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OPTIMAL CONTROL FOR LINEAR DISCRETE SYSTEMS WITH OVERTAKING CRITERION

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1. Introduction.

The problem of tracking periodic signals has been studied by Artstein and Leizarowitz [1] for a time invariant linear continuous system. They have introduced the notion of overtaking optimality and obtained a unique optimal control under controllability and observability. It is given by the feedback law involving the solution of an algebraic Riccati equation.

We have recently considered the average quadratic control problem for infinite dimensional linear periodic systems with periodic inputs [4]. Under the stabilizability and detectability condition one can show the existence of a unique nonnegative periodic solution to the usual Riccati equation. We have shown that the optimal control is a feedback law given by this periodic solution. This argument can also be applied to the tracking problem with average criterion [6]. We have also shown that the optimal law is a unique overtaking optimal control [7].

This paper is concerned with the discrete time version of the problem considered in [7] and similar results are obtained. A two dimensional simple example is given to illustrate the theory.

2. Deterministic Tracking Problem.

Consider the controlled system :

$$x(k+1) = Ax(k) + Bu(k), \quad x_0, \text{ given} \tag{2.1}$$

where $x \varepsilon R^n$, $u \varepsilon R^m$, $A \varepsilon R^{n\times n}$ and $B \varepsilon R^{n\times m}$. We associate with (2.1) a family of cost functionals

$$J_L(u)=\sum_{k=0}^{L}[\,|M(x(k)-h(k))|^2+u(k)^T N u(k)] \tag{2.2}$$

and the set of admissible controls

$$U_{ad}=\{u=\{u(k)\}: \sum_{k=0}^{L}|u(k)|^2<\infty \text{ for any } L \quad \text{such that } \sup_k |x(k)|<\infty\}, \tag{2.3}$$

where $M\varepsilon R^{p\times n}$, $N\varepsilon R^{m\times m}$ with $N>0$, $h\varepsilon R^p$ is an N-periodic sequence ($h(k+N)=h(k)$) and $|\ |$ denotes the usual Euclidean norm.
As in [1], [8] a control $u^*=\{u^*(k)\}$ is said to be overtaking optimal if it overtakes any other admissible control i.e., if for each admissible u there exists a time L_0 such that $J_L(u^*)<J_L(u)$ for all $L>L_0$.
This problem was considered by Artstein and Leizarowitz [1] for a time invariant continuous systems. They assumed that (A,B) is controllable and (A,M) observable. We extended their results to time-varying periodicsystems which are stabilizable and detectable [7].

We first consider periodic solutions of a linear system:

$$x(k+1)=Ax(k)+f(k), \tag{2.4}$$

where $f=\{f(k)\}$ is N-periodic. Let $\lambda(A)$ be the set of eigenvalues of the matrix A. We call A stable if $|\lambda|<1$ for any λ in $\lambda(A)$. If A is stable, there exists a unique N-periodic solution to (2.4) given by

$$x_p(k)=\sum_{j=-\infty}^{k-1}A^{k-1-j}f(j). \tag{2.5}$$

In fact we have

$$x_p(k)=A^k x_p(0)+\sum_{j=0}^{k-1}A^{k-1-j}f(j),$$

where $x_p(0)=\sum_{j=-\infty}^{-1}A^{-1-j}f(j)$ and $x_p(k)$ satisfies (2.4). Moreover,

$$x_p(k+N)=\sum_{j=-\infty}^{k+N-1}A^{k+N-1-j}f(j)$$

$$= \sum_{i=-\infty}^{k-1} A^{k-1-i} f(i+N)$$

$$= \sum_{i=-\infty}^{k-1} A^{k-1-i} f(i)$$

$$= x_p(k).$$

Next we recall the following:

(i) (A,B) is stabilizable if there exists $K \varepsilon R^{m\times n}$ such that $A-BK$ is stable.

(ii) (A,M) is detectable if there exists $J \varepsilon R^{n\times p}$ such that $A-JM$ is stable.

If $A-BK$ is stable, then the control law

$$u(k)=-Kx(k) + f(k)$$

is admissible for any N-periodic f.

To solve the control problem (2.1), (2.2) we need the solution of the Riccati equation (see [10])

$$Q=A^TQA + M^TM-A^TQB(N + B^TQB)^{-1}B^TQA. \qquad (2.6)$$

Proposition 2.1.(i) Suppose (A,B) is stabilizable. Then there exists a non-negative solution to the Riccati equation (2.6).

(ii) If (A,M) is detectable, then there exists at most one such solution. Moreover, if Q is a solution, then $A-B(N + B^TQB)^{-1}B^TQA$ is stable.

