数理解析研究所講究録
第 680 巻 1989 年 250-264

協力ゲームの解の間の一致性と線形性
(Coincidence of and Linearity between Game Theoretic Solutions)

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1. The egalitarian nonseparable contribution (ENSC-) method

The mathematical model of a cooperative n-person game in characteristic function form is described by the finite player set $N = \{1,2,...,n\}$ and a characteristic function $v: 2^N \to \mathbb{R}$ satisfying $v(\emptyset) = 0$. Here the worth $v(S)$ of coalition $S \subset N$ in the game v represents the savings due to cooperation between the members of S.

The purpose of the solution part in cooperative game theory is to prescribe for any game (at least) one justifiable distribution of the total savings $v(N)$ among the n players. Such distributions are represented by n-tuples $x = (x_1,x_2,...,x_n) \in \mathbb{R}^n$ of real numbers where the i-th coordinate x_i is interpreted as the resulting payoff to player $i \in N$. Generally, it is required that a payoff vector $x \in \mathbb{R}^n$ meets the efficiency principle for the n-person game v, i.e., the equality $\sum_{j\in N} x_j = v(N)$ holds. Since the introduction of the notion of a cooperative game in characteristic function form, many solution concepts for these games have been proposed. This paper is mainly devoted to the study of a solution concept that is based on the effectiveness of the (n-1)-person coalitions in an n-person game.

The (n-1)-person coalitions are said to be effective in an n-person game v whenever the resulting distribution of the total savings $v(N)$ is derived from the separable contributions of the single players to the grand coalition N. Here the separable contribution of any player $i \in N$ is given by $SC_i(v) := v(N) - v(N-\{i\})$. Usually, the efficiency principle is not met by these separable contributions. Consequently, the remaining nonseparable contribution $NSC(v) := v(N) - \sum_{j\in N} SC_j(v)$

has to be divided among the n players. At this stage, we adopt the equal division principle as the most naive division rule. The resulting solution concept is called the egalitarian nonseparable contribution (ENSC-) method. Thus, the ENSC-solution of an n-person game v is given by

$$ENSC_i(v) := SC_i(v) + n^{-1}NSC(v) \qquad \text{for all } i \in N.$$

Several well-known solution concepts are based on the idea of excess. Here the excess of coalition S with respect to an efficient payoff vector $x \in \mathbb{R}^n$ in the n-person game v is defined to be $e^v(S,x) := v(S) - \sum_{j \in S} x_j$. A nonnegative (nonpositive respectively) excess of S at x in the game v represents the gain (loss) to the coalition S if its members withdraw from the payoff vector x in order to form their own coalition. In the context of excess, we say that the (n-1)-person coalitions are effective at an efficient payoff vector $x \in \mathbb{R}^n$ if the maximal excesses at x are determined by the (n-1)-person coalitions, i.e.,

$$e^v(S,x) \le e^v(N-\{i\},x) \qquad (1.1)$$
$$\text{for all } i \in N \text{ and all } S \subset N-\{i\},\ S \neq \emptyset.$$

The set of efficient payoff vectors, at which the (n-1)-person coalitions are effective, is investigated in the next section. Due to a close determination of the intersection of the set involved with the prekernel, we formulate a sufficient condition for the coincidence of the prenucleolus and the ENSC-solution. The main result of Section 2 states that both solutions coincide whenever the ENSC-solution belongs to the set involved.

In Section 3 we characterize the effectiveness condition (1.1), applied to the excesses at the ENSC-solution itself, in terms of the gap function corresponding to the game v. The induced class of games is compared with three related subclasses of games which are taken from the literature on the ENSC-method. The theory of Section 3 is illustrated by a practical example concerning Japan.

In Section 4 we introduce the class of k-person coalitional games and describe the locus of the Shapley value for this type of games. The main results of Section 4 state that the Shapley value of a k-person coalitional game can be written as a linear or convex combination of the ENSC-solution and the centre of the set of all nonnegative efficient payoff vectors.
In this paper we omit all the proofs with the exception of the proof of the main result of Section 2. The proofs will be presented in a more detailed paper.

