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A Nonlinear Lattice and Volterra's System

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A new kind of nonlinear lattice is presented. This is not the usual dynamical system. It is asymmetric with respect to momentum, and consequently the motion is asymmetric in space. The equations of motion can be interpreted as a special case of Lotka-Volterra's equation of competing species forming a chain of preys and predators.

§1. Hamiltonian

The Hamiltonian of the exponential lattice is written as

$$H(x,p) = \sum_n \frac{p_n^2}{2} + \sum_n \left\{ e^{-(x_n - x_{n-1})} - 1 + (x_n - x_{n-1}) \right\}.$$

The potential $\phi(r) = e^{-r} - 1 + r$ reduces to quadratic $r^2/2$ when r is small. Similarly for small p, the kinetic term can be approximated by $e^{-p} - 1 + p$.

We may rather consider a system with the Hamiltonian

$$H(x,p) = \alpha^2 \sum_n \{e^{-p_n} - 1 + p_n\} + \sum_n \left\{e^{-(x_n - x_{n-1})} - 1 + (x_n - x_{n-1})\right\}, \tag{1}$$

where x_n and p_n are canonical conjugate variables, coordinate and momentum, and α is a constant. We may introduce some constants to change the potential term $e^{-r} - 1 + r$ to $\frac{a}{b}(e^{-br} - 1 + br)$ and to modify similarly the kinetic term $e^{-p} - 1 + p$. However by rescaling of coordinate, momentum and energy we can reduce the Hamiltonian to the above form, with a single parameter α.

Let us suppose (1) to hold for the infinite range of p_n and x_n ($-\infty < p_n < \infty, -\infty < x_n < \infty$). Then the canonical equations of motion are given as

$$\dot{x}_n = \frac{\partial H}{\partial p_n} = \alpha^2 \left(1 - e^{-p_n}\right), \tag{2}$$

$$\dot{p}_n = -\frac{\partial H}{\partial x_n} = e^{-(x_n - x_{n-1})} - e^{-(x_{n+1} - x_n)}. \tag{2'}$$

If we introduce

$$r_n = x_n - x_{n-1}, \tag{3}$$

we obtain the equations of motion in the form

$$\dot{r}_n = \alpha^2 \left(e^{-p_{n-1}} - e^{-p_n}\right), \tag{4}$$

$$\dot{p}_n = e^{-r_n} - e^{-r_{n+1}}, \tag{4'}$$

which are nearly symmetric with respect to r_n and p_n.

If we further write

$$\alpha \left(e^{-p_n} - 1\right) = I_n, \tag{5}$$

$$\alpha^{-1} \left(e^{-r_n} - 1\right) = V_n, \tag{5'}$$

or

$$p_n = -\log\left(1 + \frac{I_n}{\alpha}\right), \tag{6}$$

$$r_n = -\log(1 + \alpha V_n) \tag{6'}$$

and

$$\alpha t = \tau, \tag{7}$$

we obtain

$$-\frac{d}{d\tau}\log(\alpha^{-1} + V_n) = I_{n-1} - I_n \tag{8}$$

$$-\frac{d}{d\tau}\log(\alpha + I_n) = V_n - V_{n+1}. \tag{8'}$$

The set of equations (8) was already studied by Hirota and Satsuma.[1)2)] Soliton solutions and periopdic solutions revealed interesting properties, especially its non-reciprocal property in the sense that forward propagation and backward propagation are different.

When $\alpha >> 1$, p_n values are limited small, and the system reduces to the usual exponential lattice. When $\alpha = 1$, solitons can propagate only to the left.

