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A C*-ALGEBRA WITH THE REPRESENTATION EXTENSION PROPERTY

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Every C*-algebra A sits inside its regular completion $\hat{A}$ very nicely. $\hat{A}$ is a monotone complete C*-algebra, which is, in general, not a von Neumann algebra. $\hat{A}$ reflects closely the structure of A, for examples, any bounded derivation of A extends to a unique <u>inner</u> derivation of $\hat{A}$, any automorphism of A extends to a unique automorphism of $\hat{A}$, and if A is simple, any <u>outer</u> automorphism of A extends to a unique <u>outer</u> automorphism of $\hat{A}$.

The following question naturally arises: Let A be any C*-algebra and let π be any irreducible representation of A.

(*) Can we extend π to a*-representation $\hat{\pi}$ from $\hat{A}$ onto $\pi(A)^{\wedge}$?

If A is abelian, every character of A can be extended to a character of $\hat{A}$ and so (*) has an affirmative answer.

Unfortunately, the answer to this question (*) is negative in general.

Definition. Let A be a C*-algebra. Then A is said to have the representation extension property (the (RE)-property) if, for every irreducible representation π of A, π can be extended to a *-homomorphism $\hat{\pi}$ from $\hat{A}$ onto $\pi(A)^{\wedge}$. We say that A has the strong representation extension property (the

(SRE)-property) if, any quotient of A has the (RE)-property.

In [6], a characterization of C*-algebras with the (SRE)-property is given in the following form:

Theorem. *Let* A *be a separable* C*-*algebra*. *Then* A *has the* (SRE)-*property if, and only if,* A *is a restricted direct sum of a sequence* $\{B_n\}$, *consisting of infinite dimensional simple* C*-*algebras or homogeneous* C*-*algebras of finite orders* (including $\{0\}$).

In this note, we would like to give you an outline of its proof with negative examples to the question (*), which, we do hope, would be helpful to understand the proof of Theorem. We also would like to give an example to show that there is a C*-algebra which has the (RE)-property, but has not the (SRE)-property.

Detailed proof will appear in the J. London Math. Soc. (see [6]).

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Note first that if A is abelian, every character of A can be extended to a character of $\hat{A}$ and so A has the (SRE)-property.

Example 1. Let A be the set of all continuous functions F on $[0,1]$, taking values in the C*-algebra $M_2(\mathbb{C})$ of all 2 by 2 matrices over $\mathbb{C}$, such that $F(0)_{2j} = F(0)_{12} = 0$ for $i,j = 1,2$. A becomes a C*-algebra with respect to the pointwise operations and the norm $\|F\| = \mathrm{Sup}\{\|F(t)\| \mid t \in [0,1]\}$. Then A has not got the (RE)-property.

In fact, the one dimensional representation of A defined by $F \in A \longrightarrow F(0)_{11}$ cannot be extended to $\hat{A} = M_2(C[0,1]^\wedge)$.

Example 2. Let H be an infinite dimensional separable Hilbert space and let $A = K(H) \otimes C[0,1]$, the C*-tensor product of the algebra of compact operators on H, by the C*-algebra $C[0,1]$. Then A has never got the (RE)-property.

Note, first, that $\hat{A} = L(H) \bar{\otimes} C[0,1]^\wedge$ and $\hat{A}$ is a properly infinite and σ-finite AW*-algebra. Let $\pi = i \otimes \pi_t$ be the irreducible representation of A on H induced by $t \in [0,1]$. Then $\pi(A) = K(H)$. If it were possible to extend π to a *-homomorphism $\hat{\pi}$ from $\hat{A}$ onto $K(H)^\wedge = L(H)$, then, since H is separable, this would imply that π is <u>normal</u> (see [1]). Let ξ be any unit vector in H, and let $\phi_t(a) = (\hat{\pi}(1 \otimes a)\xi, \xi)$ for any $a \in C[0,1]^\wedge$. Then $\phi_t(1) = 1$, and so ϕ_t is a state on $C[0,1]^\wedge$, which is completely additive on projections. This were, however, a contradiction, because $C[0,1]^\wedge$ has no normal states.

Example 3. Let F be the Fermion algebra acting irreducibly on an infinite dimensional separable Hilbert space H.

