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On the Linear Classification of Singular Quartic Curves

群馬高専　高橋　正
(Tadashi Takahashi)

The linear classification of complex projective plane cubic curves is well known, we can list the normal forms of them.

In this paper, we try to impose a condition to constract a unique normal form. With a definition of normal forms, we find the linear classification of complex projective plane singular quartic curves.

§1. Singular point

We review some theorems and definitions about singular points and quasihomogeneous polynomials which are given in [2].

Definition 1.1 *Let* $f(z_0,\cdots,z_n)$ *be a polynomial in* $\mathbb{C}^{n+1}$ *and let* V *be an analytic set such that* $V=\{(z_0,\cdots,z_n)\,|\,f(z_0,\cdots,z_n)=0\}$. *Then a point* $(z_0,\cdots,z_n)$ *in* $\mathbb{C}^{n+1}$ *is a singular point if* $f(z_0,\cdots,z_n)$ $=0$ *and* $\dfrac{\partial f(z_0,\cdots,z_n)}{\partial z_i} = 0,\ i=0,\cdots,n.$

Definition 1.2 Suppose that $(r_0, \cdots, r_n)$ are fixed positive rational number. A polynomial $f(z_0, \cdots, z_n)$ is said to be quasihomogeneous of type$(r_0, \cdots, r_n)$ if it can be expressed as a linear combination of monomials $z_0^{i_0} z_1^{i_1} \cdots z_n^{i_n}$ for which $i_0 r_0 + i_1 r_1 + \cdots + i_n r_n = 1$.

Let d denote the smallest positive integer so that

$$r_0 = \frac{q_0}{d},\ r_1 = \frac{q_1}{d},\ \cdots,\ r_n = \frac{q_n}{d}$$

are integers. Then $f(t^{q_0} z_0, \cdots, t^{q_n} z_n) = t^d f(z_0, \cdots, z_n)$.

Theorem 1.3 Let $f(z_0, z_1, z_2)$ be a polynomial in $\mathbb{C}^3$ and let V be an anlytic set such that $V = \{(z_0, z_1, z_2) \mid f(z_0, z_1, z_2) = 0\}$ which has an isolated singular point at the origin. Then, for any $i(i=0,1,2)$,

(i) There exists an integer a_i so that $a_i \geq 2$, and f has a monomial $z_i^{a_i}$

or

(ii) There exists an integer $a_i \geq 1$ and $j(i \neq j)$ and f has a monomial $z_i^{a_i} z_j$.

Corollary 1.4 Let $g(z_0, z_1, z_2)$ be a quasihomogeneous polynomial in $\mathbb{C}^3$ and let V be an analytic set such that $V = \{(z_0, z_1, z_2) \mid g(z_0, z_1, z_2) = 0\}$ which has an isolated singular point at the origin. Then g has at least one of the followins sets (family **I~VIII**) of monomials:

family	set of monomials	r_0	r_1	r_2
I	$z_0^{a_0}$ $z_1^{a_1}$ $z_2^{a_2}$	$\frac{1}{a_0}$	$\frac{1}{a_1}$	$\frac{1}{a_2}$
II	$z_0^{a_0}$ $z_1^{a_1}$ $z_1 z_2^{a_2}$	$\frac{1}{a_0}$	$\frac{1}{a_1}$	$\frac{a_1-1}{a_1 a_2}$
III	$z_0^{a_0}$ $z_1^{a_1} z_2$ $z_1 z_2^{a_2}$ $a_1 \geq 2, a_2 \geq 2$	$\frac{1}{a_0}$	$\frac{a_2-1}{a_1 a_2-1}$	$\frac{a_1-1}{a_1 a_2-1}$
IV	$z_0^{a_0}$ $z_0 z_1^{a_1}$ $z_1 z_2^{a_2}$	$\frac{1}{a_0}$	$\frac{a_0-1}{a_0 a_1}$	$\frac{a_0 a_1-a_0+1}{a_0 a_1 a_2}$
V	$z_0^{a_0}$ $z_0 z_1^{a_1}$ $z_0 z_2^{a_2}$	$\frac{1}{a_0}$	$\frac{a_0-1}{a_0 a_1}$	$\frac{a_0-1}{a_0 a_2}$
VI	$z_0^{a_0} z_1$ $z_0 z_1^{a_1}$ $z_0 z_2^{a_2}$ $a_0 \geq 2, a_1 \geq 2$	$\frac{a_1-1}{a_0 a_1-1}$	$\frac{a_0-1}{a_0 a_1-1}$	$\frac{(a_0-1)a_1}{(a_0 a_1-1)a_2}$
VII	$z_0^{a_0} z_1$ $z_1^{a_1} z_2$ $z_0 z_2^{a_2}$	$\frac{a_1 a_2-a_2+1}{a_0 a_1 a_2+1}$	$\frac{a_2 a_0-a_0+1}{a_0 a_1 a_2+1}$	$\frac{a_0 a_1-a_1+1}{a_0 a_1 a_2+1}$
VIII	$z_0^{a_0}$ $z_1 z_2$	$\frac{1}{a_0}$	r_1	r_2

Table 1

§2. Normal form

2.1 Normal form to be unique

So-called "normal form" defining equations were not unique. For example, the defining equation of non-singular elliptic curve in Weierstrass normal form is different from it in Hesse normal form. (see [3]) What is the normal form? We define the normal form for the homogeneous polynomials in projective space to be unique. To begin with, we give a following order to the monomials of homogeneous polynomial $f = \sum a_i X^{K_i}$.