2. The coincidence of the egalitarian nonseparable contribution method and the prenucleolus concept

In the context of excess, the effectiveness of the (n-1)-person coalitions is defined by means of the condition (1.1). Here we present a characterization of the condition (1.1) in terms of minimal contributions of coalitions and subsequently, the new characterization of (1.1) gives rise to a comparison between the ENSC-solution and the so-called prenucleolus of a game.

Let $\Delta^v(T,S)$ denote the contribution of coalition S to a disjoint coalition T in the game v. That is,

$$\Delta^v(T,S) := v(T \cup S) - v(T) \qquad \text{whenever } T \cap S = \emptyset.$$

Clearly, we have that $\Delta^v(T,\emptyset) = 0$ for all $T \subset N$ and further, the separable contributions of the single players to the grand coalition are determined by $SC_i(v) = \Delta^v(N-\{i\},\{i\})$ for all $i \in N$. Now we define the smallest contribution of coalition S to single players in the game v by

$$m^v(S) := \min[\Delta^v(\{j\},S) \mid j \in N-S] \qquad \text{for all } S \subset N,\ S \neq N,$$
$$m^v(N) := v(N).$$

It is evident that $m^v(\emptyset) = 0$ and $m^v(N-\{i\}) = v(N) - v(\{i\})$ for all $i \in N$. Of particular interest is the contribution of coalition S to the complementary coalition N-S in the game v which is given by

$$v^*(S) := \Delta^v(N-S,S) = v(N) - v(N-S) \qquad \text{for all } S \subset N.$$

Obviously, we have $v^*(\emptyset) = 0$ and $v^*(N) = v(N)$. Due to the equality $(v^*)^*(S) = v(S)$ for all $S \subset N$, the characteristic function $v^*: 2^N \to \mathbb{R}$ is called the <u>dual game</u> of v. The two notions m^v and v^* are combined in the announced characterization of the effectiveness of the (n-1)-person coalitions for the excesses at a given payoff vector.

PROPOSITION 2.1. Let $x \in \mathbb{R}^n$.
Then $e^v(S,x) \leq e^v(N-\{i\},x)$ for all $i \in N$ and all $S \subset N-\{i\}$, $S \neq \emptyset$
iff $\sum_{j\in S} x_j \leq m^{(v^*)}(S)$ for all $S \subset N$ with $1 \leq |S| \leq n-2$.

Here $|S|$ denotes the number of players in coalition S. In view of the above equivalence, we define for any n-person game v the associated set $U(v) \subset \mathbb{R}^n$ by

$$U(v) := \{x \in \mathbb{R}^n \mid \sum_{j\in N} x_j = v(N) \text{ and } \sum_{j\in S} x_j \leq m^v(S) \text{ for all } S \subset N \text{ with } 1 \leq |S| \leq n-2\}.$$

Thus, the set U(v) consists of efficient payoff vectors that give rise only to payoffs not greater than the smallest contributions for all nonempty coalitions containing at most n-2 players. According to Proposition 2.1, an efficient payoff vector x satisfies the effectiveness condition (1.1) in the game v if and only if the vector x belongs to the set $U(v^*)$. Next we determine the part of the set $U(v^*)$ inside two well-known set-valued solution concepts, namely the core and the prekernel.

The core of an n-person game v consists of efficient payoff vectors that give rise only to nonpositive excesses. So, the <u>core</u> $C(v) \subset \mathbb{R}^n$ is given by

$$C(v) := \{x \in \mathbb{R}^n \mid e^v(N,x) = 0 \text{ and } e^v(S,x) \leq 0 \text{ for all } S \subset N \text{ with } S \neq N,\emptyset\}.$$

The inclusion $C(v) \subset O(v)$ always holds where the comprehensive orthant $O(v)$ is bounded above by the separable contributions of the players, i.e.,

$$O(v) := \{x \in \mathbb{R}^n \mid x_i \leq SC_i(v) \text{ for all } i \in N\}.$$

The next proposition states that the parts of the set $U(v^*)$ inside the core $C(v)$ and the orthant $O(v)$ coincide.