Let $B = F + K(H)$ (note that $F \cap K(H) = \{0\}$). Let π be the irreducible representation of B which is canonically defined by $B/K(H)$ ($\cong F$). Then π cannot be extended to a *-homomorphism from $\hat{B}$ ($= L(H)$) onto (into) $\hat{F}$.

In fact, if it were possible, then, since $\hat{B}$ is properly infinite and $\hat{F}$ is σ-finite, π would be a direct summand. This is, however, a contradiction, because $\hat{B}$ is a factor and π is not faithful.

The fact that the regular completion $\hat{A}$ of a separable C*-algebra A has no type II direct summand is used to prove our Theorem. See for details ([3], Theorem 2).

Also we use the following Theorem proved by Feldman and Fell ([1]).

Lemma 1. Let A, B be two AW*-algebras. Let A be properly infinite and σ-finite. Suppose that B is σ-finite. Then any *-homomorphism from A into B is compltely additive on projections.

Some routine calculations in the regular completions of C*-algebras (see [5], [7], [10] and [11]) give us the following:

Lemma 2. Let $\{B_n\}$ be any sequence of separable C*-algebras and let $A = \sum_{n=1}^{\infty}{}' B_n$ be the restricted direct sum of $\{B_n\}$. Then $\hat{A} = \sum_{n=1}^{\infty} \hat{B}_n$.

To prove this, we only have to check the following two properties by the uniqueness of the regular completions.

Let $\tilde{A} = \sum_{n=1}^{\infty} \hat{B}_n$. Then $\tilde{A}$ has the following two properties:

1) $C^*(A,1)_h$ is order dense in $\tilde{A}_h$,

and

2) $C^*(A,1)_h$ σ-generates $\tilde{A}_h$,

where $C^*(A,1)$ is the C*-algebra obtained from the adjunction of a unit to A if A is non-unital, otherwise, $C^*(A,1) = A$ and B_h is the self-adjoint part of a C*-algebra B.

By an homogeneous C*-algebra of order n ($n < \infty$), we mean a C*-algebra A such that $A/J \cong M_n(\mathbb{C})$ for every primitive ideal J of A ([2], [8]).

By using the fact that the Borel envelop of a separable homogeneous C*-algebra of order n is *-isomorphic to the n by n matrix algebra over the C*-algebra $\mathcal{B}$(PrimA) of all bounded complex Borel functions on the 2nd countable Hausdorff locally compact space PrimA, where PrimA is the primitive ideal space of A, we can show the following lemma (see [2], [4] and [5]).

Lemma 3. *Let A be any separable homogeneous C*-algebra of finite order n, then $\hat{A}$ is an AW*-algebra of type I_n, whose centre is *-isomorphic to the regular completion $C_b(\mathrm{PrimA})^{\wedge}$ of the ideal centre $C_b(\mathrm{PrimA})$ of A, where $C_b(\mathrm{PrimA})$ means the C*-algebra of all bounded complex continuous functions on PrimA.*

A careful examination of an irreducible representation of an homogeneous algebra of a finite order tells us that every separable homogeneous C*-algebra of a finite order has the (SRE)-property. By using this fact, together with Lemma 2 and Lemma 3, we can show the following proposition, which states the "if" part of Theorem is valid.

Proposition 1. Let A be a restricted direct sum of a sequence $\{B_n\}$ of simple C*-algebras or homogeneous C*-algebras of finite orders, then A has the (SRE)-property.

Next, we shall sketch a proof of "only if" part of Theorem. First of all, we shall start with the following proposition, which treats a C*-algebra whose regular completion is properly infinite.

Proposition 2. Let B be a separable C*-algebra whose regular completion $\hat{B}$ is properly infinite. Suppose that B has the (RE)-property. Then PrimB is discrete, and so B is a restricted direct sum of an at most countable family of separable infinite dimensional simple C*-algebras.