Now suppose $A-B(N + B^TQB)^{-1}B^TQA$ is stable. Then

$$r(k+1)-[A^T-A^TQB(N + B^TQB)^{-1}B^T]r(k) + M^TMh(k) = 0 \qquad (2.7)$$

has a unique N-periodic solution

$$r(k) = -\sum_{j=k+1}^{\infty} A^{j-k-1}M^TMh(j). \qquad (2.8)$$

Thus if (A,B) is stabilizable and (M,A) is detectable, the control law

$$\bar{u}(k) = -(N + B^TQB)^{-1}B^T[QAx(k) + r(k)] \qquad (2.9)$$

is admissible and is in fact optimal as we see below. The closed system associated with (2.9) is given by

$$x(k+1) = [A - B(N + B^TQB)^{-1}B^TQA]x(k) - B(N + B^TQB)^{-1}B^Tr(k). \qquad (2.10)$$

This system has a unique N-periodic solution $\bar{x}_p(k)$ and $x(k;x_0) - \bar{x}_p(k) \to 0$ for any x_0 as $k \to \infty$ where $x(k;x_0)$ is the solution with $x(0) = x_0$.

The following lemma is useful.

Lemma 2.1. Consider the control law

$$u(k) = -(N + B^TQB)^{-1}B^T[QAx(k) + r(k)] + v(k)$$

and its responce $x(k)$. If

$$\sum_{k=0}^{\infty} |v(k)|^2 < \infty,$$

then $|\bar{x}_p(k) - x(k)| \to 0$ as $k \to \infty$. Thus if $|\bar{x}_p(k) - x(k)| \not\to 0$ as $k \to \infty$, then

$$\sum_{k=0}^{\infty} |v(k)|^2 = \infty.$$

Proof. The solution x corresponding to u satisfies

$$x(k+1) = [A - B(N + B^TQB)^{-1}B^TQA]x(k) - B(N + B^TQB)^{-1}B^Tr(k) + v(k), \quad x(0) = x_0$$

and is given by

$$x(k) = A_c^k x_0 + \sum_{j=0}^{k-1} A_c^{k-1-j}[Bv(j) - B(N + B^TQB)^{-1}B^Tr(k)]$$

$$= A_c^{k-L}x(L) + \sum_{j=L}^{k-1} A_c^{k-1-j}[Bv(j) - B(N + B^TQB)^{-1}B^Tr(k)]$$

$$= \tilde{x}(L) + \sum_{j=L}^{k-1} A_c^{k-1-j}Bv(j),$$

where $A_c = A - B(N + B^TQB)^{-1}B^TQA$ and

$$\tilde{x}(L) = A_c^{k-L}x(L) - \sum_{j=L}^{k-1} A_c^{k-1-j}B(N + B^TQB)^{-1}B^Tr(k).$$

Now assume $\sum_{k=0}^{\infty} |v(k)|^2 < \infty$. Then for a given ε we can find an L such that

$$\left| \sum_{j=L}^{k-1} A_c^{k-1-j}Bv(j) \right| < \varepsilon/2 \quad \text{for all } k>L.$$

From the remark below (2.10) we can find an $L_0 > L$ such that $k > L_0$ implies $|x(k) - \bar{x}_p(k)| < \varepsilon/2$. Thus $|x(k) - \bar{x}_p(k)| < \varepsilon$ if $k > L_0$.

Theorem 2.1. Suppose (A,B) is stabilizable and (A,M) detectable. Then the feedback law (2.9) is overtaking optimal and is unique. Moreover, for any other $u \in U_{ad}$ there exist L_0 and ε such that

$$J_L(\bar{u}) + \varepsilon < J_L(u) \text{ for all } L > L_0. \tag{2.11}$$

Proof. Let $u = u(k)$ be any admissible control and $x(k)$ its responce. Summing the following from k=0 to k=L

$$x(k+1)^TQx(k+1) + 2r(k)^Tx(k+1) - x(k)^TQx(k) - 2r(k-1)^Tx(k)$$

and using (2.6), (2.7) and (2.1) we obtain

$$J_L(u) = \sum_{k=0}^{L} [|M(x(k) - h(k))|^2 + u(k)^TNu(k)]$$

$$= x_0^TQx_0 + 2r(-1)^Tx_0 - x(L+1)^TQx(L+1) - 2r(L)^Tx(L+1)$$

$$+ \sum_{k=0}^{L} [|Mh(k)|^2 - r(k)^TB(N + B^TQB)^{-1}B^Tr(k)]$$

$$+ \sum_{k=0}^{L} v(k)^TNv(k),$$

where

$$v(k) = u(k) + (N + B^TQB)^{-1}B^T[QAx(k) + r(k)].$$

Hence

$$J_L(u) - J_L(\bar{u})$$
$$= \bar{x}(L+1)^T Q \bar{x}(L+1) + 2r(L)^T \bar{x}(L+1)$$
$$- x(L+1)^T Q x(L+1) - 2r(L)^T x(L+1) \qquad (2.12)$$
$$+ \sum_{k=0}^{L} v(k)^T N v(k),$$

