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Str. of the Moduli Sp. of SL-eg's on a Riem. Surface and Monod. Preserving Deformation.

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§1. SL-eg's on a Riem. surface.

$M$ : cpt Riem. surface of genus  $g \geq 0$ .

$\xi \in H^1(M, \mathcal{O}^*)$ : holo. line b'dle over  $M$ .

$\mathcal{U} = \{(U_j, x_j)\}$ : a coord. covering of  $M$ .

$(\xi_{jk}) \in Z^1(\mathcal{U}, \mathcal{O}^*)$ : a repr. 1-cocycle for  $\xi$ .

Def. (1) a GL-eg. on  $\mathcal{U}$  for  $\xi$  is a collection  $\star = \{\star_j\}$  of loc. def. diff. eg's

$$\star_j \quad \frac{d^2 f_j}{dx_j^2} + P_j \frac{df_j}{dx_j} + Q_j f_j = 0 \quad \text{in } U_j \quad (\text{loc. expression})$$

$$\text{s.t.} \quad f_j = \xi_{jk} f_k \quad \text{in } U_j \cap U_k$$

$$\Rightarrow \quad \star_j = \star_k \quad \text{as diff. eg's in } U_j \cap U_k.$$

$$(2) \quad \left\{ \underline{\text{GL-eg's on } M \text{ for } \xi} \right\} := \varinjlim_{\mathcal{U}} \left\{ \text{GL-eg's on } \mathcal{U} \text{ for } \xi \right\}$$

Def. an SL-eg. on  $M$  for  $\xi$  is a GL-eg. on  $M$  for  $\xi$  s.t.

$\exists$  repr. GL-eg.  $\star$  on  $\mathcal{U}$  s.t.

$$P_j \equiv 0 \quad \text{in } U_j \quad (\forall j).$$

Lemma  $\xi \in H^1(M, \mathcal{O}^*)$

$$\exists \text{ SL-eg's for } \xi \iff c_1(\xi) = 1 - g.$$

From now on, fix  $\xi \in H^1(M, \mathcal{O}^*)$  s.t.  $c_1(\xi) = 1 - g$ . 55

$$\theta_{jk} := \{x_j; x_k\} \text{ in } U_j \cap U_k$$

$$(\theta_{jk}) \in Z^1(\mathcal{U}, \mathcal{O}(K^2)).$$

$$\mathcal{Q} := \{ Q = (Q_j) \in C^0(\mathcal{U}, \mathcal{M}(K^2)) ; \delta(Q_j) = (\theta_{jk}) \}$$

Lemma

$$\begin{array}{ccc} \{ \text{SL-eg's (or op's)} \} & \xrightarrow{\quad \quad} & \mathcal{Q} \\ \downarrow \psi & & \downarrow \psi \\ L = (L_j) & \longleftrightarrow & Q = (Q_j) \end{array}$$

where

$$L_j = -\frac{d^2}{dx_j^2} + Q_j \text{ in } U_j.$$

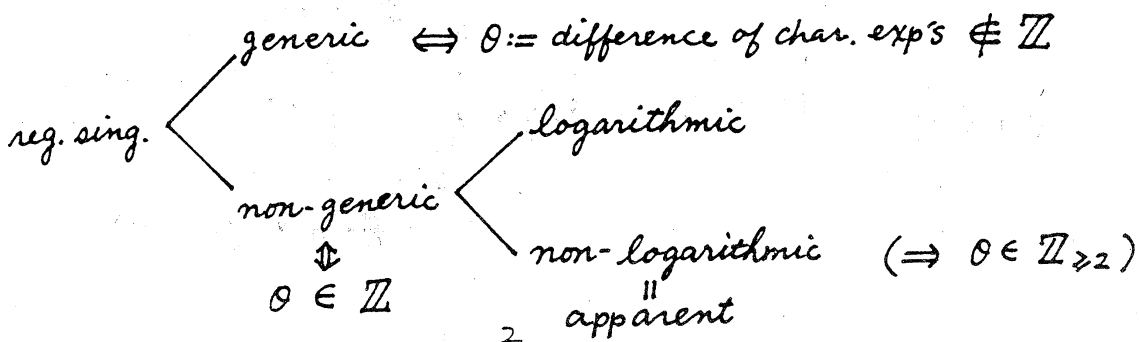
Rem an SL-op.  $L$  is considered as a diff. op.  $L: \mathcal{M}(\xi) \rightarrow \mathcal{M}(\xi \otimes K^2)$ .

Rem The sp.  $\mathcal{Q}$  parametrizing SL-op's is an  $H^0(M, \mathcal{M}(K^2))$ -affine sp. However, if a nice (= proj.) coord. cover.  $\mathcal{U} = \{(U_j, x_j)\}$  is chosen, then  $\mathcal{Q}$  is viewed as a lin. sp. i.e.