Definition 2.1.1 *For the exponents $K_i = k_{i_1}, \cdots, k_{i_n}$ and $K_j = k_{j_1}, \cdots, k_{j_n}$ ($i \neq j$), X^{K_i} is greater than X^{K_j} if there exists an integer $s (1 \leq s \leq n)$ such that $k_{i_\mu} = k_{j_\mu}$ for $\mu = 1, \cdots, s-1$ and $k_{i_s} > k_{j_s}$.*
(Lexicographic linear order)

Next, we carry out the following manipulations in turn from the maximal X^{K_n} ($K_n = m, 0, 0, \cdots, 0$) to the minimal $X^{K_{n'}}$ ($K_{n'} = 0, 0, \cdots, 0, m$) for the homogeneous polynomial $x_n{}^m + a_1 x_1{}^m + a_2 x_1{}^{m-1} x_2 + a_3 x_1{}^{m-1} x_3 + \cdots$.

Manipulation 2.1.2 *We try a monomial X^{K_i} to eliminate by suitable linear transformation. Then if we can make the monomial X^{K_i} to eliminate without generating the monomial X^{K_j} which is greater than X^{K_i}, we do so. Otherwise, we don't use the linear transformation and go to next manipulation.*

Manipulation 2.1.3 *If we can make the coefficient of the monomial X^{K_i} equal to 1 by the magnification of the coordinates without generating new dimension of coefficient of monomial X^{K_j} which is greater than X^{K_i}, we do so. Otherwise, there is nothing to be done.*

We define the normal form by the following definition.

Definition 2.1.4 *We call the results the normal forms of homogeneous polynomials of degree m in $(n-1)$-dimensional complex projective space.*

We consider it natural that the normal form should be easy to write and remember; that is, the normal form should have the fewest monomials, and each monomial should be simple. The normal forms defined in Definition 2.1.4 meet the above conditions.

2.2 Classification theory

The results of §15 in [1] reduce the classification of orbits of of the action of the group of quasihomogeneous diffeomorphisms on the space of quasihomogeneous functions with fixed quasihomogeneity type and with fixed coefficients for certain distinguished monomials. The run of the classifications is as follows. First we compute the exponents of homogeneity and with respect to them the monomial vector fields that generate over $\mathbb{C}$ the Lie algebra of the group of all quasihomogeneous diffeomorphisms. Fixing the coefficients of the distinguished monomials gives an affine plane in the space of quasihomogeneous functions. We find the isotropy algebra of this plane. The calculations show (unexpectedly) that the actions of the Lie algebras within the limits of each of our series of singularities are affinely equivalent to each other. Thus, our classification reduces to a finite number of steps. The linear classifications of 3-forms in $\mathbb{C}^2$ and $\mathbb{C}^3$ and of 4-forms in $\mathbb{C}^2$ are well known.

We introduce some definitions and notation.

Let $\alpha=(\alpha_1,\cdots,\alpha_n)$ be a set of positive rational numbers. We consider the arithmetic space $\mathbb{C}^n$ with fixed coordinates $x_1,\cdots,x_n$.

Definition 2.2.1 *A quasihomogeneous function of degree d and type α is a polynomial $\sum a_{\mathbf{m}}x^{\mathbf{m}}\in \mathbb{C}[x_1,\cdots,x_n]$ for which $(\mathbf{m},\alpha)=d$ for all $\mathbf{m}$.*

Definition 2.2.2 *The group of quasihomogeneous diffeomorphisms of type α is the group of germs of diffeomorphisms of $\mathbb{C}^n$ at 0 taking any quasihomogeneous function of type α and degree d into a quasihomogeneous function of the same degree. The Lie algebra of this group is called the quasihomogeneous algebra and is denoted by* $\mathfrak{a}(\alpha)$.

Definition 2.2.3 *The space of exponents of functions of $x_1,\cdots,x_n$ is the arithmetic space $\mathbb{C}^n$ whose points $\mathbf{m}$ with non-negative integer coordinates are the exponents of the monomials $\mathbf{x}^{\mathbf{m}}$.*

Definition 2.2.4 *The support of quasihomogeneous functions of degree d and type α is the set of all non-negative integer points $\mathbf{m}$ on the plane $(\mathbf{m},\alpha)=d$. The support is said to be complete if it is not contained in an affine subspace of $\mathbb{C}^n$ of dimension less than $n-1$.*

Quasihomogeneous functions can be regarded as functions given on the support($\sum a_m \mathbf{x}^m$ has the value a_m at m). All such functions form a linear space $\mathbb{C}^\nu$, ν is the number of points of the support. The group of quasihomogeneous diffeomorphisms and the quasihomogeneous algebra $\mathfrak{a}(\alpha)$ act on this space $\mathbb{C}^\nu$. From the definitions it follows immediately that the Lie algebra $\mathfrak{a}(\alpha)$ is generated as a $\mathbb{C}$-linear space by all the monomial vector fields $\mathbf{x}^p \partial_i$ for which $(p,\alpha)=\alpha_i$ (here and later on, $\partial_i=\partial/\partial x_i$).