§2. Lagrangean

The time rate of change of x_n, or the "velocity" v_n is given from (2) as

$$v_n = \dot{x}_n = \alpha^2\left(1 - e^{-p_n}\right). \tag{9}$$

Therefore we see that upper bound of v_n is limited ($-\infty < v_n < \alpha^2$). We have from (9)

$$p_n = \log\frac{\alpha^2}{\alpha^2 - \dot{x}_n}. \tag{10}$$

The Lagrangean of the system is given as

$$L(x,\dot{x}) = \sum_n \dot{x}_n p_n(\dot{x}_n) - H(x, p(\dot{x})), \tag{11}$$

which is

$$L(x,\dot{x}) = \sum_n \left\{ -(\alpha^2 - \dot{x}_n)\log\frac{\alpha^2}{\alpha^2 - \dot{x}_n} + \dot{x}_n \right\}$$
$$-\sum_n \left\{ e^{-(x_n - x_{n-1})} - 1 + (x_n - x_{n-1}) \right\}. \tag{12}$$

From this Lagrangean we have the momentum

$$p_n = \frac{\partial L}{\partial \dot{x}_n} = \log\frac{\alpha^2}{\alpha^2 - \dot{x}_n}, \tag{13}$$

which is the same to (10). The Lagrange equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_n} - \frac{\partial L}{\partial x_n} = 0 \tag{14}$$

gives

$$\frac{d}{dt}\log\frac{\alpha^2}{\alpha^2 - \dot{x}_n} - \left\{ e^{-(x_n - x_{n-1})} - e^{-(x_{n+1} - x_n)} \right\} = 0. \tag{15}$$

In view of (13), we see that (15) is the same to (2′) as it should be.

§3. Lotka-Volterra's System

The system under consideration has intimate connection to the Lotka-Volterra system of competing species,

$$\frac{dN_i}{dt} = \epsilon_i N_i + \frac{1}{\beta_i}\sum_j \alpha_{ji} N_j N_i \tag{16}$$

where N_i is the population of the i-th species, and ϵ_i, β_i and α_{ji} are parameters specifying the system. ϵ_i and α_{ji} are plus or negative, or zero. It is assumed that N_i has certain equilibrium value q_i, so that

$$\epsilon_i + \frac{1}{\beta_i}\sum_j \alpha_{ji} q_j = 0. \tag{17}$$

Further, we assume that (α_{ji}) is skew seymmetric,

$$\alpha_{ji} = -\alpha_{ij},$$

then it is known that the system has a conserved quantity $(\frac{dG}{dt} = 0)$

$$G = \sum_i q_i\beta_i \left(e^{-v_i} - 1 + v_i\right), \tag{18}$$

where v_i is defined by

$$N_i = q_i e^{-v_i}. \tag{19}$$

It is to be noted that (18) consists of the familiar functions of the form $e^{-v} - 1 + v$, used in (2) for momentum and interaction terms.

Lotka-Volterra's equation (16) can be written as

$$\frac{dv_i}{dt} = \frac{1}{\beta_i}\sum_j \alpha_{ji} q_j (1 - e^{-v_j}). \tag{20}$$

We may put

$$\begin{aligned} v_{2n} &= p_{n-1} \\ v_{2n+1} &= r_n. \end{aligned} \tag{21}$$

If $\alpha_{ji} = 0$ except that

$$\begin{aligned} \frac{1}{\beta_{2n+1}}\alpha_{2n,2n+1} q_{2n} &= -\alpha^2 \\ \frac{1}{\beta_{2n+1}}\alpha_{2n+2,2n+1} q_{2n+2} &= \alpha^2 \\ \frac{1}{\beta_{2n}}\alpha_{2n-1,2n} q_{2n-1} &= -1 \\ \frac{1}{\beta_{2n}}\alpha_{2n+1,2n} q_{2n+1} &= 1, \end{aligned} \tag{22}$$

then Lotka-Volterra's equation (20) reduces to our equations (4) and (4′), and therefore to (8) and (8′).

References

1) R.Hirota and J.Satsuma, J. Phys. Soc. Japan **40** (1976) 891.

2) R.Hirota and J.Satsuma, Prog. Theor. Phys. Suppl. **59** (1976) 64.