Fix a proj. str. subord. to the complex str. of  $M \Rightarrow \theta_{jk} \equiv 0$ .

$$\begin{array}{ccc} \{ \text{SL-op's} \} & \xrightarrow{\quad \quad} & H^0(M, \mathcal{M}(K^2)) \\ \downarrow \psi & & \downarrow \psi \\ L = \left(-\frac{d^2}{dx_j^2} + Q_j\right) & \longleftrightarrow & Q = (Q_j) \end{array}$$

Rem Loc. properties of a GL-eg. (eg. reg. sing. pt. char. exp's. e.t.c.) are def. through its loc. expr.'s.



Def.  $L$ : a GL-*eq.*  $p \in M$ : an app. sing. pt. of  $L$

$$\textcircled{1} \quad p: \text{multiplicity } N \in \mathbb{N} \stackrel{\text{def}}{\iff} \theta = N+1 \quad (= \text{differ. of char. exp's})$$

$$p: \text{ground state} \iff \text{mult.} = 1.$$

$$\textcircled{2} \quad p_1, \dots, p_k \in M: \text{mutually differ. app. sing. pts of } L$$

$$N_1, \dots, N_k \in \mathbb{N}: \text{mult's}$$

$$\Rightarrow n := \sum_{i=1}^k N_i : \text{the \# of app. sing. pts of } L \text{ counted with multiplicities}$$

$$L: \text{ground state} \stackrel{\text{def}}{\iff} p_j: \text{g.s. } (\forall j).$$

## §2. Analytic sp's of SL-*eqs* on a Riem. surface (Part 1).

FSL-*eq.* = Fuchsian SL-*eq.*

$m$  = # of reg. sing. pts

$k$  = # of app. sing. pts

$n$  = # of app. sing. pts c.w. mult's.

$$\star E(\ell) \stackrel{\text{def}}{:=} \left\{ \begin{array}{l} \text{FSL-*eqs* on } M \text{ with exactly } \ell \\ \text{ordered reg. sing. pts.} \end{array} \right\}$$

Study analytic space str's of  $E(m+k)$  and its subsp's.

$$C(\ell) \stackrel{\text{def}}{:=} \{ (p_1, \dots, p_\ell) \in M^\ell ; \exists i, j \text{ st. } i \neq j, p_i = p_j \}$$

$$B(\ell) := M^\ell \setminus C(\ell).$$

$$(D.G.1) \quad M^m \times M^k \supset B(m+k) \begin{array}{l} \xrightarrow{\pi} E(m+k) \\ \xrightarrow{p} B(m) \subset M^m \end{array} \quad \begin{array}{l} \downarrow \omega \\ B(m) \subset M^m \end{array}$$

$$\pi: \begin{array}{ccc} E(m+k) & \longrightarrow & B(m+k) \\ \downarrow & & \downarrow \\ \mathcal{Q} & \longrightarrow & \text{the ordered reg. sing. pts of } \mathcal{Q} \end{array} \quad 57$$

$$p: \begin{array}{ccc} B(m+k) & \longrightarrow & B(m) \quad : \text{ surj.} \\ \downarrow & & \downarrow \\ (P_1 \dots P_m, \beta_1 \dots \beta_k) & \longrightarrow & (P_1, \dots, P_m) \quad \text{proj. into first } m \text{ factors} \end{array}$$

### Complex str. of $E(\ell)$

Given  $P = (P_1, \dots, P_\ell) \in B(\ell)$ , let

$$F(\ell)_P := \Gamma(M, \mathcal{O}(k^2 \otimes [2P_1 + \dots + 2P_\ell]))$$

$$\dim F(\ell)_P = (2\ell + 3g - 3)^+ : \text{ indep. of } P \in B(\ell) \quad (:: \text{ Riem. Rock})$$

where

$$a^+ = \begin{cases} a & (a > 0) \\ 1 & (a = 0) \\ 0 & (a < 0) \end{cases}$$

$$F(\ell) \stackrel{\text{def}}{=} \bigcup_{P \in B(\ell)} F(\ell)_P$$

Lemma  $F(\ell)$  is naturally a complex vect. b'dle over  $B(\ell)$  of rank  $(2\ell + 3g - 3)^+$   $(:: \text{ Kodaira - Spencer})$

$$\pi_{\ell,j} : B(\ell) \rightarrow B(\ell-1), (P_1, \dots, P_\ell) \mapsto (P_1, \dots, \hat{P}_j, \dots, P_\ell)$$

Prop.  $E(\ell) = F(\ell) \setminus \bigcup_{j=1}^{\ell} \pi_{\ell,j}^* F(\ell-1) : \text{ a complex mfd}$

$$\dim E(\ell) = (2\ell + 3g - 3)^+ + \ell.$$

$$\star E(m,k) \stackrel{\text{def}}{=} \left\{ Q \in E(m+k); \begin{array}{l} \text{the last } k \text{ sing. pts of } Q \\ \text{are app. sing. pts} \end{array} \right\}$$

Given  $N = (N_1, \dots, N_k) \in \mathbb{N}^k$ , let

$$\star E(m,k;N) \stackrel{\text{def}}{=} \left\{ Q \in E(m+k); \begin{array}{l} \text{the } (m+j)^{\text{th}} \text{ sing. pt. of } Q \\ \text{is an app. sing. pt of} \\ \text{mult. } N_j \quad (1 \leq j \leq k) \end{array} \right\}$$

