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Fuchsian systems associated with the $P^2(F_2)$-arrangement

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§.0 Introduction

In this paper, we construct a family E(s) of Fuchsian differential equations, depending on two dimensional parameter s, defined on the complex projective plane $M=\mathbb{C}P^2$ with regular singularities along

$$H:\ xyz(x-y)(y-z)(z-x)\{(x+y-z)^2-4xy\} = 0$$

where [x,y,z] is a system of homogeneous coordinates on M.

We call this arrangement H, of six lines and one conic, the $P^2(\mathbb{F}_2)$-arrangement, because the set of lines in the projective plane $P^2(\mathbb{F}_2)$ over the finite field $\mathbb{F}_2=\{0,1\}$ consists of seven lines corresponding to the seven components of H. The arrangement H relates the Weyl group $W(F_4)$ of type F_4 as follows. The 24 mirrors of the reflections in the Coxeter group F_4 defines a hyperplane arrangement in $\mathbb{C}^4$. Passing to $\mathbb{C}P^3$, this arrangement defines a plane arrangement in $\mathbb{C}P^3$, which is called the $W(F_4)$-arrangement. The restriction $\tilde{H}$ of the $W(F_4)$-arrangement to any projective plane N in the arrangement consists of thirteen lines in $\mathbb{C}P^2$ which can be given by the equation

$$\tilde{H}:\ XYZ(X^2-Y^2)(Y^2-Z^2)(Z^2-X^2)(X+Y+Z)(-X+Y+Z)(X-Y+Z)(X+Y-Z)=0,$$

where [X,Y,Z] is a system of homogeneous coordinates on N. The arrangement H is the image of $\tilde{H}$ under the map $\pi:N\longrightarrow M$ given by

$$\pi:[X,Y,Z]\longrightarrow[x,y,z]=[X^2,Y^2,Z^2].$$

The map π is the quotient map by the group $K(\simeq \mathbb{Z}_2+\mathbb{Z}_2)$ generated by $[X,Y,Z]\longrightarrow[-X,Y,Z]$ and $[X,Y,Z]\longrightarrow[X,-Y,Z]$. (see figure 1)

The two subarrangements

$$H': xyz(x-y)(y-z)(z-x)=0$$

and

$$H'': xyz\{(x+y-z)^2-4xy\}=0$$

of the arrangement H are well-known as the singular loci of the Appell's hypergeometric differential equations F_1 and F_4, respectively,(see figure 2). Our equation interpolates the equation F_1 and the modified equation F_4' (see § 2) of F_4. More precisely, for some special values of the parameter s, our equation E(s) turns out to be the equation F_1 and for some other special values of s, it turns out to be the equation F_4'. Moreover, the principal parts of the equation E(s), after some normalization, are *linear* combinations of those of the equations F_1 and F_4'. We must say that it is a surprising result if we recall the non-linearity of the integrability condition (see §.1). This mysterious phenomenon is also reported in [Yos.2].

For some special values of the parameter s, the equation E(s) happens to give the uniformizing equations (see § 3) of the hyperbolic orbifolds found by Hunt and Höfer ([Hun],[Höf]), where a hyperbolic orbifoild is an orbifold whose universal uniformization is the complex unit ball $\mathbb{B}^2=\{(z_1,z_2)\in\mathbb{C}^2|\ |z_1|^2+|z_2|^2<1\}$.