Definition 2.2.5 *The roots of the quasihomogeneous algebra* $a(\alpha)$ *are all the non-zero vectors* m *of the space of exponents that lie in the plane* $(m,\alpha)=0$ *and have the form* $m=p-1_i$ *(where* 1_i *is the vector whose i-th component is equal to 1 and the remainder equal to 0, and the vector p has non-negative integer components).*

We observe that i can be regained from a root m, since m has precisely one negative coordinate $m_i=-1$ (not all the components of m can be non-negative because $(m,\alpha)=0$).

Theorem 2.2.6 *We assume that the support is complete. Then the action of the Lie algebra* $\mathfrak{a}(\alpha)$ *on the space of functions on the support is uniquely determined with respect to the affine equivalence class of the pair (spport, root system).* (see [1])

Theorem 2.2.7 *The quasihomogeneous Lie algebra* $\mathfrak{a}(\alpha)$ *is determined up to finitely many variants, by its root system (as a subspace of the linear space spanned by the roots) and by its* *dimension.* (see [1])

Remark. We must emphasize that the affine equivalences of supports and linear equivalences of root systems in Theorems 2.2.6 & 2.2.7 are not at all necessary to preserve either the coordinate simplex $m_i \geq 0$ on the plane $(m,\alpha)=d$ or the lattice of non-negative integral m in $\mathbb{C}^n$. We also observe that under the hypotheses of Theorems 2.2.6 and 2.2.7 the groups of quasihomogeneous diffeomorphisms and their orbits in spaces of quasihomogeneous functions do not necessarily coincide, however, the connected components of the orbits coincide. (see [1])

Lemma 2.2.8 *The space* $\mathfrak{b} = H \oplus \mathbb{C}^{\nu}$ *can be given the following Lie algebra structure:*

(1) $[h_1,h_2]=0$ ($\forall h_1,h_2 \in H$);

(2) $[h,e_m]=(h,m)e_m$ ($\forall h \in H,\ m \in M$);

(3) $[e_{m_1},e_{m_2}]=N_{m_1,m_2}$, *where* $m_1+m_2 \neq 0$,

$$N_{m_1,m_2}\begin{cases} = 0 \text{ if } m_1+m_2 \text{ is not a root;} \\ = -\max\{\lambda:\ m_1+\lambda m_2 \text{ is a root}\} \text{ if this maximum is } > 1; \\ = +\max\{\lambda:\ m_2+\lambda m_1 \text{ is a root}\} \text{ if this maximum is } > 1; \\ = \pm 1 \text{ if both maxima} = 1 \text{ (the case when both maxima are} \\ \quad > 1 \text{ is impossible).} \end{cases}$$

(4) $[e_m,e_{-m}]=h_m$, *where the function* $h_m \in H$ *changes sign the reflection of* $\mathbb{C}^r$ *that preserves M and takes m into -m, normalized by the condition* $h_m(m)=2$ *(such a reflection exists and is unique for any pair of opposite roots).*

The quasihomogeneous Lie algebra $\mathfrak{a}(\alpha)$ is isomorphic to the direc sum of the Lie algebra $\mathfrak{b}$ (for some choice of the sign ± in (3)) and a trivial (commutative) algebra: $\mathfrak{a}(\alpha) \simeq \mathfrak{b} \oplus \mathbb{C}^{n-r}$. (see [1])

Corollary 2.2.9 *Suppose that the set of weithts α and the degree d are such that there exists a quasihomogeneous function with an isolated critical point 0 with zero 2-jet. Then the root system and the support uniquely determine the Lie algebra $\mathfrak{a}(\alpha)$ and its action φ.* (see [1])

Corollary 2.2.10 *Let γ: $\mathbb{C}_1^{n-1} \to \mathbb{C}_2^{n-1}$ be an affine isomorphism of a complete support plane S_1 into a complete support plane S_2, taking S_1 into part of S_2, the roots of $\mathfrak{a}_1$ into part of the roots of $\mathfrak{a}_2$, and the bases of the roots $\mathfrak{a}_1$ into part of the bases of the corresponding roots of $\mathfrak{a}_2$. Then γ induces an isomorphism of the action φ of $\mathfrak{a}_1$ onto functions on S_1 and an isomorphism of the action of a subalgebra of $\mathfrak{a}_2$ onto the space of functions on S_2 that vanish outside $\gamma(S_1)$.*

Corollary 2.2.11 *Suppose that under the hypotheses of Corollary several points in the support S_1 are distinguished and the values of the functions are fixed at them. All the functions on S_1 with fixed values at these points from an affine plane P in the space of functions on S_1. Let $\mathfrak{a}_p$ be the isotopy algebra of this plane P.* Then the isomorphism γ of Corollary 2.2.10 induces an isomorphism of *the action of $\mathfrak{a}_p$ on P with the action of some subalgebra $\mathfrak{a}_2$, preserving the plane $\gamma^{-1*}P$ in the space of functions on S_2.* (see [1])

