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Weakly k-linked graphs

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1. はじめに

多重辺は含んでもよいが、ループは含まない有限無向グラフを考える。Gをグラフとし、$V(G)$はGの点の集合、$E(G)$はGの辺の集合とする。パスまたはサイクルは、1つの辺を高々1回しか通れないが、同じ点を2回以上通ってもよいことにする。グラフGの辺連結度を$\lambda(G)$であらわす。

定義1. Gが weakly k-linked

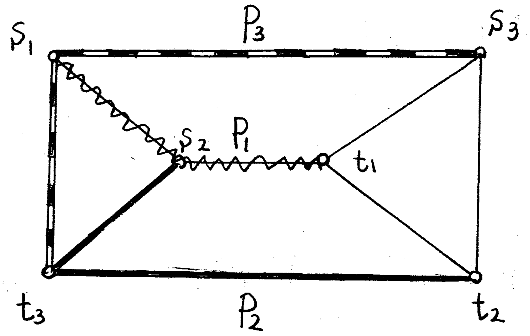
$\Longleftrightarrow$ Gの任意のk組の点の対(重複点を含んでもよい) $(s_1, t_1), \cdots, (s_k, t_k)$に対して、辺素な(i.e. 互いに辺を共有しない)パス$P_1, P_2, \cdots, P_k$があって、P_iはs_iとt_iを結ぶ($1 \le i \le k$)。

定義2. $g(k) := \min\{ n \mid もし\ \lambda(G) \ge n \Rightarrow G は\ \text{weakly } k\text{-linked} \}$。

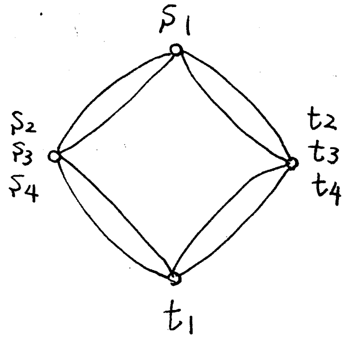
次の不等式がすぐに導ける。

(1.1)
$$g(k) \ge \begin{cases} k & (kが\ \text{odd}) \\ k+1 & (kが\ \text{even}) \end{cases}$$
。

例 1.



$\lambda(G)=3$, G は weakly 3-linked 。



$\lambda(G)=4$、G は weakly 4-linked ではない

次のことが既知である。

(1.2) $g(2)=g(3)=3$ (Okamura[4]),

$g(4)=5$ (Hirata, Kubota, Saito [1], Mader[3]),

$g(2k+1)\le 3k$, $g(2k)\le 3k-1$ $(k\ge 2)$ (Okamura[8]).

次の結果がえられた。

<u>定理1.</u> $g(3k)\le 4k$, $g(3k+1)\le g(3k+2)\le 4k+2$ $(k\ge 2)$

定理1を証明するために、定理2が必要である。

<u>定理2.</u> $k\ge 4$は偶数, G は k-辺連グラフ、$\{s,t\}\in V(G)$, $f\in E(G)$ とする。このとき、s と t を結ぶ f を通らないパス P があって、$\lambda(G-E(P)-f)\ge k-2$。