PROPOSITION 2.2.

(i) $U(v^*) \cap C(v) = U(v^*) \cap O(v)$

(ii) $U(v^*) \subset C(v)$ iff $U(v^*) \subset O(v)$.

The prekernel of an n-person game v consists of efficient payoff vectors for which any two players are equally powerful concerning their mutual maximal excesses. The <u>prekernel</u> $\mathcal{K}^*(v) \subset \mathbb{R}^n$ is given by

$$\mathcal{K}^*(v) := \{x \in \mathbb{R}^n \mid e^v(N,x) = 0 \text{ and for all } i,j \in N,\ i \neq j,$$
$$\max[e^v(S,x) \mid i \in S \subset N-\{j\}] = \max[e^v(S,x) \mid j \in S \subset N-\{i\}]\ \}.$$

It appears that the part of the set $U(v^*)$ inside the prekernel of the game v is at most a singleton consisting of the ENSC-solution. Moreover, the intersection of the set $U(v^*)$ with the prekernel coincides with the ENSC-solution if and only if the ENSC-solution belongs to the set $U(v^*)$.

THEOREM 2.3.

(i) $U(v^*) \cap \mathcal{K}^*(v) \subset \{ENSC(v)\}$

(ii) $U(v^*) \cap \mathcal{K}^*(v) = \{ENSC(v)\}$ iff $ENSC(v) \in U(v^*)$.

Closely related to the prekernel is the prenucleolus which is defined as follows. For any efficient payoff vector $x \in \mathbb{R}^n$ in an n-person game v, we define the associated complaint vector $\theta(x) \in \mathbb{R}^{2^n}$ as the vector whose components are the excesses $e^v(S,x)$, $S \subset N$, arranged in non-increasing order. The <u>prenucleolus</u> of the game v consists of efficient payoff vectors that minimize the complaint function $\theta(x)$ in the lexicographic order on $\mathbb{R}^{2^n}$ over the set of all efficient payoff vectors. It is well-known that the prenucleolus is a singleton and the prenucleolus is included in the prekernel (cf. Schmeidler, 1969). The unique point in the prenucleolus of a game v is denoted by $\eta^*(v)$. A first main theorem states that the prenucleolus of the game v belongs to the set $U(v^*)$ if and only if the ENSC-solution belongs to $U(v^*)$. Further, each of the two equivalent conditions is sufficient for the coincidence of the prenucleolus concept and the ENSC-method.

THEOREM 2.4.

(i) $\eta^*(v) \in U(v^*)$ iff $ENSC(v) \in U(v^*)$

(ii) If $ENSC(v) \in U(v^*)$, then $\eta^*(v) = ENSC(v)$.

PROOF. Suppose $\eta^*(v) \in U(v^*)$. We always have $\eta^*(v) \in K^*(v)$. From both $\eta^*(v) \in U(v^*) \cap K^*(v)$ and Theorem 2.3(i), we conclude that $ENSC(v) = \eta^*(v) \in U(v^*)$. This proves the "only if" part of the statement (i).
In order to prove the remaining parts, suppose $z := ENSC(v) \in U(v^*)$. We show the equality $\eta^*(v) = z$. We have that for all $i \in N$

$$e^v(N-\{i\},z) = v(N-\{i\}) - \sum_{j\in N-\{i\}} z_j = z_i - SC_i(v) = n^{-1}NSC(v).$$

Together with $z \in U(v^*)$ and Proposition 2.1, this yields

$$e^v(S,z) \le e^v(N-\{i\},z) = n^{-1}NSC(v)$$
$$\text{for all } i \in N \text{ and all } S \subset N-\{i\},\ S \ne \emptyset.$$