The reduction of the classification steps is carried out by means of Corollary 2.2.11. We indicate the monomials of the support by the following signs: those distinguished by the sign O have the coefficient 0, those by the sign ① have the coefficient 1, those by the sign ⊛ have the non-zero coefficient and the others by the sign ● have the parametric coefficient. And we place the signs into a equilateral triangle. (Fig. 2.2)

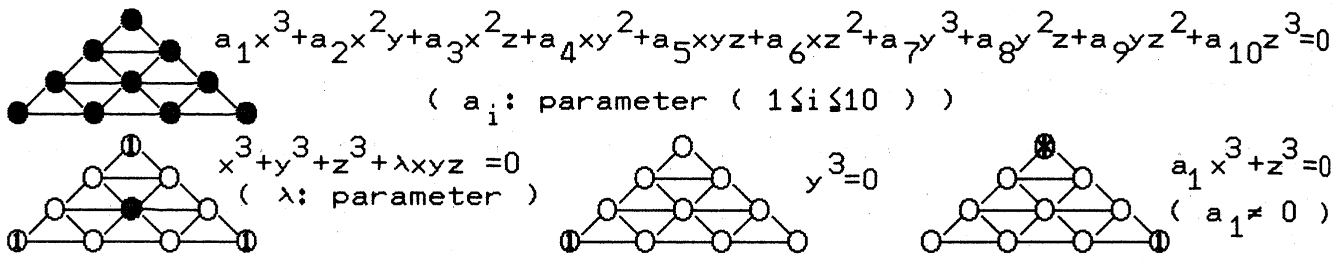

Fig. 2.2

This diagram is very convenient for the computations. According to Corollary 2.2.11, the orbits of the identity component of the group of quasihomogeneous diffeomorphisms that fix the plane P (consisting of functions on the support with the values 0 and 1 at the points distinguished by the signs O and ①) are transformed under a mapping onto the support of the homogeneous functions into orbits of the corresponding group of linear transformations. (This circumstance is a nice property of the roots in the relevant cases, in general, the images of the roots for $a(\alpha)$ in the homogeneous support need not also be roots of the linear group.)

The Lie algebra of this group of linear transformations is easy to describe: it is generated by a torus part, acting on a function on the support as multiplication by affine functions that are zero at the points of the support distinguished by the sign ⓵ (at which a non-zero value of the function is fixed) and by the images of those root vectors e_m that do not take other points of the support into points distinguished by the sign O.

§3. Normal forms of singular quartic curves

If the analytic set have singular points, we can take the singular point at [1,0,0] in $\mathbf{P}^2$ by the arbitrary linear transformation. Then the defining equation takes following form:

$$f_2(y,z)x^2+f_3(y,z)x+f_4(x,y)=0$$

where f_i denotes a homogeneous polynomial of degree i (i=2,3,4). $f_2(y,z)=0$ gives two points on a projective line. By the Definition 2.1.4 for normal forms we obtain the following classification.

$f_2(y,z)\sim yz \to$ Step 1 , $f_2(y,z)\sim z^2 \to$ Step 2 , $f_2(y,z)\equiv 0 \to$ Step 3

Step 1

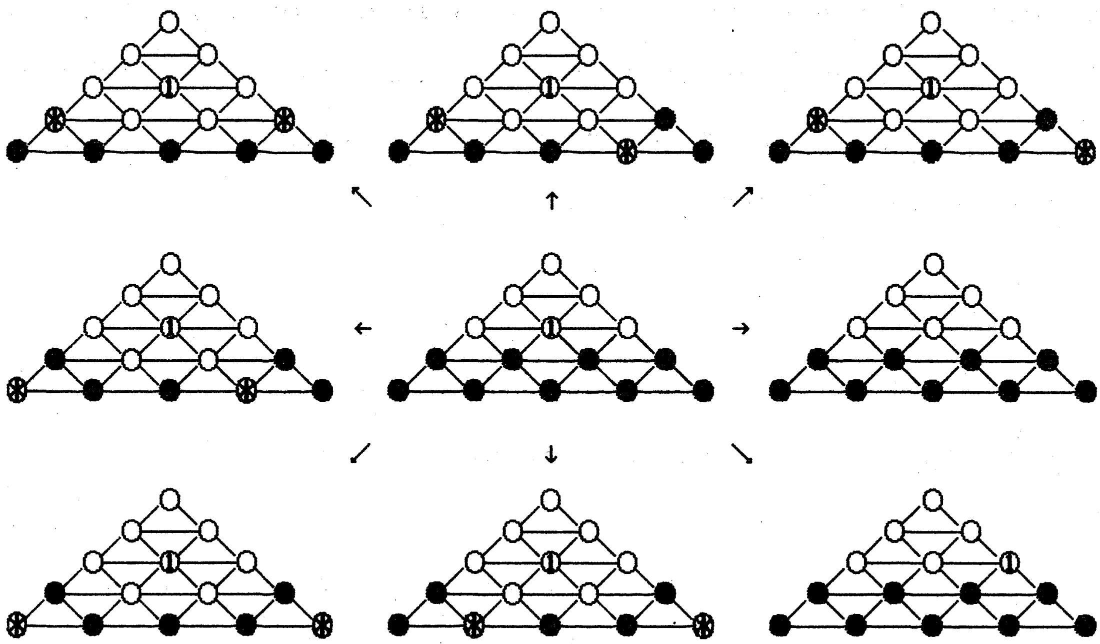

Fig. 1

$x^2yz+a_1xy^3+a_2xy^2z+a_3xyz^2+a_4xz^3+a_5y^4+a_6y^3z+a_7y^2z^2+a_8yz^3+a_9z^4:=0.$