関連する話題が、[5], [6], [7] でも議論されている。

Notations and definitions

Let $X,Y,\{x,y\}\subset V(G)$, $f\in E(G)$ and $X\cap Y=\emptyset$. We often denote $\{x\}$ by x. $V(f)$ denotes the set of end vertices of f. We denote by $\partial(X,Y;G)$ the set of edges with one end in X and the other in Y, and set $\partial(X;G):=\partial(X,V(G)-X;G)$, $e(X,Y;G):=|\partial(X,Y;G)|$ and $e(X;G):=|\partial(X,V(G)-X;G)|$. $\lambda(x,y;G)$ denotes the maximal number of edge-disjoint paths between x and y. We set $\bar{X}:=V(G)-X$, $N(x;G):=\{a\in V(G)-x \mid e(a,x)>0\}$, $N(X;G):=\bigcup_{x\in X} N(x;G)$, and $\Gamma(G,k):=\{Z\subset V(G) \mid \text{for each } a,b\in Z,\ \lambda(a,b;G)\geq k\}$. In all notations, we often omit G. G/X denotes the graph obtained from G by contracting X, and for $a\in X$, we denotes the corresponding vertex in G/X by $\hat{a}$. A path $P=P[x,y]$ denotes a path between x and y, and for $a,b\in V(P)$, $P(a,b)$ denotes a subpath of P between a and b. We call $X\subset V(G)$ a k-set if $|X|\geq 2$, $|\bar{X}|\geq 2$ and $e(X)=k$, and a k-set X is called minimum if for each $Y\subsetneq X$ with $|Y|\geq 2$, $e(Y)\geq k+1$. For $a,b\in N(x)$ with $a\neq b$, $f\in\partial(x,a)$ and $g\in\partial(x,b)$, $G_x^{a,b}$ denotes the graph $(V(G),(E(G)\cup h)-\{f,g\})$, where h is a new edge between a and b and is called a lifting of G at x arising from the lifting of f and g at x. We call $G_x^{a,b}$ admissible if for each $y,z\in V(G)-x$ with $y\neq z$, $\lambda(y,z;G_x^{a,b})=\lambda(y,z;G)$.

2. Preliminaries

In this section we assume that $k\geq 1$ is an integer and G is a graph.

Lemma 2.1 (Mader [3] and [5]). If $k\geq 2$, $\lambda(G)\geq k$, $s\in V(G)$, and $\{f_1,f_2\}\subset\partial(s)$, then there exists a cycle C such that $\{f_1,f_2\}\subset E(C)$ and $\lambda(G-E(C))\geq k-2$.

Lemma 2.2 (Mader [2]). If $x\in V(G)$, $e(x)\geq 4$, $|N(x)|\geq 2$, and x is not a cut-vertex, then there exists an admissible lifting of G at x.

Lemma 2.3 ([8,Lemma 3]). If $k\geq 3$, $V(G)=W_1\cup W_2$, $W_1\cap W_2=\emptyset$, $W_1\in\Gamma(G,k)$, and each $x\in W_2$ has even degree, then we can obtain a k-edge-connected graph $G(W_1,k)$ from G such that $W_1\subset V(G(W_1,k))$ by sequences of vertex-deletions and edge-liftings.

Lemma 2.4 ([8]). If $k\geq 4$ is even , $\lambda(G)\geq k$, and $s,t,a\in V(G)$ ($s=t$ or $s\neq t$), then there exists a path $P[s,t]$ such that $a\in V(P)$ and $\lambda(G-E(P))\geq k-2$.

Lemma 2.5. If $X\subset V(G)$, $e(X)=k$, and $\lambda(G/X)=\lambda(G/\bar{X})=k$, then $\lambda(G)=k$.

Lemma 2.6. If $\lambda(G)\geq k$, $X,Y\subset V(G)$, $X-Y$, $Y-X$, $X\cap Y$, and $\overline{X\cup Y}$ are not empty, and $e(X)=e(Y)=k$, then k is even and $e(X-Y)=e(Y-X)=e(X\cap Y)=k$.

Proof. By simple counting we have

$$e(X-Y)+e(Y-X)=e(X)+e(Y)-2e(X\cap Y,\overline{X\cup Y}),$$
$$e(X\cap Y)+e(X\cup Y)=e(X)+e(Y)-2e(X-Y,Y-X).$$

Thus $e(X-Y)=e(Y-X)=e(X\cap Y)=k$, and $k=e(X)=e(X-Y)+e(X\cap Y)=0 \pmod 2$.

Lemma 2.7. Suppose that $\lambda(G)=k\geq 3$ and $|V(G)|\geq 4$. Then

(1) If k is odd, G is k-regular, and $x\in V(G)$, then $|N(x)|\geq 3$.