Put $x := \eta^*(v)$. In the lexicographic comparison between the two complaint vectors $\theta(x)$ and $\theta(z)$, we may ignore the excesses of N and $\emptyset$ since $e^v(S,x) = e^v(S,z) = 0$ whenever $S = N$ or $S = \emptyset$. Therefore, we deduce $\theta_1(z) = n^{-1}NSC(v)$, whereas $\theta_1(x) \le \theta_1(z)$ because of $x = \eta^*(v)$. Now it follows that for all $i \in N$

$$x_i - SC_i(v) = e^v(N-\{i\},x) \le \theta_1(x) \le \theta_1(z) = n^{-1}NSC(v).$$

So, $x_i \le z_i$ for all $i \in N$. Due to efficiency, the equality $x = z$ holds. □

3. The prenucleolus in comparison with the ENSC-solution on four related classes of games

This section deals with several conditions which are sufficient for the coincidence of the prenucleolus concept and the ENSC-method. The main sufficient condition mentioned in Theorem 2.4(ii) is based on the effectiveness of the (n-1)-person coalitions for the excesses at the ENSC-solution. The condition that determines whether or not the ENSC-solution belongs to the set $U(v^*)$ is chiefly of theoretical importance, but unsuited for computational purposes. For computation' sake, we characterize the effectiveness condition (1.1) in terms of the gap function corresponding to the characteristic function v.

The gap of coalition S in the game v is defined to be $g^v(S) := \sum_{j \in S} SC_j(v) - v(S)$. A nonnegative (nonpositive respectively) gap of S in the game v represents the loss (gain) to the coalition S if its worth is compared with the total amount of the separable contributions of its members. Clearly, the corresponding gap function $g^v: 2^N \to \mathbb{R}$ satisfies $g^v(\emptyset) = 0$, $g^v(N) = -NSC(v)$ and $g^v(N-\{i\}) = g^v(N)$ for all $i \in N$. The notion of the gap function is useful for a third characterization of the effectiveness of the (n-1)-person coalitions for the excesses at the ENSC-solution.

THEOREM 3.1. The following statements are equivalent.

(i) $z := ENSC(v) \in U(v^*)$

(ii) $e^v(S,z) \leq e^v(N-\{i\},z)$ for all $i \in N$ and all $S \subset N-\{i\}$, $S \neq \emptyset$

(iii) $n^{-1}g^v(N) \leq (|S|+1)^{-1}g^v(S)$ for all $S \subset N$, $S \neq N,\emptyset$. (3.1)

In the context of cooperative game theory, the ENSC-method has been considered in several recent papers, e.g., Funaki (1986), Legros (1986) and Driessen (1985; 1988, pages 94-104). All three of them describe a subclass of games on which the prenucleolus concept agrees with the ENSC-method. It appears that any of their conditions is stronger than the condition (3.1). For the sake of a uniform treatment, we summarize the three related conditions in terms of the gap function.

Funaki's condition for the corresponding gap function is specified as follows: for all $i \in N$ and all $S \subset N-\{i\}$, $S \neq \emptyset$,

$$g^v(S \cup \{i\}) - g^v(S) \leq n^{-1}g^v(N). \qquad (3.2)$$

Legros required that the gap function satisfies the following condition: for all $i \in N$ and all $S \subset N-\{i\}$,

$$g^v(S \cup \{i\}) \geq g^v(S) \quad \text{and} \quad g^v(\{i\}) \geq n^{-1}(n-1)g^v(N). \qquad (3.3)$$

Driessen's condition for the gap function is as follows:

$$0 \leq g^v(N) \leq g^v(S) \qquad \text{for all } S \subset N,\ S \neq \emptyset. \qquad (3.4)$$

Notice that the condition (3.3) includes the monotonicity condition for the gap function and hence, the gap $g^v(N)$ of the grand coalition N should be at the top level. In contradistinction to (3.3), the condition (3.4) requires that the gap of the grand coalition should be at the bottom level. In any case, each of the three conditions (3.2) - (3.4) guarantees the validity of the condition (3.1).