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Rem  $E(m, k) = \bigsqcup_{N \in \mathbb{N}^k} E(m, k; N)$

Given  $P \in B(m)$ , let

☆  $E(P, k; N) \stackrel{\text{def}}{=} \{ Q \in E(m, k; N) ; \omega(Q) = P \}$

Given  $\theta = (\theta_1, \dots, \theta_m) \in \mathbb{C}^m$ , let

☆  $E(m, k; \theta, N) \stackrel{\text{def}}{=} \left\{ Q \in E(m, k; N) ; \begin{array}{l} \text{the differ. of char. exp's at} \\ \text{the } j\text{th sing. pt is } \theta_j \ (1 \leq j \leq m) \end{array} \right\}$

☆  $E(P, k; \theta, N) := E(P, k; N) \cap E(m, k; \theta, N)$

Theorem 1 Let  $m \geq \max(2-g, 0)$ ,  $k \geq 1$ ,  $N \in \mathbb{N}^k$ . Then

(1)  $E(m, k; N)$  : pure dim. analytic subsp. of  $E(m+k)$

$$\dim E(m, k; N) = 3(m+g-1) + k$$

(2)  $\pi : E(m, k; N) \rightarrow B(m+k)$  : surj.

Cor 1  $E(P, k; N)$  : pure dim. analytic sp.

$$\dim E(P, k; N) = 2m + k + 3g - 3$$

Cor 2  $E(m, k; \theta, N)$  : pure dim. analytic sp.

$$\dim E(m, k; \theta, N) = 2m + k + 3g - 3$$

$\pi : E(m, k; \theta, N) \rightarrow B(m+k)$  : surj.

Cor 3  $E(P, k; \theta, N)$  : pure dim. analytic sp.

$$\dim E(P, k; \theta, N) = m + k + 3g - 3.$$

§3. Analytic sp's of SL-eg's on a Riem. surface (Part 2).

Given  $\Pi = (P_1, \dots, P_m, g_1, \dots, g_k) \in B(m+k)$ ,  $N = (N_1, \dots, N_k) \in \mathbb{N}^k$ , let

$$\eta(\mathbb{R}; N) := K^{-1} \otimes [N_1 \xi_1 + \dots + N_k \xi_k - (P_1 + \dots + P_m)] \in H^1(M, \mathcal{O}^*)$$

Rem  $c_1(\eta(\mathbb{R}; N)) = |N| - m + 2 - 2g$ .

$$D(m, k; N) := \{ \mathbb{R} \in B(m+k) ; h^0(M, \mathcal{O}(\eta(\mathbb{R}; N))) \geq 1 \}$$

$$B(m, k; N) := B(m+k) \setminus D(m, k; N).$$

Rem  $|N| \leq m + 2g - 3 \Rightarrow D(m, k; N) = \emptyset$ .

Lemma  $|N| \leq m + 3g - 3$  ( $\because$  Grauert)

$\Rightarrow D(m, k; N)$  : a proper analytic subset of  $B(m+k)$ , i.e.

$B(m, k; N)$  : a nonempty analytic  $\mathbb{Z}$ -open subset of  $B(m+k)$ .

Lemma  $p: B(m, k; N) \rightarrow B(m)$  : surj.

$$\star \mathbb{E}(m, k; N) \stackrel{\text{def}}{=} \{ Q \in E(m, k; N) ; \pi(Q) \in B(m, k; N) \}$$

$$\star \mathbb{E}(P, k; N) \stackrel{\text{def}}{=} \{ Q \in E(P, k; N) ; \pi(Q) \in B(m, k; N) \}$$

$$\star \mathbb{E}(m, k; \theta, N) \stackrel{\text{def}}{=} \{ Q \in E(m, k; \theta, N) ; \pi(Q) \in B(m, k; N) \}$$

$$\star \mathbb{E}(P, k; \theta, N) \stackrel{\text{def}}{=} \{ Q \in E(P, k; \theta, N) ; \pi(Q) \in B(m, k; N) \}$$

$$\begin{array}{ccc}
 & \mathbb{E}(m, k; N) & \\
 \pi \swarrow & & \downarrow \omega \\
 M^m \times M^k \supset B(m, k; N) & & B(m) \subset M^m \\
 p \searrow & & \\
 & & 
 \end{array} \quad (\text{D.G. 2})$$

Theorem 1 If  $E(\dots)$  and  $B(m+k)$  are replaced by  $\mathbb{E}(\dots)$  and  $B(m, k; N)$ , resp., then Th & Cor's in § 2 are still O.K.