(2) if k is even, $(x,y)\subset V(G)$, $e(x)=k$, and $e(y)\leq k+1$, then $e(x,y)\leq k/2$.

Proof. (1) If $|N(x)|\leq 2$, then for some $y\in N(x)$,

$e(x,y)\geq(k+1)/2$ and $e(\{x,y\})\leq k-1$.

(2) If $e(x,y)\geq k/2+1$, then $e(\{x,y\})=e(x)+e(y)-2(k/2+1)\leq k-1$.

3. Proof of Theorem 2

If $V(f)=\{s,t\}$, then by Lemma 2.1, for a $g\in\partial(s)-f$, G has a cycle C such that $\{f,g\}\subset E(C)$ and $\lambda(G-E(C))\geq k-2$, and the result holds. If $V(f)=\{a,s\}$ and $a\neq t$, then by Theorem 1(1) in [5] G has a path P[a,t] such that $f\in E(P)$ and $\lambda(G-E(P))\geq k-2$. Thus let $V(f)\cap\{s,t\}=\emptyset$, and set $T:=V(f)\cup\{s,t\}$. We may assume (see the proof of Theorem 2 and Figures 2,3 in [8])

(3.2) For each $x\in T$, $e(x)=k$, and for each $x\in V(G)$, $e(x)=k$ or $k+1$.

We proceed by induction on $|E(G)|$. We assume that the result does not hold in G. Then

(3.3) $e(s,t)=0$.

(3.4) $V(G)-T\neq\emptyset$.

Proof. Assume $V(G)=T$. Let $V(f)=\{a,b\}$. Then

$$2e(s,t)=e(s)+e(t)-e(\{s,t\})=e(a)+e(b)-e(\{a,b\})=2e(a,b)>0.$$

(3.5) If $X\subset V(G)-T$ and $|X|\geq 2$, then $e(X)\geq k+1$.

Proof. Assume $e(X)=k$ and $x\in X$. By induction G/X has a required path P[s,t]. If $\tilde{x}\notin V(P)$, then P is a required path for G, thus let $\tilde{x}\in V(P)$ and $E(P)\cap\partial(\tilde{x};G/X)=\{g_1,g_2\}$. By Lemma 2.1 $G/\bar{X}$ has a cycle C such that $\{g_1,g_2\}\subset E(C)$ and $\lambda(G/\bar{X}-E(C))=k-2$. By combining P and C in G, we get a required path for G (see Lemma 2.5).

(3.6) If $x \in V(G)-T$, then $e(x)=k+1$.

Proof. Assume $e(x)=k$. By Lemma 2.2 there is an admissible lifting G_x of G at x. Set $G_1 := G_x(V(G_x)-x,k)$ (see Lemma 2.3). Then $\lambda(G_1)=k$, and by induction G_1 has a required path $P[s,t]$. Let P_1 be the corresponding path in G, and let P_2 be a simple subpath of P_1 between s and t. Then P_2 is a required path for G

(3.7) If $\{x,y\} \subset V(G)-T$ and $g \in \partial(x,y)$, then $\lambda(G-g)<k$.

(3.8) If $X \subset V(G)$ is a minimum k-set and $\{x,y\} \subset X-T$, then $e(x,y)=0$.

Proof. Assume $e(x,y)>0$. By (3.7) there is a k-set Y such that $|Y \cap \{x,y\}|=1$. Then $Y-X \neq \emptyset \neq \overline{X-Y}$, since X is minimum. Then by Lemma 2.6 $e(X \cap Y)=k$. Thus $X \cap Y=\{x\}$ or $\{y\}$, contrary to (3.6).

(3.9) If $X \subset V(G)$ is a k-set, then $|X \cap T|=2$.