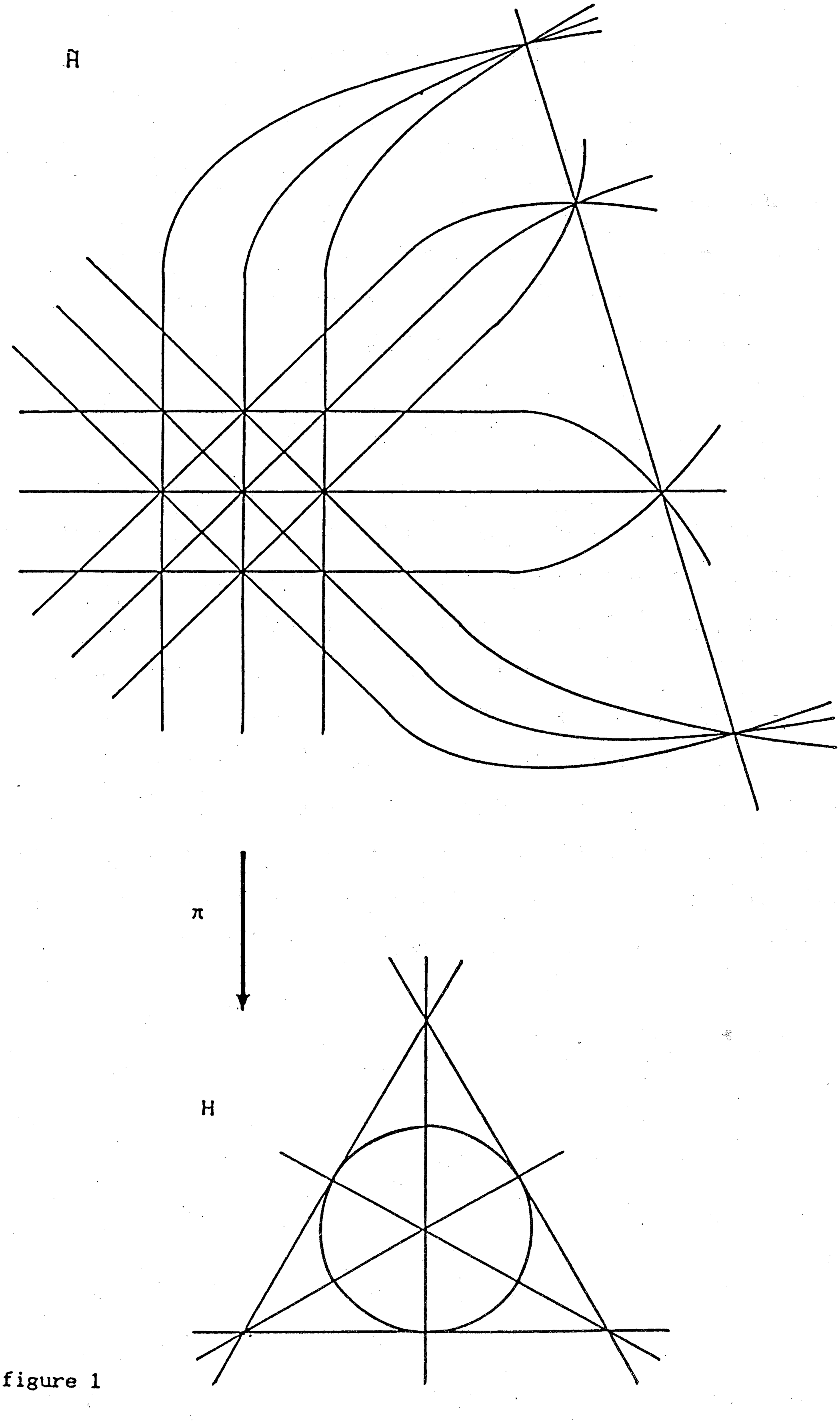
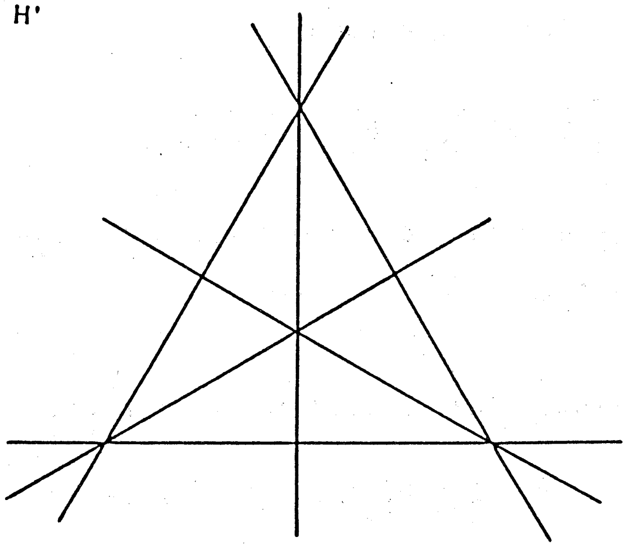


figure 1

H'



H"

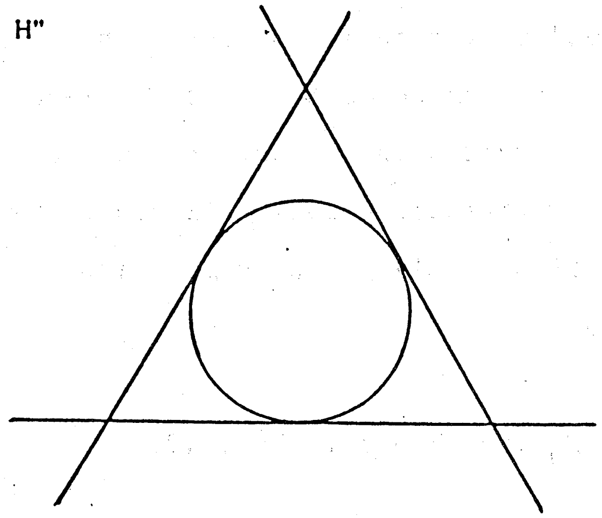


figure 2

§1. The main theorem

We consider a system in the form

$$\text{(E)} \quad \frac{\partial^2 w}{\partial x_i \partial x_j} = \Sigma_{k=1}^{2} P_{ij}^{k}(x) \frac{\partial w}{\partial x_k} + P_{ij}^{0}(x) w \quad i,j=1,2$$

defined on $M=\mathbb{C}P^2$, where w is the unknown and $x=(x_1,x_2)$ is a system of inhomogeneous coordinates on M.

Definition: A system (E) is said to be *in normal form* if

$$\text{(N)} \quad \Sigma_{j=1}^{2} P_{ij}^{j}(x) = 0 \quad \text{for } i=1,2$$

Definition: The system (E) is said to be *completely integrable* if (E) has three linearly independent solutions.