#### § 4. Reducible and irreducible SL-eg's.

$$\tau_{jk} := \frac{1}{2} \frac{d}{dz_j} \log K_{jk} \quad \text{in } U_j \cap U_k$$

$$(\tau_{jk}) \in Z^1(\mathcal{U}, \mathcal{O}(K))$$

$$\mathcal{P} := \{ P = (P_j) \in C^0(\mathcal{U}, \mathcal{M}(K)) ; \delta(P_j) = (\tau_{jk}) \}$$

$$\star V(\ell) \stackrel{\text{def}}{=} \{ P \in \mathcal{P} ; P \text{ has exactly } \ell \text{ ordered simple poles} \}$$

$$A(\ell) := \{ \alpha = (\alpha_1, \dots, \alpha_\ell) \in \mathbb{C}^\ell ; \alpha_1 + \dots + \alpha_\ell = g-1, \alpha_1, \dots, \alpha_\ell \neq 0 \}$$

$$\pi : V(\ell) \rightarrow A(\ell) \times B(\ell), \quad P \mapsto (\alpha, P)$$

$P = (P_1, \dots, P_\ell) : m\text{-tuple of ordered simple poles of } P$

$\alpha = (\alpha_1, \dots, \alpha_\ell) : \alpha_j = \text{Res } P \text{ at } P_j.$

$$\Phi : V(\ell) \rightarrow E(\ell), \quad P = (P_j) \mapsto Q = \left( \frac{dP_j}{dx_j} + P_j^2 \right)$$

$$\begin{array}{ccc} V(\ell) & \xrightarrow{\Phi} & E(\ell) \\ \pi \downarrow & & \downarrow \pi \\ A(\ell) \times B(\ell) & \searrow & B(\ell) \end{array} \quad (\text{D.G.3})$$

Lemma  $V(\ell)$  has a natural complex mfd str. s.t.

$\pi : V(\ell) \rightarrow A(\ell) \times B(\ell)$  is a holo. affine b'dle of rk  $g$ .

$$\dim V(\ell) = 2\ell + g - 1. \quad (\because K-S)$$

Rem  $\Phi$  is a holo and closed map.

Lemma  $\#$  (each fiber of  $\Phi$ )  $\leq 2^\ell$ .

$\Phi$  is a finite holo. map. (we can apply FMT)

Def  $L = (L_j) \leftrightarrow Q \in E(\ell)$  is reducible

$$\Leftrightarrow L_j = \text{a product of first order op's } (\forall j).$$

$$\star E(\ell)_{\text{red}} := \{ Q \in E(\ell) ; Q \text{ is reducible} \}$$

$$E(\ell)_{\text{irr}} := \{ Q \in E(\ell) ; Q \text{ is irreducible} \}$$

Given  $X \subset E(l)$ , let

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$$X_{\text{red}} := X \cap E(l)_{\text{red}}, \quad X_{\text{irr}} := X \cap E(l)_{\text{irr}}$$

Prop. Let  $l \geq \max(1, 2-g)$ , then

$$E(l)_{\text{red}} = \Phi(V(l)) : \text{an analytic subsp. of } E(l)$$

$$\text{codim } E(l)_{\text{red}} = l + 2g - 2$$

Theorem 1 Let  $m \geq \max(1, 2-g)$ ,  $k \geq 0$ ,  $N \in \mathbb{N}^k$ . Then

$$E(m, k; N)_{\text{red}} : \text{an analytic subsp. of } E(m, k; N)$$

$$\text{codim } E(m, k; N)_{\text{red}} = m + 2g - 2$$

Cor 1  $m, k, N$ : as above.  $P \in B(m)$ .

$$E(P, k; N)_{\text{red}} : \text{an analytic subsp. of } E(P, k; N)$$

$$\text{codim } E(P, k; N)_{\text{red}} = m + 2g - 2$$

Def  $\theta = (\theta_1, \dots, \theta_m) \in \mathbb{C}^m$ ,  $N = (N_1, \dots, N_k) \in \mathbb{N}^k$ : given.

$$(\theta, N) : \text{generic} \stackrel{\text{def}}{\iff} \forall \delta_i, \epsilon_j \in \{\pm 1\} \quad (1 \leq i \leq m, 1 \leq j \leq k)$$

$$\sum_{i=1}^m \delta_i \theta_i + \sum_{j=1}^k \epsilon_j (N_j + 1) \neq 2g - 2 - (m + k)$$

Cor 2  $m, k, N$ : as in Th.

$$(\theta, N) : \text{generic} \implies E(m, k; \theta, N)_{\text{red}} = \emptyset$$

Cor 3  $(\theta, N) : \text{non-generic}$

$$E(m, k; \theta, N)_{\text{red}} : \text{an analytic subsp. of } E(m, k; \theta, N)$$

$$\text{codim } E(m, k; \theta, N)_{\text{red}} \geq m + 2g - 3$$

Cor 4  $(\theta, N) : \text{non-generic}$ ,  $P \in B(m)$ .

$$E(P, k; \theta, N)_{\text{red}} : \text{an analytic subsp. of } E(P, k; \theta, N)$$

$$\text{codim } E(P, k; \theta, N)_{\text{red}} \geq m + 2g - 3$$

Rem  $E(\dots)$  can be replaced by  $[E(\dots)]$  in the above Th & Cor's.