Let $J \in$ PrimB. Let $\hat{\pi}_J$ be a *-homomorphism from $\hat{B}$ onto $(B/J)^\wedge$ which is an extension of π_J. Since $\hat{B}$ is σ-finite, properly infinite and $(B/J)^\wedge$ is σ-finite, $\hat{\pi}_J$ is normal by Lemma 1, that is, there is a unique central projection e_J in $\hat{B}$ such that

$$J = \hat{B}e_J \cap B.$$

Moreover, $\hat{B}(1 - e_J) \cong (B(1 - e_J))^{\wedge} \cong (B/J)^{\wedge}$, which implies that $\hat{B}(1 - e_J)$ is a non-zero factor and so every $J \in \mathrm{Prim}B$ is <u>maximal</u> in PrimB.

For any $J \in \mathrm{Prim}B$, let $J^c = \hat{B}(1 - e_J) \cap B$ ($\neq \{0\}$), which is a closed two-sided ideal of B.

Let $K \subset \mathrm{Prim}B$. Let $J_0 \in \overline{K}$, that is, $J_0 \supset \cap \{ J | J \in K \}$. We shall show that there is $J \in K$ such that $J_0 + J \neq B$.

Suppose that $J_0 + J = B$ for all $J \in K$. Then, $J^c \subset J_0$ for all $J \in K$. In fact, for any fixed $J \in K$, if $y \in J^c$ ($y \geqq 0$), then one can find $x_0 \in J_0$ ($x_0 \geqq 0$) and $x_1 \in J$ ($x_1 \geqq 0$) such that $y = x_0 + x_1$. Since $y \in J^c$, $e_J y = 0$ follows and so $e_J x_0 e_J + x_1 = 0$, that is, $x_1 = 0$. This means that $y \in J_0$. Hence $1 - e_{J_0} \leqq e_J$ for all $J \in K$.

On the other hand, $J_0 \supset \cap\{J | J \in K\}$ tells us that $1 - e_{J_0} \leqq \mathrm{Sup}\{1 - e_J | J \in K\}$ and hence $1 = e_{J_0}$. This is a contradiciton and so, if $J_0 \in \overline{K}$, then there is $J \in K$ such that $J_0 + J \neq B$. Since J and J_0 are maximal, this means that $J_0 = J \in K$, that is, $\overline{K} = K$ for all $K \subset \mathrm{Prim}B$. Hence PrimB is discrete. This completes the proof.

Let A be a separable C*-algebra and let

$$X_0 = \{J \in \mathrm{Prim}A \mid (A/J)^{\wedge} \text{ is properly infinite}\},$$

$$Y = \{J \in \mathrm{Prim}A \mid (A/J)^{\wedge} \text{ is finite}\},$$

and for each k ($k \geqq 1$),

$$X_k = \{J \in \mathrm{Prim}A \mid (A/J)^{\wedge} \cong M_k(\mathbb{C})\}.$$

Since $(A/J)^\wedge$ has no type II-direct summand for each $J \in \mathrm{Prim}A$,

$$Y = \bigcup_{k=1}^{\infty} X_k \quad (\text{some } X_k \text{ may be } \emptyset).$$

Suppose that A has the (SRE)-property. Then, by a rather lengthy speculation, we have the following:

Lemma 4. Keep the notations and assumptions above in mind. X_k $(k \geqq 0)$ is open and closed in PrimA.

The following F.B. Wright's theorem is used to prove Lemma 4, but we shall omit the details (see [6] and [9]).

Lemma 5([9]). Let B be a finite AW*-algebra of type I and let X be the set of all maximal ideals of B. Then B/M is a finite AW*-factor of type I for every M except possibly for a closed nowhere dense set in X. The exceptional set is empty if, and only if, the number of homogeneous summands of B is finite. If this set is non-empty, then B/M is a von Neumann factor of type II_1 for every such M.

So one can find closed two-sided ideals B and C such that

$$A = B \oplus C$$

and $\mathrm{Prim}B \approx X_0$, $\mathrm{Prim}C \approx \bigcup_{k=1}^{\infty} X_k$.

By our assumption, B has the (RE)-property and so, by Proposition 2, PrimB is discrete. So either $B = \{0\}$ or else

B is a restricted direct sum of at most countable, infinite dimensional simple C*-algebras.