Replacing x by $x'-\frac{a_2}{2}y-\frac{a_3}{2}z$, we reduce the form to

$x^2yz+g_1xy^3+g_2xz^3+g_3y^4+g_4y^3z+g_5y^2z^2+g_6yz^3+g_7z^4=0.$

If $(g_1,g_3,g_4)=(0,0,0)$ or $(g_2,g_6,g_7)=(0,0,0)$, changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$ or $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$ respectively, we reduce the form to $g_5x^2z^2+g_4xz^3+g_1yz^3+g_3z^4=0.$

$g_5\neq0 \rightarrow$ Step 2 , $g_5=0 \rightarrow$ Step 3

Step 2

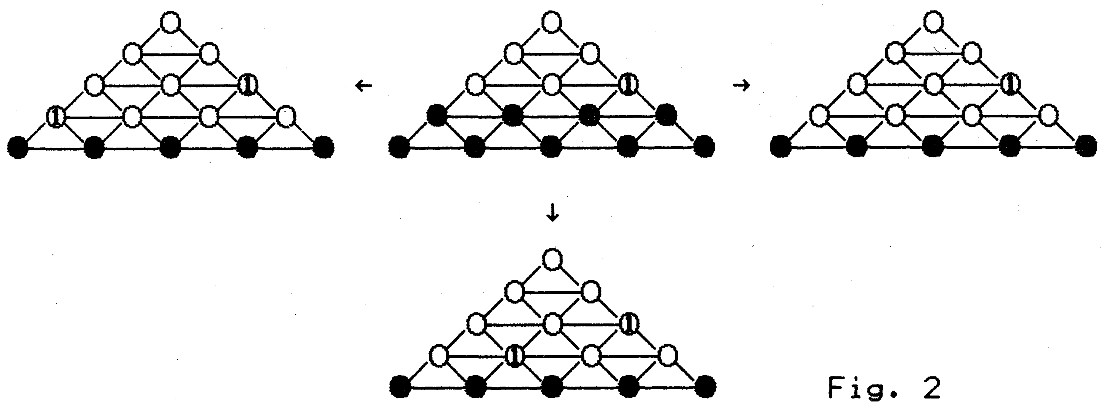

Fig. 2

Case 1: $x^2z^2+xy^3+a_1xy^2z+a_2xyz^2+a_3xz^3+a_4y^4+a_5y^3z+a_6y^2z^2+a_7yz^3+a_8z^4:=0$.

Case 2: $x^2z^2+xy^2z+a_1xyz^2+a_2xz^3+a_3y^4+a_4y^3z+a_5y^2z^2+a_6yz^3+a_7z^4:=0$.

Case 3: $x^2z^2+a_1xyz^2+a_2xz^3+a_3y^4+a_4y^3z+a_5y^2z^2+a_6yz^3+a_7z^4:=0$.

Case 1:

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&\alpha&\beta\\0&1&\gamma\\0&0&1\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $x^2z^2+xy^3+g_1xy^2z+g_2xyz^2+g_3xz^3+g_4y^4+g_5y^3z+g_6y^2z^2+g_7yz^3+g_8z^4=0$. Here, we set as folloes:

$\alpha=\dfrac{a_1{}^2-3a_2}{6}$, $\beta=\dfrac{-2a_1{}^3+9a_1a_2-27a_3}{54}$, $\gamma=\dfrac{-a_1}{3}$. Then we reduce the form to $x^2z^2+xy^3+h_1y^4+h_2y^3z+h_3y^2z^2+h_4yz^3+h_5z^4=0$.

Case 2:

Replacing x by $x'-\dfrac{a_1}{2}y-\dfrac{a_2}{2}z$, we reduce the form to $x^2z^2+xy^2z+g_1y^4+g_2y^3z+g_3y^2z^2+g_4yz^3+g_5z^4=0$.

Case 3:

Replacing x by $x'-\dfrac{a_1}{2}y-\dfrac{a_2}{2}z$, we reduce the form to $x^2z^2+g_1y^4+g_2y^3z+g_3y^2z^2+g_4yz^3+g_5z^4=0$. Go to Step 4.