Any completely integrable system in the form (E) can be transformed into the uniquely determined system in normal form by replacing the unknown w by its product with a non-zero function of x. The consequent system is called the *normal form* of (E) ([Yos.1]).

It is known [Yos.3] that the equation (E) in normal form is completely integrable if and only if the coefficients $\{P_{ij}^{k}\}$ satisfy the following equations.

$$P_{ij}^{k}(x) = P_{ji}^{k}(x) \quad i,j=1,2, \ k=0,1,2,$$

$$P_{11}^{0}(x) = -\frac{\partial P_{11}^{1}(x)}{\partial x_1} - \frac{\partial P_{11}^{2}(x)}{\partial x_2} + 2\{P_{11}^{1}(x)\}^2 - 2P_{22}^{2}(x) P_{11}^{2}(x),$$

$$P^0_{12}(x) = \frac{\partial P^2_{22}(x)}{\partial x_1} + \frac{\partial P^1_{11}(x)}{\partial x_2} + P^1_{22}(x)P^2_{11}(x) - P^1_{11}(x)P^2_{22}(x),$$

$$P^0_{22}(x) = -\frac{\partial P^2_{22}(x)}{\partial x_2} - \frac{\partial P^1_{22}(x)}{\partial x_1} + 2\{P^2_{22}(x)\}^2 - 2P^1_{11}(x)P^1_{22}(x),$$

$$(IC)_1 := -2\frac{\partial^2 P^1_{11}(x)}{\partial x_1 \partial x_2} - \frac{\partial^2 P^2_{11}(x)}{\partial x_2^2} + 6P^1_{11}(x)\frac{\partial P^1_{11}(x)}{\partial x_2} - 3P^2_{11}(x)\frac{\partial P^2_{22}(x)}{\partial x_2}$$

$$-3P^2_{22}(x)\frac{\partial P^2_{11}(x)}{\partial x_2} + 3P^1_{11}(x)\frac{\partial P^2_{22}(x)}{\partial x_1} - \frac{\partial^2 P^2_{22}(x)}{\partial x_1^2}$$

$$-2P^2_{11}(x)\frac{\partial P^2_{11}(x)}{\partial x_1} - P^1_{22}(x)\frac{\partial P^2_{11}(x)}{\partial x_1} = 0,$$

$$(IC)_2 := -2\frac{\partial^2 P^2_{22}(x)}{\partial x_2 \partial x_1} - \frac{\partial^2 P^1_{22}(x)}{\partial x_1^2} + 6P^2_{22}(x)\frac{\partial P^2_{22}(x)}{\partial x_1} - 3P^1_{22}(x)\frac{\partial P^1_{11}(x)}{\partial x_1}$$

$$-3P^1_{11}(x)\frac{\partial P^1_{22}(x)}{\partial x_1} + 3P^2_{22}(x)\frac{\partial P^1_{11}(x)}{\partial x_2} - \frac{\partial^2 P^1_{11}(x)}{\partial x_2^2}$$

$$-2P^1_{22}(x)\frac{\partial P^2_{11}(x)}{\partial x_2} - P^2_{11}(x)\frac{\partial P^1_{22}(x)}{\partial x_2} = 0.$$

Lemma 1 ([Yos.3]): Let $Q^k_{ij}(\xi)$ $(i,j,k=1,2)$ be the coefficients of the normal form of the transformed system of (E) by the coordinate change $\xi=\xi(x)$. Then we have

$$Q^{k}_{ij}(\xi)=\sum_{l=1}^{2}\frac{\partial^2\xi}{\partial x_i\partial x_j}\frac{\partial x_k}{\partial \xi_l}-\frac{1}{3}(\delta^k_i\frac{\partial}{\partial x_j}+\delta^k_j\frac{\partial}{\partial x_i})\log(\det(\frac{\partial\xi}{\partial x}))$$

$$+\sum_{p,q,r=1}^{2}P^{r}_{pq}(x)\frac{\partial\xi_p}{\partial x_i}\frac{\partial\xi_q}{\partial x_j}\frac{\partial x_k}{\partial\xi_r}$$

where δ^k_j is the Kroneker symbol.

If $Q^k_{ij}(x)=P^k_{ij}(x)$ $(i,j,k=1,2)$, then the system (E) is said to be *invariant* under the transform $x \longrightarrow \xi$.

Definition: *A projective solution* of a completely integrable system (E) is the pair $z=(z_1,z_2)$ of ratios $z_i=w_i/w_0 (i=1,2)$ of three linearly independent solutions w_0, w_1 and w_2 of (E).

Definition: A projective solution of (E) is said to be *ramifying at $O=(0,0)$ along $x_1=0$ with exponent* α if there exists a projective solution $z=(z_1,z_2)$ which has the following expression :

$$z_1(x)=x_1^{\alpha}v_1,\quad z_2(x)=v_2,\quad \det(\partial z/\partial x)=x_1^{\alpha-1}u$$

for some $\alpha\in\mathbb{C}$, where v_1,v_2 and u are holomorphic at O , not divisible by x_1.