THEOREM 3.2. If the condition (3.2), (3.3) or (3.4) holds, then the condition (3.1) holds and $\eta^*(v) = ENSC(v) \in C(v)$.

We conclude the section with a practical example of a game which only satisfies the main sufficient condition (3.1) for the coincidence of the prenucleolus concept and the ENSC-method.

EXAMPLE 3.3. We consider the model of the cooperative water resource development as described in Suzuki and Nakayama (1976). Especially, we pay attention to their model applied to the case of Kanagawa prefecture in Japan. There are two agricultural associations (of the Rivers Sakawa and Sagami, denoted as 1 and 2) who view the existing water supplies as being more than adequate for their own irrigation needs, and three city water service authorities (of the cities Kanagawa, Yokohama and Kawasaki, denoted as 3, 4 and 5) whose current or future water needs are not met by existing sources. Each city might acquire the quantity of additional water it needs in two ways: construct a dam on the River Sakawa with or without the cooperation of other cities and/or arrange with the agricultural associations for the direct diversion of water from them to the city. The characteristic cost function c is derived from the minimum cost c(S) of meeting the additional water needs of the cities in S, on the understanding that there is no cooperation from those agents outside S.

The data concerning the cost figures is taken from Suzuki and Nakayama (1976, page 1085) and is listed in Table 1. With the cost function $c: 2^N \to \mathbb{R}$ we associate the cost savings function $v: 2^N \to \mathbb{R}$ by means of $v(S) := \sum_{j \in S} c(\{j\}) - c(S)$ for all $S \subset N$. The corresponding saving figures is also listed in Table 1. In the same table we observe that the associated savings game v satisfies neither of the three conditions (3.2) - (3.4), but nevertheless, the condition (3.1) is satisfied by the game v. Therefore, we obtain $\eta^*(v) = \text{ENSC}(v)$ and as such, the prenucleolus cost allocation is determined by

S	c(S)	v(S)	$g^v(S)$	$\frac{g^v(S)}{\lvert S\rvert+1}$
∅	0	0	0	—
1	0	0	116.4	58.2
2	0	0	116.7	58.3
3	489.7	0	122.7	61.3
4	747.6	0	243.5	121.7
5	749.8	0	307.3	153.6
12	0	0	233.1	77.7
13	440.0	49.7	189.4	63.1
14	700.5	47.1	312.8	104.3
15	694.0	55.8	367.9	122.6
23	486.4	3.3	236.1	78.7
24	546.3	201.3	158.9	53.0
25	512.2	237.6	186.4	62.1
34	1106.5	130.8	235.4	78.5
35	1108.3	131.2	298.8	99.6
45	1209.0	288.4	262.4	87.5

S	c(S)	v(S)	$g^v(S)$	$\frac{g^v(S)}{\lvert S\rvert+1}$
123	440.0	49.7	306.1	76.5
124	518.1	229.5	247.1	61.8
125	490.2	259.6	280.8	70.2
134	998.0	239.3	243.3	60.8
135	961.5	278.0	268.4	67.1
145	1069.2	428.2	239.0	59.7
234	948.2	289.1	193.8	48.4
235	950.7	288.8	257.9	64.5
245	1069.2	428.2	239.3	59.8
345	1554.3	432.8	240.7	60.2
1234	865.4	371.9	227.4	45.5
1235	803.8	435.7	227.4	45.5
1245	940.9	556.5	227.4	45.5
1345	1424.6	562.5	227.4	45.5
2345	1424.3	562.8	227.4	45.5
12345	1307.9	679.2	227.4	—