Step 3

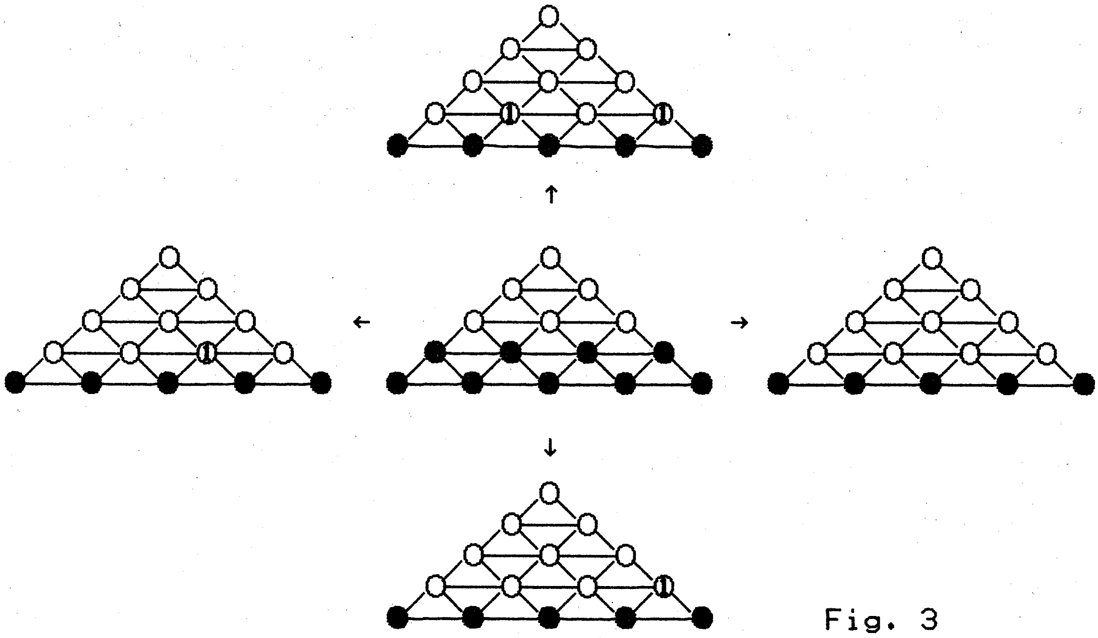

Fig. 3

In this case, the form is as follows: $f_3(y,z)x+f_4(y,z)=0$ where $f_i(y,z)$ denotes a homogeneous polynomial of degree i. $f_3(y,z)=0$ gives three points on a projective line. By the Definition 2.1.4 for normal forms we obtain the following classification.

$f_3(y,z)\sim y^2z+z^3$ → Step 5 , $f_3(y,z)\sim yz^2$ → Step 6 ,

$f_3(y,z)\sim z^3$ → Step 7 , $f_3(y,z)\equiv 0$ → Step 8

Step 4

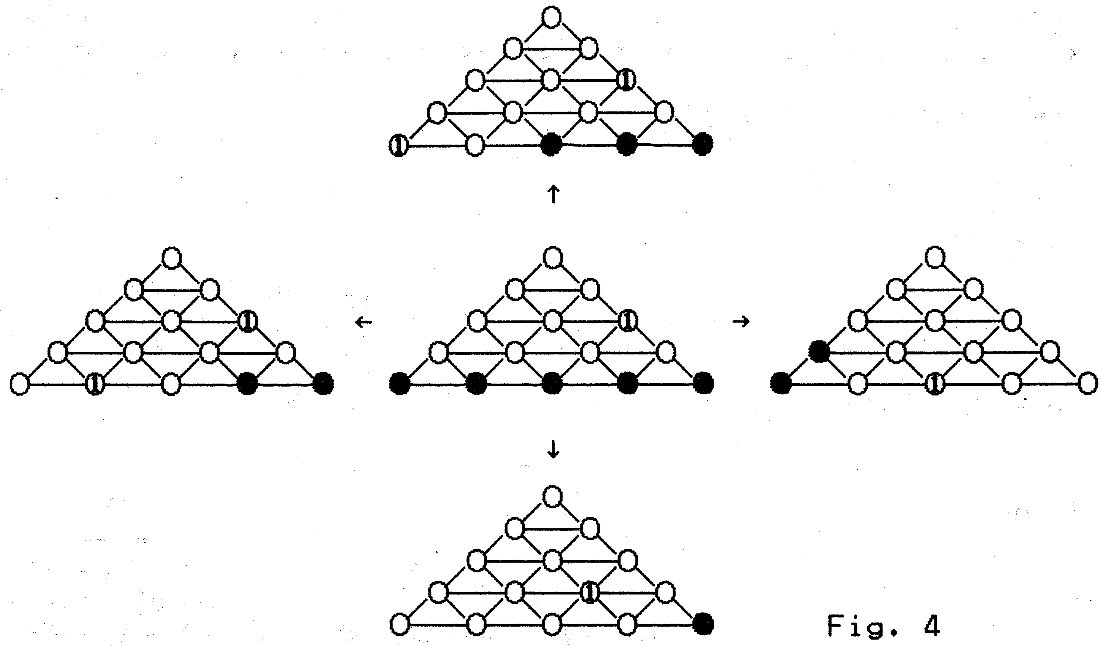

Fig. 4

Case 1: $x^2z^2+y^4+a_1y^3z+a_2y^2z^2+a_3yz^3+a_4z^4:=0.$

Case 2: $x^2z^2+y^3z+a_1y^2z^2+a_2yz^3+a_3z^4:=0.$

Case 3: $x^2z^2+y^2z^2+a_1yz^3+a_2z^4:=0.$

Case 4: $x^2z^2+a_1yz^3+a_2z^4:=0.$

Case 1:

Replacing y by $y' - \frac{a_1}{4}z$, we reduce the form to $x^2z^2+y^4+g_1y^2z^2+g_2yz^3+g_3z^4=0.$

Case 2:

Replacing y by $y' - \frac{a_1}{3}z$, we reduce the form to $x^2z^2+y^3z+g_1yz^3+g_2z^4=0.$

Case 3:

Replacing y by $y' - \frac{a_1}{2}z$, we reduce the form to

$x^2z^2+y^2z^2+g_1z^4=0$. And replacing x by $\frac{x'+y'}{2}$, y by $\frac{x'-y'}{2i}$, we reduce the form to $xyz^2+g_1z^4=0$.

Case 4:

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $a_1xy^3+a_2y^4+y^2z^2=0$. Go to Step 3.

Step 5

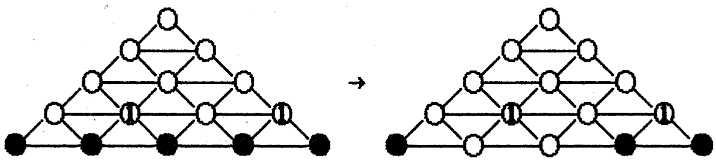

Fig. 5

$xy^2z+xz^3+a_1y^4+a_2y^3z+a_3y^2z^2+a_4yz^3+a_5z^4:=0$.

Replacing x by $x'-a_2y-a_3z$, we reduce the form to $xy^2z+xz^3+g_1y^4+g_2yz^3+g_3z^4=0$.

Step 6

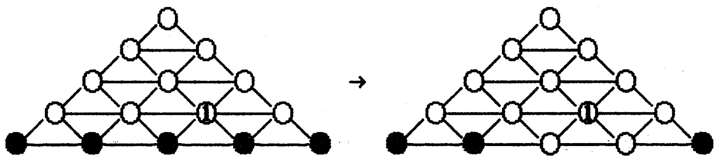

Fig. 6

$xyz^2+a_1y^4+a_2y^3z+a_3y^2z^2+a_4yz^3+a_5z^4:=0$.

Replacing x by $x'-a_3y-a_4z$, we reduce the form to $xyz^2+a_1y^4+a_2y^3z+a_5z^4=0$.

Step 7

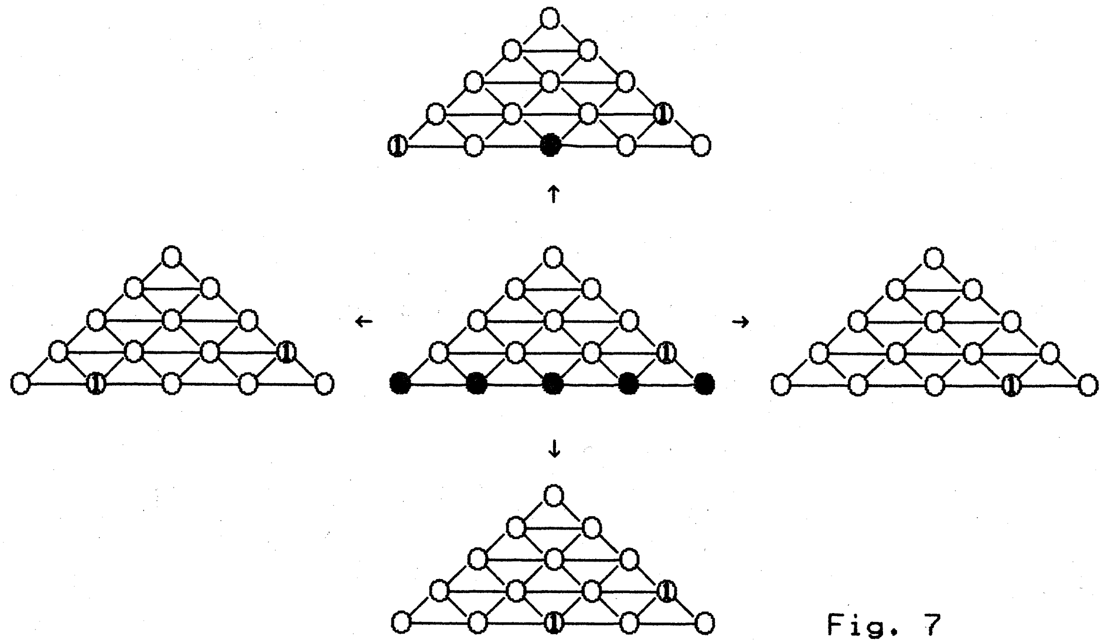

Fig. 7

Case 1: $xz^3+y^4+a_1y^3z+a_2y^2z^2+a_3yz^3+a_4z^4:=0$.

Case 2: $xz^3+y^3z+a_1y^2z^2+a_2yz^3+a_3z^4:=0$.

Case 3: $xz^3+y^2z^2+a_1yz^3+a_2z^4:=0$.

Case 4: $xz^3+a_1yz^3+a_2z^4:=0$.