Lemma 2 ([Yos.2]):If a projective solution of (E) in normal form is ramifing along $x_1=0$ with exponent α, then the coefficients P^k_{ij} of (E) have the following properties:

(R) $\quad P^2_{22}(x),\ x_1P^2_{11}(x),\ \frac{1}{x_1}P^1_{22}(x)$ and $P^1_{11}(x)-\frac{\alpha-1}{3x_1}$

are holomorphic.

Definition: A system (E) in the normal form is said to have *ramifing singularities* along $x_1 = 0$ with exponent α if the condition (R) holds.

To state the theorm we prepare some notations. Let $[x,y,z]$ be a system of homogeneous coordinates on M related to (x_1,x_2) by $x_1 = \frac{x}{z}$ and $x_2 = \frac{y}{z}$. Let H_i $(i=1,\ldots,7)$ denote the following curves:

$H_1:\{x=0\}$, $H_2:\{y=0\}$, $H_3:\{z=0\}$, $H_4:\{x=y\}$, $H_5:\{y=z\}$, $H_6:\{z=x\}$,

$H_7: \{(x+y-z)^2-4xy = 0\}$,

so we have $H = \cup_{i=1}^{7} H_i$. Let G be the transformation group on M generated by $[x,y,z] \longrightarrow [z,x,y]$ and $[x,y,z] \longrightarrow [y,x,z]$. Note that G is isomorphic to the symmetric group in three letters.

THEOREM: For given complex numbers $s_i \neq 1 (i=1,\ldots,7)$, there is a completely integrable differential equation E(s), in normal form, with ramifying singularities along H_i with exponent s_i if and only if

$$s_1=s_2=s_3, \quad s_4=s_5=s_6 \quad \text{and } 6s_1-3s_4-2s_7+2=0.$$

(In particular E(s) is G-invariant.) The four coefficients of the system (E(s)) are explicitly given as follows.

$$P^1_{11}(x_1,x_2) = \frac{S^1}{x_1} + \frac{S^4(x_1-2x_2+1)}{2(x_1-1)(x_1-x_2)} + \frac{4S^7 x_2}{(x_1+x_2-1)^2-4x_1x_2},$$

$$P^2_{11}(x_1,x_2) = \frac{-3S^4 x_2(x_2-1)}{2x_1(x_1-1)(x_1-x_2)} + \frac{4S^7 x_2(x_2-1)}{x_1\{(x_1+x_2-1)^2-4x_1x_2\}},$$

$$P^1_{22}(x_1,x_2)= P^2_{11}(x_2,x_1), \qquad P^2_{22}(x_1,x_2)= P^1_{11}(x_2,x_1),$$

where $S^i= \frac{1}{3}(s_i-1)$.

Remark: Other coefficients P^k_{ij} of (E) are uniquely determined by the equalities (N) and by the assumption that (E) is completely integrable. Therefore, in the sequel, in order to describe the system (E) in normal form we give only the four coefficients P^1_{11}, P^2_{11}, P^2_{22} and P^1_{22} of (E).

§.2 Relation between E(s) and Appell's hypergeometric differential equations

Appell's hypergeometric equation $F_1(a,b,b',c)$ is a differential equation with regular singularities on $\cup_{i=1}^{6}H_i$, while it is nonsingular along H_7. The normal form of $F_1(a,b,b',c)$ is given by

$$P^1_{11}(a,b,b',c;x_1,x_2)$$
$$= \frac{1}{3}\frac{x_2(x_2-1)}{f_1}\{(c-b')x_2+ (2b-c)x_1+ (b'-(a+b+1)x_1x_2+ (a-b+1)x_1^2\}$$

$$P^2_{11}(a,b,b',c;x_1,x_2) = \frac{(x_2(x_2-1))^2}{f_1} b$$

$$P^2_{22}(a,b,b',c;x_1,x_2) = P^1_{11}(a,b',b,c;x_2,x_1)$$

$$P^1_{22}(a,b,b',c;x_1,x_2) = P^2_{11}(a,b',b,c;x_2,x_1)$$

where $f_1=x_1x_2(x_1-1)(x_2-1)(x_1-x_2)$.

Proposition 1: The equation $E(s)$ with $s_7=1$ coincides with the normal form of Appell's F_1 which is G-invariant. Precise correspondence between $E(s)=E(s_1,s_2,s_3)$ and $F_1(a,b,b',c)$ is given by

$$E(s_1,s_4,1)= F_1(\cancel{-\tfrac{1}{2}(2s_1+s_4-1)}\ {-s_1-s_4+1},- \tfrac{1}{2}(s_4-1),- \tfrac{1}{2}(s_4-1),- \tfrac{1}{2}(2s_1+s_4-3))$$

This identity is valid for all s_1, $s_4 \in \mathbb{C}$.