TABLE 1. Cost and saving figures in 10^8 yen for the cooperative water resource development in Japan with two agricultural associations 1, 2 and three city water service authorities 3,4,5.

$$\eta_i^*(c) = c(\{i\}) - \eta_i^*(v) = c(\{i\}) - \mathrm{ENSC}_i(v) \qquad \text{for all } i \in N.$$

In view of this, straightforward calculations yield

$$\eta^*(c) = (-70.92, -71.22, 412.48, 549.58, 487.98).$$

4. The Shapley value in comparison with the ENSC-solution on the class of k-person coalitional games

The previous two sections were devoted to the coincidence of the prenucleolus and the ENSC-solution. Henceforward, we study the relationship between the ENSC-solution and another game theoretical solution concept, the so-called Shapley value (Shapley, 1953). To be exact, we establish that the Shapley value for a specific type of games can be obtained as a linear or convex combination of the ENSC-solution and a certain reference point inside the set of all efficient payoff vectors.

Throughout this section we suppose that any n-person game v is zero-normalized, i.e., $v(\{i\}) = 0$ for all $i \in N$ where $n \geq 3$. The reference point inside the set of all efficient payoff vectors is determined by the egalitarian distribution of the total savings $v(N)$ among the n players in the game v. So, the chosen reference point coincides with the centre of the set of all nonnegative efficient payoff vectors. The central payoff vector $(n^{-1}v(N),\dots,n^{-1}v(N)) \in \mathbb{R}^n$ is shortly written as $n^{-1}v(N)1_n$.

The Shapley value of an n-person game v is usually seen as an efficient payoff vector which meets a certain expectation principle. That is, the expected payoff to any player $i \in N$ is derived from his marginal payments $\Delta^v(S,\{i\})$ for joining coalitions S not containing player i, and the probability distribution $\{n^{-1}\binom{n-1}{|S|}^{-1} \mid S \subset N-\{i\}\}$ over the collection of all such coalitions. Formally, the <u>Shapley value</u> $\phi(v) \in \mathbb{R}^n$ is given by

$$\phi_i(v) := \sum_{S \subset N-\{i\}} n^{-1}\binom{n-1}{|S|}^{-1} \Delta^v(S,\{i\}) \qquad \text{for all } i \in N.$$

The main purpose is to investigate the locus of the Shapley value on the class of k-person coalitional games. By means of the integer k satisfying $1 \le k \le n-1$, we divide the coalitions in an n-person game into three kinds:

- essential coalitions which contain precisely k players
- small coalitions which consist of less than k players
- large coalitions which consist of more than k players and different from the grand coalition N.

For k-person coalitional games, the worth of any large coalition is based on the profits of the essential coalitions that can be formed within the large coalition. In fact, each large coalition is supposed to achieve the sum of the profits of the essential subcoalitions. Here we denote by $\alpha_T \in \mathbb{R}$ the maximal profit obtainable from the formation of the essential coalition T within any large coalition. Moreover, it is supposed that the worth of any small coalition depends on the coalition size instead of the players in the coalition. In other words, the game is required to be symmetric with respect to the coalition size up to k. Thus, an n-person game v is called a <u>k-person coalitional game</u> if the game v satisfies the next two conditions:

(i) symmetry up to k, i.e.,

$$v(S) = v(T) \qquad \text{for all } S,T \subset N \text{ with } |S| = |T| < k.$$

(ii) there exist $\alpha_T \in \mathbb{R}$ for all $T \subset N$, $|T| = k$, such that

$$v(S) = \sum_{\substack{T\subset S,\\ |T|=k}} \alpha_T \qquad \text{for all } S \subset N \text{ with } k < |S| < n. \tag{4.1}$$

First of all, we describe the locus of the Shapley value for k-person coalitional games in the separate cases $k = n-1$ and $k = 1$. In case $k = n-1$, then the condition (4.1) is superfluous. So, an n-person game is an (n-1)-person coalitional game if and only if the game is symmetric with respect to the coalition size up to n-1. According to the next theorem, the Shapley value of an (n-1)-person coalitional game v is the convex combination of the ENSC-solution and the central payoff vector $n^{-1}v(N)1_n$ with coefficients $(n-1)^{-1}$ and $(n-1)^{-1}(n-2)$. From this we deduce that the Shapley value of an (n-1)-person coalitional game lies near the central payoff vector and far off the ENSC-solution.