Proof. Let $X_1 \subset X$ be a minimum k-set (X_1 might equal X). By (3.5) $|X_1 \cap T| \geq 1$. Assume $X_1 \cap T=\{a\}$. By (3.1), (3.2), and (3.6) $|X_1-a| \geq 2$. Let $x \in X_1-a$, $y \in \bar{X}_1$, and set $G_1 := G/\bar{X}_1$. Then $|V(G_1)| \geq 4$ and by (3.8) $N(x;G_1) \subset \{a,\tilde{y}\}$. By Lemma 2.7(2) $e(x,a) \leq k/2$ and $e(x,\tilde{y};G_1) \leq k/2$, contrary to (3.6). Thus $|X_1 \cap T| \geq 2$, and similarly $|\bar{X} \cap T| \geq 2$. Hence $|X \cap T|=2$.

(3.10) G has no k-set.

Proof. Assume that G has a k-set X. Let $X_1 \subset X$ and $X_2 \subset \bar{X}$ be minimum k-sets. By (3.9) $|X_i \cap T|=2$ $(i=1,2)$.

(3.10.1) If Y is a k-set and $X_1 \cap Y \neq \phi$, then $X_1 \subset Y \subset \bar{X}_2$.

For, if $X_1 - Y \neq \phi$, then $Y - X_1 \neq \phi$ and $\overline{Y \cup X_1} \neq \phi$, since X_1 is minimum. By Lemma 2.6 $e(X_1-Y)=e(X_1 \cap Y)=e(Y-X_1)=k$. Thus $|X_1|=2$ and by (3.6) $X_1 \subset T$. Let $X_1=\{a_1,a_2\}$ and $a_1 \in Y$. If $|Y-X_1| \geq 2$, then by (3.9) $|(Y-X_1) \cap T|=2$, and so $|Y \cap T|=3$, contrary to (3.9). Thus $|Y-X_1|=1$, and by (3.6) $Y-X_1 \subset T$. Let $Y-X_1=\{a_3\}$. $Y \cap X_2=\{a_3\}$, and so $X_2-a_3 \in T$ as above, let $X_2-a_3=\{a_4\}$. Now $e(a_1,a_2)=e(a_1,a_3)=e(a_3,a_4)=k/2$. $e(\bar{T}) \leq k$ and by (3.4) $\bar{T} \neq \phi$, contrary to (3.6) or (3.9). Hence $X_1 \subset Y$. If $Y \cap X_2 \neq \phi$, then similarly $X_2 \subset Y$, contrary to (3.9). Therefore $Y \subset \bar{X}_2$.

(3.10.2) $V(G)=X_1 \cup X_2$.

For, assume $V(G) \neq X_1 \cup X_2$. Then there is a $Y \subset \bar{X}_2$ such that $X_1 \subsetneq Y$ and $e(Y)=k$. We choose Y such that $|Y|$ is minimal (Y might equal $\bar{X}_2$). Let $x \in Y-X_1$. If $N(x) \subset X_1 \cup \bar{Y}$, then $e(x,X_1) \geq (k+1)/2$ or $e(x,\bar{Y}) \geq (k+1)/2$, and so $e(X_1 \cup x)<k$ or $e(Y-x)<k$. Thus for some $y \in Y-X_1$, $e(x,y)>0$. By (3.7) there is a k-set Z such that $|Z \cap \{x,y\}|=1$. We may let $X_1 \cap Z \neq \phi$ (if not, then we take $\bar{Z}$ as Z). Then by (3.10.1) $X_1 \subsetneq Z \subset \bar{X}_2$. By choice of Y, $Z-Y \neq \phi$. By Lemma 2.6 $e(Z-Y)=k$, contrary to (3.6) or (3.9).