Appell's hypergeometric equation $F_4(a,b,c,c')$ is a differential equation with four linearly independent solutions and has regular singularities on $\cup_{i=1}^{3}H_i \cup H_7$, while it is nonsingular along $\cup_{i=4}^{6}H_i$. The solution space of $F_4(a,b,c,c')$ has three dimensional invariant subspace if $b=c+c'+1$ [Kat]. The corresponding equation is called the modified F_4 and denoted by $F_4'(a,b,c,c')$. The four coefficients of the normal form of F_4' (a,b,c,c') are given by

$$P_{11}^{1}(a,b,c,c';x_1,x_2)$$
$$= -\frac{1}{3}\frac{1}{f_4}\{c(x_2-1)^2+(a-(b+c+2c'-1))x_1$$
$$+(a+5b-c-2c'+1)x_1x_2+(a-(b+2c'-1))x_1^2\}$$

$$P_{11}^{2}(a,b,c,c';x_1,x_2)= \frac{x_2^2}{f_4}\{(a-b-c'+1)x_1+(a+b-c'+1)(1-x_2)\}$$

$$P_{22}^{2}(a,b,c,c';x_1,x_2) = P_{11}^{1}(a,b,c',c;x_2,x_1)$$

$$P_{22}^{1}(a,b,c,c';x_1,x_2) = P_{11}^{2}(a,b,c',c;x_2,x_1)$$

where $f_4=x_1x_2\{(x_1+x_2-1)^2-4x_1x_2\}$.

Proposition 2: The equation $E(s)$ with $s_4=1$ coinsides with the normal form of Appell's modified F_4 which is G-invariant. Precise correspondence between E(s) and $F_4'(a,b,c,,c')$ is give by

$$E(s_1,1,3s_1-\frac{1}{2})=F_4'(-3s_1+1,2s_1+1,-s_1+1,-s_1+1).$$

§.3 Uniformizing equations of some hyperbolic orbifolds

We briefly recall the definitions of orbifolds and their uniformizations. Let X be a complex manifold, S be a hypersurface of X, $S=\cup_j S_j$ be its decomposition into irreducible components, and let b_j be either infinity or an integer called the weight attached to the corresponding S_j. The triple (X,S,b) is called an *orbifold* if for every point in $X-\cup\{S_j|\ b_j=\infty\}$ there is an open neighborhood U and a covering manifold which ramifies along $U\cap S$ with the given indices b. It is called *uniformizable* if there is a global covering manifold (called a *uniformization*) of X with the given ramification datum (S,b). If X is uniformizable, there exists an uniformization $\tilde{X}$, which is simply connected, called the *universal uniformization*. Let X be an orbifold and $\tilde{X}$ be its universal uniformization. The multivalued inverse map $X\longrightarrow\tilde{X}$ of the projection $\tilde{X}\longrightarrow X$ is called the *developing map*.

If the universal uniformization of an orbifold (X,S,b) is isomorphic to the complex ball (we call such an orbifold hyperbolic), there exists a unique Fuchsian differential equation in normal form such that its projective solution gives the developing map. The equation is called the *uniformizing differential equation* of the orbifold (X,S,b). For more detail see [Yos.2].

In his theses [Hun], B.Hunt studied N-dimensional hyperbolic orbifolds. He discovered a 3-dimensional hyperbolic orbifold, attached to the $W(F_4)$-arrangement. Restricting to a plane in the $W(F_4)$-arrangement, we have a 2-dimensional hyperbolic orbifold, attached to a line arrangement in $\mathbb{C}P^2$ – the $P^2(F_2)$-arrangement – . Furthermore, Höfer[Höf] showed that there are only four hyperbolic orbifolds over this arrangement. These orbifolds are given as follows. Consider the arrangement $\tilde{H}$ in N. Define the weight function b on N as follows (see figure 3).

case	($\cup_{i=1}^{3}\tilde{H}_i$,	$\cup_{i=4}^{6}\tilde{H}_i$,	$\tilde{H}_7$,	p,	q,	r)
1	(∞,	2,	4,	-4,	2,	2)
2	(6,	2,	2,	-4,	6,	3)
3	(-6,	6,	2,	4,	2,	1)
4	(-3,	3,	∞,	∞,	1,	1)

Colollary: Uniformizing differential equations of the above orbifolds are obtained by pulling back the equation E(s) under the map π, where the values of the parameter s are given as follows.

case	(s_1,	s_4,	s_7)
1	(0,	$\frac{1}{2}$,	$\frac{1}{4}$)
2	($\frac{1}{12}$,	$\frac{1}{4}$,	$\frac{1}{4}$)
3	($-\frac{1}{12}$,	$\frac{1}{6}$,	$\frac{1}{4}$)
4	($-\frac{1}{6}$,	$\frac{1}{6}$,	0)

Remark: In ·· case $s_7=1$, the equation E(s) reduces to Appell's

F_1, which is studied by Terada [Ter] and Deligne-Mostow [D-M].

§3. Proof of the results

Let the projective planes M, N, arrangements $\tilde{H}$, H, the group K and the projection $\pi:N \longrightarrow M$ be as above. The arrangement $\tilde{H}$ in N consists of thirteen lines:

$$\tilde{H}_1=\{X=0\},\ \tilde{H}_2=\{Y=0\},\ \tilde{H}_3=\{Z=0\},\ \tilde{H}_4=\{X^2-Y^2=0\},\ \tilde{H}_5=\{Y^2-Z^2=0\},$$

$$\tilde{H}_6=\{Z^2-X^2=0\},\ \tilde{H}_7=\{(X+Y+Z)(-X+Y+Z)(X-Y+Z)(X+Y-Z)=0\}.$$

Note that $\pi(\tilde{H}_i)=H_i(i=1,\cdots,7)$ and $\pi(\tilde{H})=H$.