THEOREM 4.1. Let the n-person game v be an (n-1)-person coalitional game, i.e., $v(S) = v(T)$ for all $S,T \subset N$ with $|S| = |T| < n-1$.
Then $\phi(v) = (n-1)^{-1}\text{ENSC}(v) + (n-1)^{-1}(n-2)n^{-1}v(N)1_n$.

The condition (4.1) applied to k = 1 reduces to the requirement

$$v(S) = \sum_{j\in S} \alpha_j \qquad \text{for all } S \subset N \text{ with } 1 < |S| < n.$$

This type of a game is also known as a quota game, in which any player $i \in N$ has a quotum $\alpha_i \in \mathbb{R}$ potentially and the players can achieve their quota solely by the formation of multiperson coalitions. It appears that the Shapley value of a 1-person coalitional game v is the convex combination of the ENSC-solution and the central payoff vector $n^{-1}v(N)1_n$ with coefficients $(n-1)^{-1}(n-2)$ and $(n-1)^{-1}$. In contradistinction to the locus of the Shapley value for (n-1)-person coalitional games, the Shapley value of a 1-person coalitional game lies near the ENSC-solution and far off the central payoff vector.

THEOREM 4.2. Let the n-person game v be a 1-person coalitional game (or equivalently, a quota game).
Then $\phi(v) = (n-1)^{-1}(n-2)\text{ENSC}(v) + (n-1)^{-1}n^{-1}v(N)1_n$.

Finally, we treat the main result concerning the Shapley value for k-person coalitional games where $2 \le k \le n-2$. In order to formulate the last theorem, we first fix some notation. The set of all essential coalitions is denoted by Γ_k, i.e.,

$$\Gamma_k := \{T \mid T \subset N,\ |T| = k\}.$$

Consider any n-person game v and any real numbers α_T, $T \in \Gamma_k$, such that the condition (4.1) holds. For any player $i \in N$, we define the set Γ_k^i of coalitions and the real numbers v^i, α^i by

$$\Gamma_k^i := \{T \mid T \in \Gamma_k,\ i \in T\},$$

$$v^i := \sum_{T\in\Gamma_k^i} v(T),$$

$$\alpha^i := \sum_{T\in\Gamma_k^i} \alpha_T .$$

Here the real number v^i (α^i respectively) represents the total amount that player i can gain by acting as a member of essential coalitions with respect to the formation of the essential coalitions within the game v itself (within larger coalitions). The real numbers $\bar{v}$ and $\bar{\alpha}$ are determined by averaging, i.e.,

$$\bar{v} := n^{-1} \sum_{j \in N} v^j \qquad \text{and} \qquad \bar{\alpha} := n^{-1} \sum_{j \in N} \alpha^j .$$

Due to the last main theorem, the locus of the Shapley value for k-person coalitional games v where $2 \le k \le n-2$ can be described as a linear combination of the ENSC-solution and the central payoff vector $n^{-1}v(N)1_n$, on the understanding that the quotient of the two deviations $\bar{v} - v^i$ and $\bar{\alpha} - \alpha^i$ is the same for all players $i \in N$.