Let $X_1 \cap T=\{a_1,a_2\}$ and $X_2 \cap T=\{b_1,b_2\}$. By (3.8) for each $x \in X_1-T$, $N(x) \subset \{a_1,a_2\} \cup X_2$, and for each $y \in X_2-T$, $N(y) \subset \{b_1,b_2\} \cup X_1$. By (3.6), for $i=1,2$, $|X_i|$ is even and $(k+1)|X_i-T| \leq 3k$, thus $|X_i-T|=0$ or 2. By (3.4) we may let $X_1-T=\{x_1,x_2\}$. For $i=1,2$, if $e(a_i,X_2) \geq k/2$, then $e(X_2 \cup a_i) \leq k$, thus $e(a_i,X_2) \leq k/2-1$. Similarly

$e(x_i,X_2)\leq k/2$ $(i=1,2)$. If $V(f)=\{b_1,b_2\}$, then $e(a_1,a_2)=0$. By Lemma 2.7(2) $e(a_1,x_i)\leq k/2$ $(i=1,2)$, and so $e(a_1,x_i)>0$ $(i=1,2)$. Similarly $e(a_2,x_i)>0$ $(i=1,2)$ and the result follows. If $V(f)=\{a_1,a_2\}$, then we may let $|X_2-T|=0$, contrary to (3.3). Thus we may let $V(f)=\{a_1,b_1\}$. Now $e(a_2,b_2)=0$. If $e(b_2,\{x_1,x_2\})>0$, let $g_1\in\partial(b_2,x_1)$, then $e(x_1,a_2)=0$, and so there are $g_2\in\partial(x_1,x_2)$ and $g_3\in\partial(a_2,x_2)$. $e(\{a_1,x_1,a_2\};G/X_2)=e(a_1)+e(x_1)+e(a_2)-2e(a_1,\{x_1,a_2\})\geq 3k+1-2(k-1)=k+3$. Thus $\lambda(G/X_2-\{f,g_1,g_2,g_3\})\geq k-2$. Therefore $e(b_2,\{x_1,x_2\})=0$, and so $|X_2-T|=2$ (note that $e(b_2,a_2)=0$). Let $X_2-T=\{y_1,y_2\}$. Similarly $e(a_2,\{y_1,y_2\})=0$. Since $\{a_1,b_1\}$ is not a separating set, $e(\{x_1,x_2\},\{y_1,y_2\})>0$. Let $g_1\in\partial(x_1,y_1)$. Then $e(x_1,a_2)=e(y_1,b_2)=0$, and for $g_2\in\partial(x_1,x_2)$, $g_3\in\partial(x_2,a_2)$, $g_4\in\partial(y_1,y_2)$ and $g_5\in\partial(y_2,b_2)$, $\lambda(G-\{f,g_1,g_2,g_3,g_4,g_5\})=k-2$.

By (3.7) and (3.10) for each $x\in V(G)-T$, $N(x)\subset T$. Let $V(f)=\{a_1,a_2\}$. $k|\bar{T}|\leq 4k-2$ and $|\bar{T}|$ is even , thus and by (3.4) $|\bar{T}|=2$. Let $\bar{T}=\{x_1,x_2\}$. By (3.10) $e(s,a_i)<k/2$ $(i=1,2)$, and so by (3.3) $e(s,\{x_1,x_2\})>0$ and $e(t,\{x_1,x_2\})>0$. We may let $e(s,x_1)>0$, then $e(t,x_1)=0$, $e(t,x_2)>0$ and $e(s,x_2)=0$. $\{a_1,a_2\}$ is not a separating set , thus $e(\{s,x_1\},\{t,x_2\})>0$, and so there is a $g_1\in\partial(x_1,x_2)$. For $i=1,2$, $e(\{s,x_2,a_i\})\geq 3k-2e(a_i,\{s,x_2\})\geq 3k-2(k-1)=k+2$. Thus for $g_2\in\partial(s,x_1)$ and $g_3\in\partial(t,x_2)$, $\lambda(G-\{f,g_1,g_2,g_2\})\geq k-2$.

3. Proof of Theorem 1

The proof of Lemma 4.1 will be given later.