We cnstruct a K-invariant differential equation $(\tilde{E})$ defined on N with ramifying singularities along $\tilde{H}_i$ with exponent $t_i(i=1,\cdots,7)$. We follow the method established in [Yos.3].

Lemma 3([Yos.3]): If the equation (E) has ramifying singularities along the line at infinitely, then the total degree of the rational function $P^k_{ij}(x)$ is negative for i,j,k=1,2.

By Lemma 3, we can put

$$\tilde{P}^1_{11} = X_2(X_2^2-1)A/F,\quad \tilde{P}^2_{11} = X_2^2(X_2^2-1)^2B/F,$$

$$\tilde{P}^2_{22} = X_1(X_1^2-1)C/F,\quad \tilde{P}^1_{22} = X_1^2(X_1^2-1)^2D/F,$$

where

$$A = \sum_{i+j\le 8} a(i,j)X_1^iX_2^j,\quad B = \sum_{i+j\le 5} b(i,j)X_1^iX_2^j,$$

$$C = \sum_{i+j\le 8} c(i,j)X_1^iX_2^j,\quad D = \sum_{i+j\le 5} d(i,j)X_1^iX_2^j,$$

and

$$F= X_1X_2(X_1^2-1)(X_2^2-1)(X_1^2-X_2^2)(X_1+X_2+1)(-X_1+X_2+1)(X_1-X_2+1)(X_1+X_2-1).$$

The assumption that the system (E) is K-invariant says that

$a(i,j)=c(i,j)=0$ unless $i\equiv j\equiv 0 \mod 2$,

$b(i,j)=0$ unless $i+1\equiv j\equiv 0 \mod 2$,

$d(i,j)=0$ unless $i\equiv j+1\equiv 0 \mod 2$.

Applying Lemma 2, along every component of $\tilde{H}$, we obtain finitely many linear equations with unknowns $a(i,j),\cdots, d(i,j)$. By solving these, all the coefficients $a(i,j),\cdots,d(i,j)$ are expressed in terms of $t_i(i=1,\cdots 7)$:

$$A(X_1,X_2)=-T^1(X_1^2-1)(X_1^2-X_2^2)\{(X_1^2+X_2^2-1)^2-4X_1^2X_2^2\}$$
$$-T^4X_1^2(X_1^2-2X_2^2+1)\{(X_1^2+X_2^2-1)^2-4X_1^2X_2^2\} -8T^7X_1^2X_2^2(X_1^2-1)(X_1^2-X_2^2),$$

$$B(X_1,X_2)=3T^4X_1\{(X_1^2+X_2^2-1)^2-4X_1^2X_2^2\} - 8T^7X_1(X_1^2-1)(X_1^2-X_2^2),$$

$$C(X_1,X_2)=-A(X_2,X_1), \qquad D(X_1,X_2)=-B(X_2,X_1).$$

where $T^i= \frac{1}{3}(t_i-1)$; Moreover these liner equations require that t_i's satisfy $T^1=T^2=T^3$, $T^4=T^5=T^6$, $3T^1-3T^4-2T^7= 0$.

Now we study the integrability condition. The integrability condition of $\tilde{E}(t)$ is given by

$$(IC)_1= -64T^7(3T^1-3T^4-2T^7)X_1^2X_2^3(X_1^2-1)^2(X_2^2-1)^3(X_1^2-X_2^2)^2/F^2,$$

$$(IC)_2= -64T^7(3T^1-3T^4-2T^7)X_1^3X_2^2(X_1^2-1)^3(X_2^2-1)^2(X_1^2-X_2^2)^2/F^2.$$

Since the parameter has the relation such that $3T^1-3T^4-2T^7=0$, we have $(IC)_1=(IC)_2=0$.

Finally, we project $\tilde{E}(t)$ by the quotient map $\pi:M \longrightarrow M/K \simeq N$, where π is given by $(X_1,X_2) \longrightarrow (x_1,x_2)=(X_1^2,X_2^2)$. By Lemma 1, coefficients P^k_{ij} of the normal form of $\pi(\tilde{E}(t))$ are given as follows.

$$P^1_{11}(x) = -\frac{1}{6x_1^2} + \frac{1}{2x_1}\tilde{P}^1_{11}(X), \qquad P^2_{22}(x) = -\frac{1}{6x_2^2} + \frac{1}{2x_2}\tilde{P}^2_{22}(X),$$

$$P^2_{11}(x) = \frac{x_2}{2x_1^2}\tilde{P}^2_{11}(X), \qquad P^1_{22}(x) = \frac{x_1}{2x_2^2}\tilde{P}^1_{22}(X).$$

Since π is branching along only the three lines $\cup_{i=1}^3 H_i$ with indices two, there are the relations among the exponents such that $t_1=2s_1$, $t_4=s_4$, and $t_7=s_7$. Thus we obtain the differential equations which we want. Easy caluculations show the remaining claims of the theorem.