Rem If  $m \geq \max(1, 3-g)$ , then all analytic subsp's app. in the above Th & Cor's have + codim's.

Theorem 2 If  $E(\dots)$  and  $\mathbb{E}(\dots)$  are replaced by  $E(\dots)_{irr}$  and  $\mathbb{E}(\dots)_{irr}$ , resp., then Th's 1 in § 2 & § 3 are still O.K.

$$\Lambda(n) := \{(k, N) ; 0 \leq k \leq n, N = (N_1, \dots, N_k) \in \mathbb{N}^k, |N| \leq n\}$$

$$\star \mathbb{E}(m|n)_{irr} = \bigsqcup_{(k, N) \in \Lambda(n)} \mathbb{E}(m, k; N)_{irr}$$

$$\star \mathbb{E}(P|n)_{irr} = \bigsqcup_{(k, N) \in \Lambda(n)} \mathbb{E}(P, k; N)_{irr}$$

$$\star \mathbb{E}(m|n; \emptyset)_{irr} = \bigsqcup_{(k, N) \in \Lambda(n)} \mathbb{E}(m, k; \emptyset, N)_{irr}$$

$$\star \mathbb{E}(P|n; \emptyset)_{irr} = \bigsqcup_{(k, N) \in \Lambda(n)} \mathbb{E}(P, k; \emptyset, N)_{irr}$$

Rem  $\Lambda(n) \ni (n, \mathbb{1}_n)$  corresp. to g.s. eq's, where  $\mathbb{1}_n = \overbrace{(1, \dots, 1)}^n$ .

§ 5. Complex mfd's of proj. representations & proj. monod. map.

$$G = PL(2, \mathbb{C}) = \text{Aut}(\mathbb{P}^1), \quad \mathfrak{g} = \text{Lie } G = \mathfrak{sl}(2, \mathbb{C}).$$

Given  $P = (P_1, \dots, P_m) \in B(m)$ , let  $|P| = \{P_1, \dots, P_m\}$  and

$[P]$ : the real blow-up of  $M$  at  $P_1, \dots, P_m$ . (see Fig. 1)

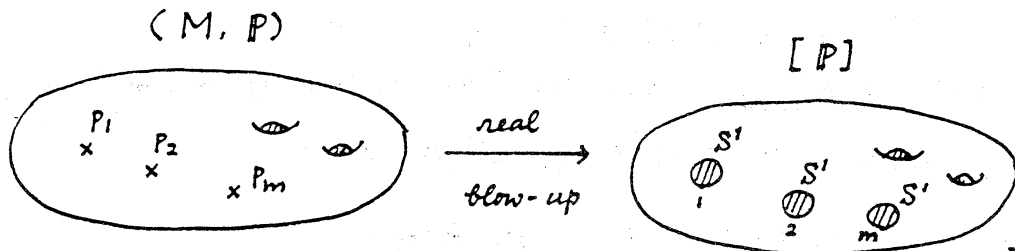


Fig. 1.

Rem  $\partial[P] = S^1 \sqcup \dots \sqcup S^1$  ( $m$  times)

$\hat{R}(P) := \text{Hom}(\pi_1(M \setminus |P|), G)$ , equipped with C-O topology

$\hat{p} \in \hat{R}(P)$ : irreducible  $\stackrel{\text{def}}{\iff} \hat{p}(\pi_1) \subset \text{Aut}(\mathbb{P}^1)$  has no fixed pt in  $\mathbb{P}^1$ .

$$\hat{R}(\mathbb{P})_{\text{irr}} := \{ \hat{p} \in \hat{R}(\mathbb{P}) ; \text{irr.} \}$$

Rem  $\hat{R}(\mathbb{P})$  : noncan. by a complex submfd of the Lie gr.  $G^{m+2g}$   
: admits an action of  $\text{Ad}(G)$

$\hat{R}(\mathbb{P})_{\text{irr}}$  : inv. under the act. of  $\text{Ad}(G)$   
nonempty analytic  $\mathbb{Z}$ -open subset of  $\hat{R}(\mathbb{P})$ .

Act. of  $\text{Ad}(G)$  on  $\hat{R}(\mathbb{P})_{\text{irr}}$  is free. ( $\because$  Schur's lemma)

$$\star R(\mathbb{P}) := \hat{R}(\mathbb{P})/\text{Ad}(G), \quad R(\mathbb{P})_{\text{irr}} := \hat{R}(\mathbb{P})_{\text{irr}}/\text{Ad}(G).$$

Rem  $R(\mathbb{P}) \xrightarrow{\quad} H^1([\mathbb{P}], G)$   
 $\downarrow \quad \quad \quad \downarrow$   
 $\mathcal{P} \longleftrightarrow L(\mathcal{P})$  : loc. system whose char. repr. is  $\mathcal{P}$

Prop.  $R(\mathbb{P})_{\text{irr}}$  carries a natural complex mfd str. s.t.