Lemma 4.1. Suppose that $k\geq 4$ is an even integer, $n\geq 3$ is an integer, G is a 2-connected graph, $V(G)=T\cup W_1\cup W_2$ (disjoint union), $T=\{s_1,\ldots,s_n,t_1,\ldots,t_n\}$, $|T|=2n$, $T\cup W_1\in\Gamma(G,k)$, $e(s_i)=e(t_i)=k$ $(1\leq i\leq k)$, for each $x\in W_1$, $e(x)=k$ or $k+1$, and for each $x\in W_2$, $e(x)<k$ is even. Then there is a subgraph $G^*\subset G$ such that

(a) for some $1\leq i<j<l\leq n$, $G-E(G^*)$ has edge-disjoint paths $P_1[s_i,t_i]$, $P_2[s_j,t_j]$ and $P_3[s_l,t_l]$.

(b) $V(G^*)=K_1\cup K_2$ and $K_1\cap K_2=\emptyset$,

(c) $T-\{s_i,t_i,s_j,t_j,s_l,t_l\}\subset K_1\in\Gamma(G^*,k-4)$,

(d) for each $x\in K_2$, $e(x;G^*)$ is even.

Proof of Theorem 1

By (1.1) it suffices to prove $g(3k)\leq 4k$ and $g(3k+2)\leq 4k+2$ $(k\geq 2)$. Let $\alpha=0$ or 1, $m\geq 2$ is an integer, $k:=4m+2\alpha$ and $n:=3m+2\alpha$. Assume that G is a k-edge-connected graph and $\{s_1,\ldots,s_n,t_1,\ldots,t_n\}:=T$ are vertices of G (not necessarily distinct). We prove that there are edge-disjoint paths $P_1,\ldots,P_n$ such that P_i joins s_i and t_i $(1\leq i\leq n)$. We may assume (see the proof of Theorem 2 and Figure 3 in [6])

(4.1) $e(s_i)=e(t_i)=k$ $(1\leq i\leq n)$ and for each $x\in V(G)$, $e(x)=k$ or $k+1$.

We proceed by induction on $|E(G)|$. If $s_1=s_2$, then by Lemma 2.4 there is a path $P[t_1,t_2]$ such that $s_1\in V(P)$ and $\lambda(G-E(P))k-2$. By induction $G-E(P)$ has edge-disjoint paths $P_3[s_3,t_3],\ldots,$ $P_n[s_n,t_n]$. Thus let $|T|=2n$. By Lemma 4.1 there is a subgraph $G^*\subset G$ such that (a),(b),(c) and (d) hold. By Lemma 2.3

$G^*(K_1,k-4)$ is $(k-4)$-edge-connected, and by induction $G^*(K_1,k-4)$ has $(n-3)$ edge-disjoint paths joining (s_r,t_r) $(1\leq r\leq n,\ r\neq i,j,l)$. Thus the result holds in G.

Proof of Lemma 4.1

Suppose that G satisfies the hypothesis of Lemma 4.1, but the result does not hold. Choose G with this property such that $|E(G)|$ is minimal.

(4.2) $W_2=\phi$.

Proof. Assume $x\in W_2$. Then $e(x)\geq 4$. By Lemma 2.2 we have an admissible lifting G_x of G at x. The result holds in G_x, and so in G.

(4.3) If $1\leq i<j\leq n$, $G_1\subset G$ is a subgraph such that $G-E(G_1)$ has edge-disjoint paths $P_1[s_i,t_i]$ and $P_2[s_j,t_j]$, $V(G_1)=K_1\cup K_2$, $K_1\cap K_2=\phi$, $T-\{s_i,t_i,s_j,t_j\}\subset K_1$, and for each $x\in K_2$, $e(x;G_1)$ is even, then $K_1\notin\Gamma(G_1,k-2)$.

Proof. Assume $K_1\in\Gamma(G_1,k-2)$. Let $1\leq l\leq n$ and $l\neq i,j$. By Lemma 2.4 G_1 has a path $P[s_l,t_l]$ such that $\lambda(G_1-E(P))\geq k-4$. Let $G^*:=G_1-E(P)$.

(4.4) If $x\in W_1$, then $e(x)=k+1$.

Proof. Assume $e(x)=k$. By Lemma 2.2 there is an admissible lifting G_x of G at x. The result holds in G_x with $V(G_x)=T\cup(W_1-x)\cup(x)$, and it also holds for G.