Propositions 1 and 2 are proved by straightfoward computation, so we omit the detail. If $s_7=1$, the equations $(IC)_1=(IC)_2=0$ are satisfied. This proves the last statement of Proposition 1.

§5. Linear structure of the set of the solutions of the non-linear differential equation (IC)

The integrability condition

$$\text{IC} : (IC)_1 = (IC)_2 = 0$$

of the system (E) with the condition (N) is a system of nonlinear differential equations with unknowns $\{P^k_{ij}\}=\{P^1_{11}=-P^2_{12},\ P^2_{11},\ P^1_{22},\ P^2_{22}=-P^1_{12}\}$. There is a one-to-one correspondence between the set of solutions of IC and the set of completely integrable systems (E) in normal form. We have a great interest in rational solutions of IC of which corresponding systems (E) have transcendental solutions. The method used in section 4 is a practical one to find such solutions.

Since the system IC is by no means linear, we can not expect that linear combinations $\{t_1R^k_{ij} + t_2Q^k_{ij}\}$ $(t_1, t_2 \in \mathbb{C})$ of two solutions $\{R^k_{ij}\}$ and $\{Q^k_{ij}\}$ of IC are also solutions of IC: indeed, it is not true in general. But sometimes miracles occur. Propositions 1 and 2 say that the coefficients $\{P^k_{ij}\}$ of the principal parts of our system $E(s)$ are linear combinations of the coefficients of the principal parts of the normal form of Appell's F_1 (with a special parameter) and those of the system F_4' (with a special parameter). In [Yos.2], it is shown that linear combinations $\{t_1R^k_{ij} + t_2Q^k_{ij}\}$ of two solutions $\{R^k_{ij}\}$ and $\{Q^k_{ij}\}$ of IC:

$$\{R\}: \quad \begin{aligned} R^1_{11}(x,y) &= 3/x + 81x^2y^3(2-x^3-y^3)/w, \\ R^2_{11}(x,y) &= 81xy(1+x^3-y^6-x^3y^3)/w, \\ R^2_{22}(x,y) &= R^1_{11}(y,x), \quad R^1_{22}(y,x) = R^2_{11}(y,x), \end{aligned}$$

$$\{Q\}: \quad \begin{aligned} Q^1_{11}(x,y) &= 3x^2(y^3-1)(1+x^3-2y^3)/2h, \\ Q^2_{11}(x,y) &= -\ 9xy(y^3-1)^2/2h, \\ Q^2_{22}(x,y) &= Q^1_{11}(y,x), \quad Q^1_{22}(x,y) = Q^2_{11}(y,x), \end{aligned}$$

where

$$w = \Pi^2_{a,b=0}(\omega^a x + \omega^b y + 1) = (x^3 + y^3 + 1)^3 - 27x^3y^3,$$

$$h = (x^3-1)(y^3-1)(x^3-y^3)$$

are solutions of IC.

These two examples suggest the existence of some linear structure of the set of solutions of IC, which is as yet veiled in mystery.

We conclude this paper by giving a useful system to test whether $t_1R^k_{ij} + t_2Q^k_{ij}$ (t_1, $t_2 \in \mathbb{C}$) are solutions of IC.

Proposition 3. Let $\{R^k_{ij}\}$ and $\{Q^k_{ij}\}$ be solutions of IC. Then

(1) For all $t \in \mathbb{C}$, $\{tR^k_{ij}\}$ are the solutions of IC if and only if $\{R^k_{ij}\}$ satisfy the following conditions:

$$2\frac{\partial^2 R^1_{11}}{\partial x_1 \partial x_2} + \frac{\partial^2 R^2_{11}}{\partial x_2^2} + \frac{\partial^2 R^2_{22}}{\partial x_1^2} = 0,$$

$$2\frac{\partial^2 R^2_{22}}{\partial x_2 \partial x_1} + \frac{\partial^2 R^1_{22}}{\partial x_1^2} + \frac{\partial^2 R^1_{11}}{\partial x_2^2} = 0.$$

(2) $\{R^k_{ij} + Q^k_{ij}\}$ is a solution of IC if and only if ($\{R^k_{ij}\}$, $\{Q^k_{ij}\}$) satisfy the following equations:

$$6\{R^1_{11}\frac{\partial Q^1_{11}}{\partial x_2} + Q^1_{11}\frac{\partial R^1_{11}}{\partial x_2}\}$$

$$-3\{R^2_{11}\frac{\partial Q^2_{22}}{\partial x_2} + Q^2_{11}\frac{\partial R^2_{22}}{\partial x_2} - Q^2_{22}\frac{\partial R^2_{11}}{\partial x_2} - Q^2_{22}\frac{\partial R^2_{11}}{\partial x_2} + R^1_{11}\frac{\partial Q^2_{22}}{\partial x_1} + Q^1_{11}\frac{\partial R^2_{22}}{\partial x_1}\}$$

$$-2\{R^2_{11}\frac{\partial Q^1_{22}}{\partial x_1} + Q^2_{11}\frac{\partial R^1_{22}}{\partial x_1}\} - \{R^1_{22}\frac{\partial Q^2_{11}}{\partial x_1} + Q^1_{22}\frac{\partial R^2_{11}}{\partial x_1}\} = 0,$$

