\ \int_t^{t+\Delta} F_N(u)du & \int_t^{t+\Delta} F_{N'}(u)du \end{vmatrix} \leq 0.$$

□

Noticing that $\bar{H}_N(x) = \int_0^\infty \frac{e^{-\Lambda(x+y)}}{e^{-\Lambda(y)}} dH_N^*(y)$, we have by Theorem 17 and Lemma 3 :

<u>Theorem 18.</u> $\frac{\Pr[N>\ell]}{\Pr[N>k]} \le \frac{\Pr[N'>\ell]}{\Pr[N'>k]}$ for $k\le\ell$, $1-e^{-\Lambda(t)}$: DMRL

$$\Rightarrow \int_0^\infty \bar{H}_N(x)dx \ge \int_0^\infty \bar{H}_{N'}(x)dx \ . \qquad \square$$

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