$\hat{R}(\mathbb{P})_{\text{irr}} \xrightarrow{\text{Ad}(G)} R(\mathbb{P})_{\text{irr}}$  is a holo. principal  $G$ -bdle.

$$\dim R(\mathbb{P})_{\text{irr}} = 3m + 6g - 6.$$

$$T_{\mathcal{P}} R(\mathbb{P})_{\text{irr}} \cong H^1([\mathbb{P}]; \text{Ad}L(\mathcal{P})), \quad \mathcal{P} \in R(\mathbb{P})_{\text{irr}}$$

Rem  $R(\mathbb{P})$  is not a complex mfd.

Given  $\theta = (\theta_1, \dots, \theta_m) \in (\mathbb{C} \setminus \mathbb{Z})^m$ , let

$$\hat{R}(\mathbb{P}; \theta)_{\text{irr}} = \left\{ \hat{p} \in \hat{R}(\mathbb{P})_{\text{irr}} ; \begin{array}{l} \text{loc. repr. around } \mathcal{P}_j \text{ induced by } \mathcal{P} \\ \text{has eigen-}v\text{'s } \exp(\pm \pi \sqrt{-1} \theta_j) \end{array} \right\}$$

$$\star R(\mathbb{P}; \theta)_{\text{irr}} := \hat{R}(\mathbb{P}; \theta)_{\text{irr}} / \text{Ad}(G).$$

Prop.  $R(\mathbb{P}; \theta)_{\text{irr}}$  is a complex submfd of  $R(\mathbb{P})_{\text{irr}}$  s.t.

$$\dim R(\mathbb{P}; \theta)_{\text{irr}} = 2(m + 3g - 3).$$

$$T_{\mathcal{P}} R(\mathbb{P}; \theta)_{\text{irr}} \cong \text{Ker} \left[ \begin{array}{c} H^1([\mathbb{P}]; \text{Ad}L(\mathcal{P})) \\ \xrightarrow{j^*} H^1(\partial[\mathbb{P}]; \text{Ad}L(\mathcal{P})|_{\partial[\mathbb{P}]}) \end{array} \right]$$

for  $\mathcal{P} \in R(\mathbb{P}; \theta)_{\text{irr}}$ .

(see the follow. Rem.)

Rem Cohom exact seq.

$$\begin{array}{ccc}
 H^0([P]; \text{Ad}L(\rho)) & \xrightarrow{j^*} & H^0(\partial[P]; \text{Ad}L(\rho)|_{\partial[P]}) \\
 \parallel & & \parallel \\
 0 & \left( \begin{array}{l} \text{irr. of } \rho \\ \text{Schur} \end{array} \right) & \mathbb{C} \oplus \dots \oplus \mathbb{C} \text{ (m times)} \\
 \xrightarrow{\delta^*} & H^1([P], \partial[P]; \text{Ad}L(\rho)) & \xrightarrow{i^*} H^1([P]; \text{Ad}L(\rho)) \\
 \xrightarrow{j^*} & H^1(\partial[P]; \text{Ad}L(\rho)|_{\partial[P]}) & 
 \end{array}$$

$$T_{\rho} R(P; \theta)_{\text{irr}} \cong \frac{H^1([P], \partial[P]; \text{Ad}L(\rho))}{H^0(\partial[P]; \text{Ad}L(\rho)|_{\partial[P]})}$$

Def the proj. monod. map PM. Given  $P \in B(m)$ , let

$$\begin{array}{ccc}
 \text{PM}: \mathbb{E}(P|n)_{\text{irr}} & \longrightarrow & R(P)_{\text{irr}} \\
 \downarrow & & \downarrow \\
 Q & \longrightarrow & \text{the proj. monod. repr. class for } Q
 \end{array}$$

$$\text{PM}: \mathbb{E}(P|n; \theta)_{\text{irr}} \longrightarrow R(P; \theta)_{\text{irr}}$$

Rem PM is a holo map.

Theorem 1  $m, k, N \in \mathbb{N}^k, \theta \in (\mathbb{C} \setminus \mathbb{Z})^m$ : given.  $|N| \leq m+3g-3$

$\Rightarrow \forall \eta \in B(P, k; N), \exists U$ : open nbd of  $\eta$  in  $B(P, k; N)$  s.t.

PM is 1 to 1 on  $\mathbb{E}(P, k; \theta, N)_{\text{irr}}|_{\pi U}$ . (see D.G. 4)

$$\begin{array}{ccc}
 \mathbb{E}(P, k; \theta, N)_{\text{irr}}|_{\pi U} \subset \mathbb{E}(P, k; \theta, N)_{\text{irr}} & & \\
 \pi \downarrow & \swarrow \pi & \downarrow \omega \\
 \eta \in U \subset B(P, k; N) & & P \in B(m)
 \end{array} \quad (\text{D.G. 4})$$

Theorem 1' Let  $P \in B(m)$ .  $n = m+3g-3 \Rightarrow$

$$\text{PM}: \mathbb{E}(P|n)_{\text{irr}} \rightarrow R(P)_{\text{irr}}, \text{ PM}: \mathbb{E}(P|n; \theta)_{\text{irr}} \rightarrow R(P; \theta)_{\text{irr}}$$

are loc. injective holo. map's.