Case 1:

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&\alpha&\beta\\0&1&\gamma\\0&0&1\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $xz^3+y^4+g_1y^3z+g_2y^2z^2+g_3yz^3+g_4z^4=0$. Here, we set as folloes:

$$\alpha = \frac{-a_1^3+4a_1a_2-8a_3}{8},\quad \beta = \frac{3a_1^4-16a_1^2a_2+64a_1a_3-256a_4}{256},\quad \gamma = \frac{-a_1}{4}.$$

Then we reduce the form to $xz^3+y^4+g_1y^2z^2=0$.

Case 2:

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&\alpha&\beta\\0&1&\gamma\\0&0&1\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $xz^3+y^3z+g_1y^2z^2+g_2yz^3+g_3z^4=0$. Here, we set as folloes:

$$\alpha = \frac{a_1^{\,2}-3a_2}{3}, \quad \beta = \frac{-2a_1^{\,3}+9a_1a_2-27a_3}{27}, \quad \gamma = \frac{-a_1}{3}.$$

Then we reduce the form to $xz^3+y^3z=0$.

Case 3:

Replacing x by $x'-a_1y-a_2z$, we reduce the form to $xz^3+y^2z^2=0$.

Case 4:

Changing the coordinates so that $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -a_1 & 1 & -a_2 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$. We reduce the form to $yz^3=0$. Go to Step 8.

Step 8

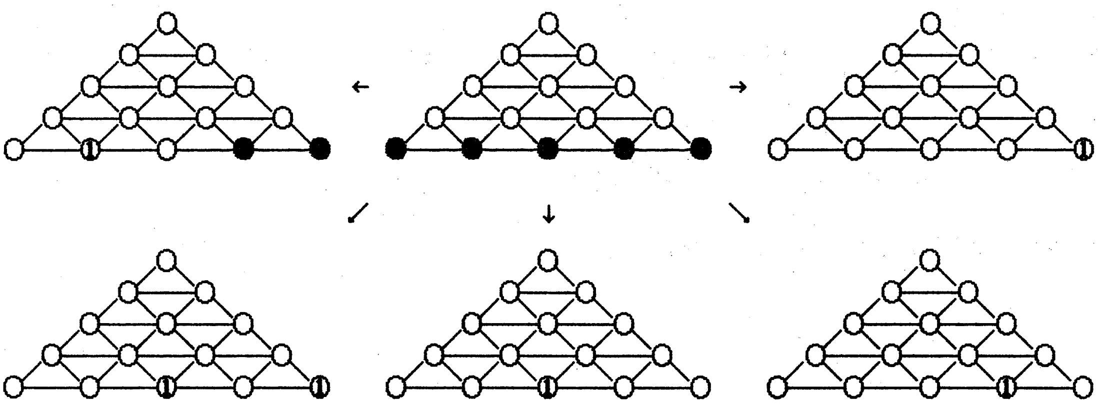

Fig. 8

In this case, the form is a homogeneous polynomial of degree 4 for two variables y,z. We denote the form by $f_4(y,z)$. Then $f_4(y,z)=0$ gives four points on a projective line. By the definition 2.1.4 for the normal forms we obtain the following classification.

$f_4(y,z) \sim y^3z+3a_1yz^3+a_2z^4 \quad (4a_1^{\,3}+a_2^{\,2}\neq 0)$,

$f_4(y,z) \sim y^2z^2+z^4$, $f_4(y,z) \sim y^2z^2$, $f_4(y,z) \sim yz^3$, $f_4(y,z) \sim z^4$

By results in Step 1~8 we obtain the linear classification of singular quartic curves in $\mathbf{P}^2$ and their normal forms.

Lemma 3 *The defining equation of singular quartic curve in* $\mathbf{P}^2$ *is linear equivalent to one of following fifteen cases:*

I : $x^2yz+a_1xy^3+a_2xz^3+a_3y^4+a_4y^3z+a_5y^2z^2+a_6yz^3+a_7z^4=0$,

$(a_1,a_3,a_4)\neq(0,0,0)$ and $(a_2,a_6,a_7)\neq(0,0,0)$

II : $x^2z^2+xy^3+a_1y^4+a_2y^3z+a_3y^2z^2+a_4yz^3+a_5z^4=0$,

III : $x^2z^2+xy^2z+a_1y^4+a_2y^3z+a_3y^2z^2+a_4yz^3+a_5z^4=0$,

IV : $x^2z^2+y^4+a_1y^2z^2+a_2yz^3+a_3z^4=0$,

V : $x^2z^2+y^3z+a_1yz^3+a_2z^4=0$,

VI : $xy^2z+xz^3+a_1y^4+a_2yz^3+a_3z^4=0$,

VII : $xyz^2+a_1y^4+a_2y^3z+a_3z^4=0$,

VIII : $xz^3+y^4+a_1y^2z^2=0$,

IX : $xz^3+y^3z=0$,

X : $xz^3+y^2z^2=0$,

XI : $y^3z+3a_1yz^3+a_2z^4=0$ $\quad(4a_1^3+a_2^2\neq0)$,

XII : $y^2z^2+z^4=0$,

XIII : $y^2z^2=0$,

XIV : $yz^3=0$,

XV : $z^4=0$.

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