where $\bar{x}$ is the responce to $\bar{u}$. Suppose $\sum_{k=0}^{\infty} |v(k)|^2 = \infty$. Then $J_L(u) - J_L(\bar{u}) \to \infty$ as $L \to \infty$ since the terms on the right hand side of (2.12) except the last are bounded. Hence in view of Lemma 2.1 we only need to consider the case :

$$\sum_{k=0}^{\infty} |v(k)|^2 = b < \infty \text{ and hence } |\bar{x}_p(L) - x(L)| \to 0 \text{ as } L \to \infty.$$

In this case we choose L_0 large enough so that for any $L > L_0$

$$b_L = \sum_{k=0}^{L} |v(k)|^2 > b - \varepsilon \text{ and}$$

$$|\bar{x}(L+1)^T Q \bar{x}(L+1) + 2r(L)^T \bar{x}(L+1)$$
$$- x(L+1)^T Q x(L+1) - 2r(L)^T x(L+1)| < b - 2\varepsilon$$

Then from (2.12) we obtain

$$|J_L(u) - J_L(\bar{u}) - b_L| < b - 2\varepsilon \text{ or } J_L(\bar{u}) + \varepsilon < J_L(u).$$

3. Stochastic Tracking Problem.

Let (Ω, F, F_k, P) be a stochastic basis. We recall that a stochastic process $x(k)$ is N-periodic if it has an N-periodic distribution. In this section we consider the stochastic system

$$x(k+1) = Ax(k) + Bu(k) + Gw(k), \quad x_0, \text{ given} \qquad (3.1)$$

where $G \varepsilon R^{n \times q}$ and $w(k)$ is a q-dimensional independent Gaussian F_k-measurable process with zero mean and N-periodic covariance $W(k)$.

Now we replace (2.2) by

$$J_L(u)= E\sum_{k=0}^{L}[\,|M(x(k)-h(k))|^2+u(k)^T N\cdot u(k)] \tag{3.2}$$

and (2.3) by

$$U_{ad}=\{u=u(k):\ u(k) \text{ is } F_k\text{-measurable, } E\sum_{k=0}^{L}|u(k)|^2<\infty$$

$$\text{for any } L \text{ such that } \sup_k E|x(k)|^2<\infty\}\ . \tag{3.3}$$

We wish to find an overtaking optimal control for (3.2).

First we consider

$$x(k+1) = A\,x(k) + f(k) + Gw(k), \tag{3.4}$$

where A is stable and $f = \{f(k)\}$ is N-periodic. Then there exists a unique N-periodic Gaussian solution to (3.4) given by

$$x_p(k) = A^k x_p(0) + \sum_{j=0}^{k-1} A^{k-1-j}[f(j) + Gw(j)],$$

where
$$E\,x_p(0) = \sum_{j=-\infty}^{-1} A^{-1-j} f(j)$$

$$\mathrm{Cov}[x_p(0)]= \sum_{j=-\infty}^{-1} A^{-j} G W(j) G^T A^{T-j}.$$

Moreover, $x(k;x_0)-x_p(k)\to 0$ a.s. and in mean square as $k\to\infty$.

As in §2 we shall show that the control law

$$\bar{u}(k) = -(N + B^T Q B)^{-1}B^T[Q A\,x(k) + r(k)] \tag{2.9}$$

is optimal. The closed system associated with (2.9) is given by

$$x(k+1)=[A-B(N + B^T Q B)^{-1}B^T Q A]x(k)$$
$$-B(N + B^T Q B)^{-1}B^T r(k) + Gw(k). \tag{3.5}$$

This system has a unique N-periodic solution $\bar{x}_p(k)$ and $x(k;x_0)-\bar{x}_p(k) \to 0$ for any x_0 as $k\to\infty$ where $x(k;x_0)$ is the solution of (3.5) with $x(0) = x_0$.