This Th1' has a lot of implications.

Rem Let  $P \in B(m)$ ,  $n = m + 3g - 3$ . Then, for  $(k, N) \in \Lambda(n)$

$$\dim \mathbb{E}(P, k; N)_{\text{irr}} \begin{cases} < \\ = \end{cases} \dim R(P)_{\text{irr}} \quad \text{if } (k, N) \begin{cases} \neq \\ = \end{cases} (n, \mathbb{1}_n)$$

$$\dim \mathbb{E}(P, k; \emptyset, N)_{\text{irr}} \begin{cases} < \\ = \end{cases} \dim R(P; \emptyset)_{\text{irr}}$$

$$\star \mathbb{E}'(P|n)_{\text{irr}} := \mathbb{E}(P|n)_{\text{irr}} \setminus \mathbb{E}(P, n; \mathbb{1}_n)_{\text{irr}}$$

$$\star \mathbb{E}'(P|n; \emptyset)_{\text{irr}} := \mathbb{E}(P|n; \emptyset)_{\text{irr}} \setminus \mathbb{E}(P, n; \emptyset, \mathbb{1}_n)_{\text{irr}}$$

Theorem 2 Let  $P \in B(m)$ ,  $n = m + 3g - 3$ .

$$(i) \quad \begin{array}{l} \text{PM}(\mathbb{E}'(P|n)_{\text{irr}}) \\ \text{PM}(\mathbb{E}'(P|n; \emptyset)_{\text{irr}}) \end{array} : \text{nowhere dense in } \begin{array}{l} R(P)_{\text{irr}} \\ R(P; \emptyset)_{\text{irr}} \end{array}$$

$$(ii) \quad \begin{array}{l} \text{PM} : \mathbb{E}(P, n; \mathbb{1}_n)_{\text{irr}} \rightarrow R(P)_{\text{irr}} \\ \text{PM} : \mathbb{E}(P, n; \emptyset, \mathbb{1}_n)_{\text{irr}} \rightarrow R(P; \emptyset)_{\text{irr}} \end{array} \quad \text{loc. inj. open holo map.}$$

(This fact will be said that the R-H prob. is loc. solvable for g.s. eq's with  $m + 3g - 3$  app. sing. pts.)

Def  $A_1, A_2$  : analytic sp's.  $f : A_1 \rightarrow A_2$  : holo. map.

$f$  : almost surjective  $\Leftrightarrow$  analytic Z-closure of  $f(A_1) = A_2$

Theorem 3 (ess. by Ôtsuki)  $P \in B(m)$ ,  $\emptyset \in (\mathbb{C} \setminus \mathbb{Z})^m$ ,  $n = m + 4g - 3$

$$\Rightarrow \text{PM} : \mathbb{E}(P|n)_{\text{irr}} \rightarrow R(P)_{\text{irr}} : \text{almost surj.}$$

$$\text{PM} : \mathbb{E}(P|n; \emptyset)_{\text{irr}} \rightarrow R(P; \emptyset)_{\text{irr}} : \text{surj.}$$

Conj.  $n = m + 3g - 3 \Rightarrow \text{PM} : \mathbb{E}(P, n; \mathbb{1}_n)_{\text{irr}} \rightarrow R(P)_{\text{irr}}$  alm. surj.

(If this is true, then one can say that the R-H prob. is glob.

solvable for SL-eg's with at most  $m + 3g - 3$  app. sing. pts c.w. m.)

Cor to Th 2  $P \in B(m)$ ,  $n = m + 3g - 3$

$\mathbb{E}(P, n; \mathbb{1}_n)_{\text{irr}}$ ,  $\mathbb{E}(P, n; \emptyset, \mathbb{1}_n)_{\text{irr}}$  : complex mfd's.

Simplification  $n = m + 3g - 3$

$$R(m; \emptyset)_{\text{irr}} := \bigsqcup_{P \in B(m)} R(P; \emptyset)_{\text{irr}}$$

$$\begin{array}{ccc}
 & \mathbb{E}(m, n; \emptyset, \mathbb{1}_n)_{\text{irr}} & \xrightarrow{PM} R(m; \emptyset)_{\text{irr}} \\
 \pi \swarrow & \downarrow \omega & \swarrow \\
 B(m, n; \mathbb{1}_n) & & B(m)
 \end{array}
 \quad (D.G.5)$$

NB.  $R(m; \emptyset)_{\text{irr}}$  is a loc. system over  $B(m)$  whose char. homom. is given by

$$Br(m) := \pi_1(B(m), P_0) \longrightarrow \text{Aut}(R(P_0; \emptyset)_{\text{irr}}), \quad l \longmapsto [P \mapsto P \circ l_*]$$

$$\text{where } Br(m) \longrightarrow \text{Aut}(\pi_1(M \setminus |P|)) \quad , \quad l \longmapsto l_*$$

Later, by using Th 2, we shall show

"Garnier distribution" is  $\begin{cases} \text{Frob. integrable} \\ \text{transverse to each fiber of } \omega \end{cases}$

§6. Cousin Prob. & the mfd of g.s. eq's.