$$6\{R^2_{22}\frac{\partial Q^2_{22}}{\partial x_1} + Q^2_{22}\frac{\partial R^2_{22}}{\partial x_1}\}$$

$$-3\{R^1_{22}\frac{\partial Q^1_{11}}{\partial x_1} + Q^1_{22}\frac{\partial R^1_{11}}{\partial x_1} - R^1_{11}\frac{\partial Q^1_{22}}{\partial x_1} - Q^1_{11}\frac{\partial R^1_{22}}{\partial x_1} + R^2_{22}\frac{\partial Q^1_{11}}{\partial x_2} + Q^2_{22}\frac{\partial R^1_{11}}{\partial x_2}\}$$

$$-2\{R^1_{22}\frac{\partial Q^2_{11}}{\partial x_2} + Q^1_{22}\frac{\partial R^2_{11}}{\partial x_2}\}-\{R^2_{11}\frac{\partial Q^1_{22}}{\partial x_2} + Q^2_{11}\frac{\partial R^1_{22}}{\partial x_2}\}=0.$$

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$\tilde{H} = \cup_{i=1}^{7} \tilde{H}_i$

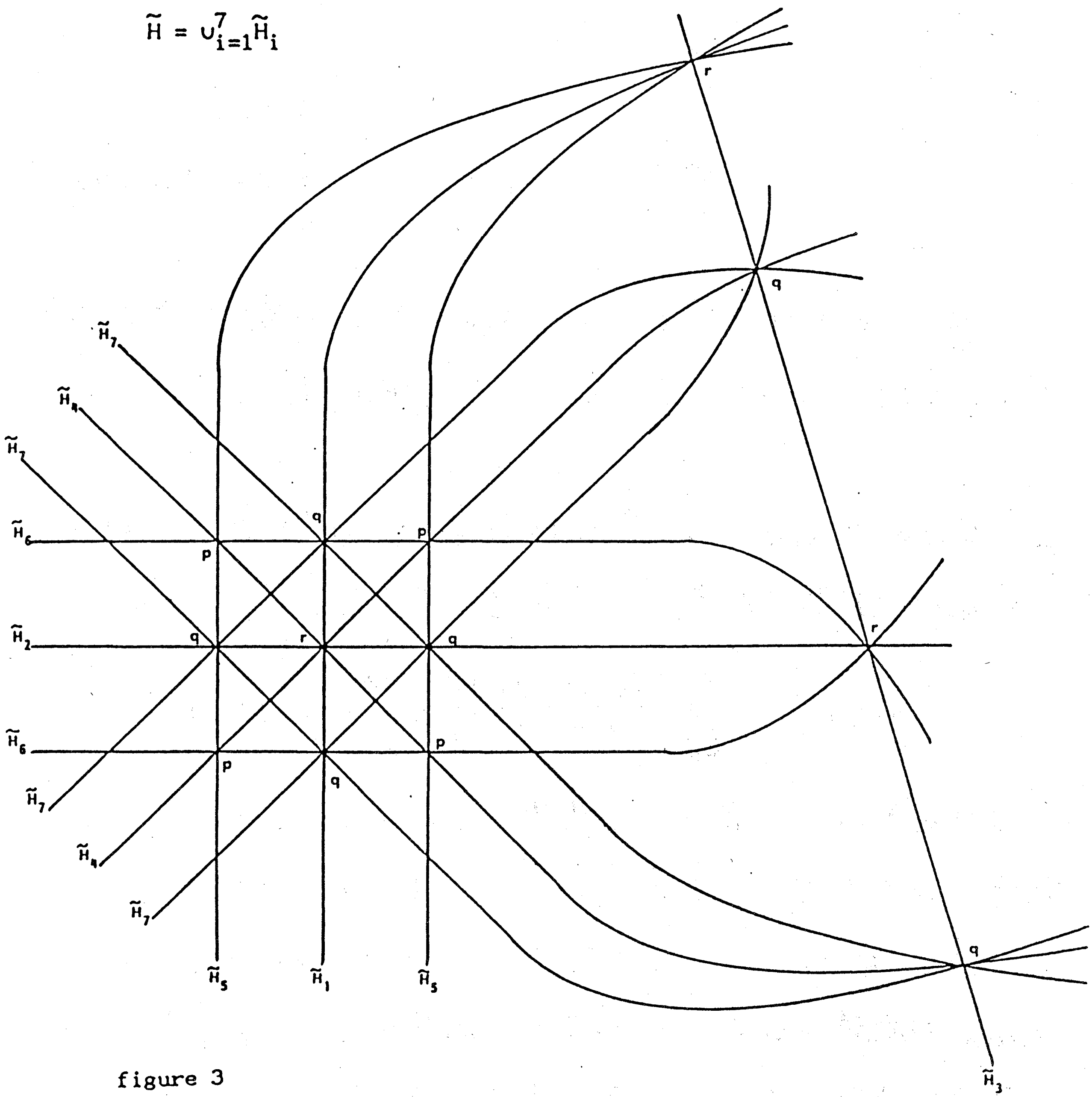

figure 3