Let f_k be the projection in the ideal centre $Z(M(C))$ of the multiplier algebra of C corresponding to the continuous characteristic function of X_k $(k \geqq 1)$ on PrimC, then, since C is separable, it is easy to check that the mapping

$$x \in C \longrightarrow (xf_k) \in \sum_{k=1}^{\infty}{}' Cf_k$$

is a *-isomorphism between C and $\sum_{k=1}^{\infty}{}' Cf_k$. Since Cf_k is k-homogeneous, this completes the proof.

Next, we shall construct an example of a C*-algebra which has the (RE)-property but not the (SRE)-property.

Let A be the set of all continuous functions F on $[0,1]$, taking values in $M_2(\mathbb{C})$ such that $F(t)_{i2} = F(t)_{2j} = 0$ for $i,j = 1,2$ and for t with $0 \leqq t \leqq 1/2$. Then A becomes a C*-algebra with respect to the pointwise operations and the norm $\| F \| = \mathrm{Sup}\{\|F(t)\| \mid t \in [0,1]\}$. Let

$$I_1 = \{F \in A \mid F(t) = 0 \quad \text{for} \quad t \in [1/2,1]\}$$

and let

$$I_2 = \{F \in A \mid F(t) = 0 \quad \text{for} \quad t \in [0,1/2]\}.$$

Then I_1 and I_2 are closed ideals of A such that

(1) $I_1 \cap I_2 = \{0\}$,

(2) $A/I_2 \cong C[0,1/2]$,

(3) $A/I_1 \cong B$

where B is the C*-algebra of all continuous $M_2(\mathbb{C})$-valued functions G on $[1/2,1]$ such that $G(1/2)_{i2} = G(1/2)_{2j} = 0$ for $i,j = 1,2$,

(4) $I_1 \cong C_0[0,1/2]$

where $C_0[0,1/2]$ is the C*-algebra of all complex-valued continuous functions on $[0,1/2]$, vanishing at $1/2$,

(5) $I_2 \cong B_0$

where B_0 is the C*-subalgebra of B, vanishing at $1/2$, and

(6) $I_1 + I_2$ is essential in A.

Lemma 6. For any $J \in \text{Prim}A$, one can find a unique $t \in [0,1]$, such that, either $t \in [0,1/2]$ and $F + J = F(t)_{11}1$ for all $F \in A$ or else, $t \in (1/2,1]$ and $F + J = F(t)$ for all $F \in A$.

Since, by (1)-(6),

$$\hat{A} \cong \begin{pmatrix} C[0,1/2]^{\wedge} & 0 \\ & \\ 0 & 0 \end{pmatrix} \oplus M_2(C[1/2,1]^{\wedge}),$$

by making use of Lemma 6, it is easy to check that A has the (RE)-property. On the other hand, since $A/I_1 \cong B$ and B has not got the (RE)-property (see example 1), A has not got the (SRE)-property.

References

[1] J. Feldman and J.M.G. Fell, Separable representations of rings of operators, Ann. of Math., 65(1957), 241-249.

[2] J.M.G. Fell, The structures of algebras of operator fields, Acta Math., 106(1961), 233-280.

[3] M. Ozawa and K. Saitô, Embeddable AW*-algebras and regular completions, J. London Math. Soc., 34(1986), 511-523.

[4] G.K. Pedersen, C*-algebras and their automorphism groups, London Math. Soc. Monographs, 14(Academic Press, London 1979).

[5] K. Saitô, A structure theory in the regular σ-completion of C*-algebras, J. London Math. Soc., 22(1980), 549-558.

[6] K. Saitô, Representations of C*-algebras and regular completions, To appear in J. London Math. Soc..

[7] K. Saitô and J.D.M. Wright, Outer automorphisms of regular completions, J. London Math. Soc., 27(1983), 150-156.

[8] M. Takesaki and J. Tomiyama, Applications of fibre bundles to their certain class of C*-algebras, Tôhoku Math. J., 13(1961), 498-523.

[9] F.B. Wright, A reduction for algebras of finite type, Ann. of Math., 60(1954), 560-570.

[10] J.D.M. Wright, Regular σ-completions of C*-algebras, J. London Math. Soc., 12(1976), 299-309.

[11] J.D.M. Wright, Wild AW*-factors and Kaplansky-Rickart algebras, J. London Math. Soc., 13(1976), 83-89.

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