<u>Lemma 3.1.</u> Consider the control law

$$u(k) = -(N + B^TQB)^{-1}B^T[QAx(k) + r(k)] + v(k)$$

and its responce $x(k)$. If

$$E\sum_{k=0}^{\infty} |v(k)|^2 < \infty,$$

then $E|\bar{x}_p(k) - x(k)| \to 0$ as $k\to\infty$. Thus if $E|\bar{x}_p(k) - x(k)| \not\to 0$ as $k\to\infty$, then

$$E\sum_{k=0}^{\infty} |v(k)|^2 = \infty.$$

Theorem 3.1. Suppose (A,B) is stabilizable and (A,M) detectable. Then the feedback law (2.9) is overtaking optimal and is unique. Moreover, for any other $u \varepsilon U_{ad}$ there exist L_0 and ε such that

$$J_L(\bar{u}) + \varepsilon < J_L(u) \text{ for all } L > L_0.$$

Proof. Let $u = u(k)$ be any admissible control and $x(k)$ its responce. Summing the following from k=0 to k=L

$$E[x(k+1)^TQx(k+1) + 2r(k)^Tx(k+1) - x(k)^TQx(k) - 2r(k-1)^Tx(k)]$$

and using (2.6), (2.7) and (3.5) we obtain

$$J_L(u) = E\sum_{k=0}^{L}[|M(x(k) - h(k))|^2 + u(k)^TNu(k)]$$

$$= E[x_0^TQx_0 + 2r(-1)^Tx_0 - x(L+1)^TQx(L+1) - 2r(L)^Tx(L+1)]$$

$$+ \sum_{k=0}^{L}[|Mh(k)|^2 - r(k)^TB(N + B^TQB)^{-1}B^Tr(k) + \mathrm{tr}.GW(k)G^TQ]$$

$$+ E\sum_{k=0}^{L} v(k)^TNv(k),$$

where

$$v(k) = u(k) + (N + B^TQB)^{-1}B^T[QAx(k) + r(k)].$$

The rest are similar to the deterministic case.

Remark 3.1. The tracking problem with overtaking criterion can be extended to more general time varying systems as in [4], [5], [7]. The stochastic problem under partial observation is considered in [8] and the extension of our approach to this case is also immediate.

4. An Example.

We take

$$A=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},\ M=(1/\sqrt{2})[1 \quad 0],\ B=\begin{bmatrix} 0 \\ 1 \end{bmatrix},\ N=1,$$

and four-periodic $h(k)$ with $h(0)=[1 \quad 0]^T$, $h(1)=[0 \quad -1]^T$, $h(2)=[-1 \quad 0]^T$, $h(3)=[0 \quad 1]^T$. Then the solution of the algebraic Riccati equation is $Q=I$. The optimal law for (2.1) and (2.2) is

$$\bar{u}(k)=-(N+B^TQB)^{-1}B^T[QAx(k)+r(k)].$$
$$=(1/2)[x_1(k)-r_2(k)]$$

where $x=[x_1, x_2]^T$, $r=[r_1, r_2]^T$.

The closed loop matrix is

$$A_c=A-B(N+B^TQB)^{-1}B^TQA=\begin{bmatrix} 0 & 1 \\ -1/2 & 0 \end{bmatrix}.$$

The four-periodic solution $r(k)$ of (2.7) is

$$r(0)=-\sum_{j=1}^{\infty} \bar{A}^{j-1}M^TMh(j)=[0 \quad 1]^T,\quad \bar{A}=A_c^T$$

$$r(1)=-\sum_{j=2}^{\infty} \bar{A}^{j-2}M^TMh(j)=[1 \quad 0]^T,$$

$$r(2)=-\sum_{j=3}^{\infty} \bar{A}^{j-3}M^TMh(j)=[0 \quad -1]^T$$

$$r(3) = -\sum_{j=4}^{\infty} \bar{A}^{j-4} M^T M h(j) = [-1 \quad 0]^T.$$

The optimal closed system is

$$x(k+1) = A_c x(k) - B(N + B^T Q B)^{-1} B^T r(k)$$

and its 4-periodic solution $x_p(k)$ is given by

$$x_p(k) = -\sum_{j=-\infty}^{k-1} A_c^{k-1-j} B(N + B^T Q B)^{-1} B^T r(j)].$$

with

$$x_p(0) = -\sum_{j=-\infty}^{-1} A_c^{-1-j} B(N + B^T Q B)^{-1} B^T r(j)] = [1 \quad 0]^T.$$

$$x_p(1) = -\sum_{j=-\infty}^{0} A_c^{-j} B(N + B^T Q B)^{-1} B^T r(j)] = [0 \quad -1]^T.$$

$$x_p(2) = -\sum_{j=-\infty}^{1} A_c^{1-j} B(N + B^T Q B)^{-1} B^T r(j)] = [-1 \quad 0]^T.$$

$$x_p(3) = -\sum_{j=-\infty}^{2} A_c^{2-j} B(N + B^T Q B)^{-1} B^T r(j)] = [0 \quad 1]^T.$$

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