THEOREM 4.3. Let the n-person game v be a k-person coalitional game such that (4.1) holds and $2 \le k \le n-2$.
If $(\bar{\alpha} - \alpha^i)^{-1}(\bar{v} - v^i) = c$ for all $i \in N$ and a certain constant $c \in \mathbb{R}$, then

$$\phi(v) = \beta \, \mathrm{ENSC}(v) + (1-\beta)n^{-1}v(N)1_n$$

where

$$\beta := \left[k\binom{n-1}{k}\right]^{-1}\left[\binom{n-1}{k} - 1 + c\right].$$

It may happen that the maximal profit α_T, obtainable from the formation of the essential coalition T within any large coalition, is equal to the original worth v(T) in the game v, i.e., $\alpha_T = v(T)$ for all $T \subset N$ with $|T| = k$. Under these circumstances, it follows immediately that $v^i = \alpha^i$ for all $i \in N$, $\bar{v} = \bar{\alpha}$ and hence, Theorem 4.3 reduces to the next corollary.

COROLLARY 4.4. Let $2 \le k \le n-2$ and the n-person game v be such that

$$v(S) = v(T) \qquad \text{for all } S,T \subset N \text{ with } |S| = |T| < k,$$
$$v(S) = \sum_{T \subset S,\ |T|=k} v(T) \qquad \text{for all } S \subset N \text{ with } k < |S| < n.$$

Then $\phi(v) = k^{-1}\mathrm{ENSC}(v) + k^{-1}(k-1)n^{-1}v(N)1_n$.

We conclude the paper with an example of a 2-person coalitional game. We compare the ENSC-solution of this game with both the Shapley value and the prenucleolus payoff vector.

EXAMPLE 4.5. We consider the 4-person glove game v whose trader set $N = \{1,2,3,4\}$ consists of two owners 1, 2 of one right-handed glove, one owner 3 of two left-handed gloves and one owner 4 of one left-handed glove. The worth v(S) of coalition S in the glove game v is given by the largest possible number of the assembled pairs of gloves within S. Then we get

$$v(N) = v(\{1,2,3\}) = 2, \qquad v(\{1,2,4\}) = v(\{1,3,4\}) = v(\{2,3,4\}) = 1,$$
$$v(\{1,3\}) = v(\{1,4\}) = v(\{2,3\}) = v(\{2,4\}) = 1,$$
$$v(S) = 0 \qquad \text{for all other } S \subset N.$$

This 4-person glove game is a 2-person coalitional game since the validity of the condition (4.1) can be obtained by choosing the profits $\alpha_{12} = \alpha_{13} = \alpha_{14} = 0.5$, $\alpha_{23} = 1$, $\alpha_{24} = \alpha_{34} = 0$. Notice that the numerical profits of the essential coalitions are not uniquely determined. Straightforward calculations yield that $v^i = 2$ for all $i \in N$ and so, $\bar{v} = 2$. Because $\bar{v} - v^i = 0$ for all $i \in N$, we conclude that Theorem 4.3 applies to the 2-person coalitional game v and as a result, the Shapley value is the convex combination of the ENSC-solution and the central payoff vector $\frac{1}{2}(1,1,1,1)$ with coefficients $\frac{1}{3}$ and $\frac{2}{3}$. In order to calculate the ENSC-solution, we remark that the separable contributions and the nonseparable contribution are given by $SC_i(v) = 1,1,1,0$ for $i = 1,2,3,4$ respectively and $NSC(v) = -1$. Thus, $ENSC(v) = \frac{1}{4}(3,3,3,-1)$ and hence, $\phi(v) = \frac{1}{12}(3,3,3,-1) + \frac{1}{3}(1,1,1,1) = \frac{1}{12}(7,7,7,3)$. It is left to the reader to verify the Shapley value payoff vector by using the classical formula for the Shapley value. Finally, we deduce from the strict inequality $\frac{1}{4}g^v(N) = \frac{1}{4} > 0 = \frac{1}{2}g^v(\{4\})$ that the game v does not satisfy the main sufficient condition (3.1) for the coincidence of the prenucleolus and the ENSC-solution. It appears that the prenucleolus is equal to the unique core-element (1,1,0,0) and so, the prenucleolus differs from the ENSC-solution.

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