By §5, it is natural to assume

$$(**) \quad \boxed{n = m + 3g - 3}$$

☆  $\mathbb{E}_m(\emptyset) \stackrel{\text{def}}{=} \mathbb{E}(m, n; \emptyset, \mathbb{1}_n)_{\text{irr}}$  ; more detailed study of  $\mathbb{E}_m(\emptyset)$ .

Given  $\mathbb{R} = (P_1, \dots, P_m, \delta_1, \dots, \delta_n) \in B(m+n)$ , let

$$\xi(\mathbb{R}) \stackrel{\text{def}}{=} K^2 \otimes [P_1 + \dots + P_m - (\delta_1 + \dots + \delta_n)] \in H^1(M, \mathcal{O}^*)$$

Rem  $\xi(\mathbb{R}) = K \otimes \eta(\mathbb{R}, \mathbb{1}_n)^{-1}$  (see §3)

$$c_1(\xi(\mathbb{R})) = g - 1.$$

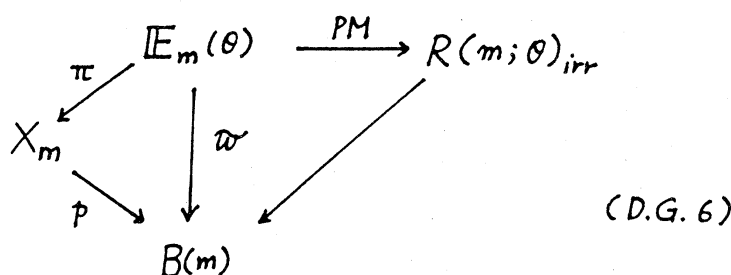
Advantage of (\*\*) = Fredholm Alternative

$h^0(M, \mathcal{O}(\xi(\mathbb{R}))) = h^1(M, \mathcal{O}(\xi(\mathbb{R})))$  for  $\mathbb{R} \in B(m+n)$  ( $\because R-R$ )

☆  $D_m := \{ \mathbb{R} \in B(m+n) ; h^0(M, \mathcal{O}(\xi(\mathbb{R}))) \geq 1 \}$

☆  $X_m := B(m+n) \setminus D_m$  : the sp. of sing. pts

Rem  $D_m = D(m, n ; \mathbb{1}_n)$ ,  $B_m = B(m, n ; \mathbb{1}_n)$  (see §3)



Example (i)  $g=0 \Rightarrow D_m = \emptyset$

(ii)  $g=1 (\Rightarrow m=n)$

*lin. eq.*

$$D_m = \{ \mathbb{R} = (P_1, \dots, P_m, \delta_1, \dots, \delta_n) \in B(m+n) ; P_1 + \dots + P_m \sim \delta_1 + \dots + \delta_n \}$$

$$\uparrow \{ \mathbb{R} \in B(m+n) ; \sum_{j=1}^m \int_{P_j}^{\delta_j} \omega \equiv 0 \pmod{\text{periods}} \}$$

*Abel th*

*holo 1-form on M*

Key Idea Given sing. pts  $\mathbb{R} \in X_m$  and "fiber coord's" (= accessory param. of a diff. eq.), then construct a diff. eq.  $\in \mathbb{E}_m(\theta)$  as a sol. to the "Cousin Problem (CP)"!

Key Lemma  $X_m$  : an analytic  $\mathbb{Z}$ -open subset ( $\neq \emptyset$ ) of  $B(m+n)$ .

(i)  $H^1(M, \mathcal{O}(\xi(\mathbb{R}))) = 0$ ,  $\leftarrow$  solvability of (CP)

(ii)  $H^0(M, \mathcal{O}(\xi(\mathbb{R}))) = 0$   $\leftarrow$  unig. of sol. to (CP)

(iii) hold for all  $\mathbb{R} \in X_m$   $\leftarrow$  holo. dep. of sol to (CP) on "data".

Theorem  $\mathbb{E}_m(\theta) \xrightarrow{\pi} X_m$  is holo. affine b'dle of rk  $m+3g-3$ .

Rem  $\dim \mathbb{E}_m(\theta) = 2(m+3g-3) + m$ .

§ 7. Fund. 2-form on the mfd of g.s. eq's.

Theorem  $\exists \Omega$ : a closed 2-form on  $\mathbb{E}_m(\theta)$  s.t.

$\Omega|_{\omega^{-1}(P)}$  is a symplectic str. on each fiber  $\omega^{-1}(P)$   
of  $\mathbb{E}_m(\theta) \xrightarrow{\omega} B(m)$ ,  $P \in B(m)$ .

Def  $\Omega$  is called the fund. 2-form on  $\mathbb{E}_m(\theta)$

§ 8. Monod. Preserving deform.

Theorem The monodromy preserving deformation is an  
 $\Omega$ -invariant foliation on  $\mathbb{E}_m(\theta)$ .