(4.5) If $x,y\in W_1$ and $f\in\partial(x,y)$, then $\lambda(G-f)\leq k-1$.

(4.6) If $a,b\in T$, then $e(a,b)=0$.

Proof. If $f\in\partial(s_1,t_1)$, then by Theorem 2 there is a path $P[s_2,t_2]$ such that $f\notin E(P)$ and $\lambda(G-E(P)-f)\geq k-2$, contrary to (4.3). If $f\in\partial(s_1,s_2)$, then by Theorem 2 G has a path $P_1[s_3,t_3]$ such that $f\notin E(P_1)$ and $\lambda(G-E(P_1)-f)\geq k-2$. By Lemma 2.4 $G-E(P_1)-f$ has a path $P_2[t_1,t_2]$ such that $s_1\in V(P_2)$ and $\lambda(G-E(P_1\cup P_2)-f)\geq k-4$.

(4.7) If $x_1,x_2\in W_1$, $a_1,a_2\in T$, $f_i\in\partial(x_i,a_1)$ (i=1,2) and $g\in\partial(x_2,a_2)$, then $V(G)-a_1\notin\Gamma(G-\{f_1,f_2\},k)$.

Proof. Set $G_1:=G-\{f_1,f_2\}$, and assume $V(G)-a_1\in\Gamma(G_1,k)$. Set $G_2:=G_1(V(G)-a_1,k)$. If $a_1=s_1$ and $a_2=t_1$, then by Theorem 2 G_2 has a path $P[s_2,t_2]$ such that $g\notin E(P)$ and $\lambda(G_2-E(P)-g)\geq k-2$, contrary to (4.3). If $a_1=s_1$ and $a_2=s_2$, then by Theorem 2 G_2 has a path $P_1[s_3,t_3]$ such that $g\notin E(P_1)$ and $\lambda(G_2-E(P_1)-g)\geq k-2$. By Lemma 2.4 $G_2-E(P_1)-g$ has a path $P_2[t_1,t_2]$ such that $a_2\in V(P_2)$ and $\lambda(G_2-E(P_1\cup P_2)-g)\geq k-4$.

(4.8) G has no k-set.

Proof. Assume X is a minimum k-set. Let $u\in\bar{X}$. If $x,y\in X\cap W_1$ and $f\in\partial(x,y)$, then $V(G)-\{x,y\}\in\Gamma(G-f,k)$. For, if not, then for some k-set Z, $|Z\cap\{x,y\}|=1$. Then $Z-X\neq\emptyset\neq\overline{X\cup Z}$ and by Lemma 2.6 $e(X-Z)=e(X\cap Z)=k$. Thus $|X|=2$ and $e(x)=e(y)=k$, contrary to (4.6). Thus by (4.5) $N(X\cap W_1;G/\bar{X})\subset T\cup\{\tilde{u}\}$, and by (4.6) $X\cap W_1\neq\emptyset\neq X\cap T$. By (4.4) $|X\cap W_1|\geq 2$, and so $|X\cap T|\geq 2$. Let $a\in X\cap T$. Since $e(a,\bar{X})<k/2$ (otherwise $e(X-a)\leq k$), by Lemma 2.2 for some $x,y\in N(a)\cap X$, $G_a^{x,y}$ is admissible. By (4.6) $\{x,y\}\subset W_1$.

Let $f_1 \in \partial(a,x)$ and $f_2 \in \partial(a,y)$, then $V(G)-a \in \Gamma(G-(f_1,f_2),k)$. Let $b \in ((N(x) \cup N(y)) \cap X)-a$, then $b \in T$, contrary to (4.7)

By (4.2),(4.5),(4.6) and (4.8) G is a bipartite graph with the partition (T,W_1). Let $a \in T$. By Lemma 2.2 for some $x,y \in N(a)$, $G_a^{x,y}$ is admissible and we can deduce a contradiction (see the proof of (4.8)).

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