数理解析研究所講究録
第 690 巻 1989 年 1-8

Introduction to real algebraic geometry

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§1. Introduction

Real algebraic geometry was studied from old. However its systematic study was difficult, and results obtained had no interaction til quite recent. It is after the appearance of [B] and [C-R] that many mathematicians began to research real algebraic geometry with the same penetrating eyes. They wanted to construct a theory of real algebraic geometry. A fundamental concept in the theory is the real spectrum of a ring such as the usual spectrum in algebraic geometry. In this paper we give a short introduction of real spectra. We refer to [B-C-R] for all foundations on real algebraic geometry. This book explains also all important results on real algebraic geometry except results on Nash manifolds. See [S_1] for Nash manifolds.

§2. Ordered field and ring

Rings are always commutative and unitary in this ring. An <u>ordered field</u> or <u>ring</u> means a field or ring with order which is compatible with the addition and the multiplication. Hence an ordered ring is an integral domain, and the order is uniquely extended to its quotient field.

An important example of an ordered field is $\mathbb{R}$ with the usual order, on the other hand $\mathbb{C}$ does not admit an order.

A field K admits an order if and only if $a_1^2+\cdots+a_n^2=0$ for $a_i \in K$ implies $a_1=\cdots=a_n=0$. Consider the case when K admits an order. Then its order is not necessarily unique. For example, the rational function field $\mathbb{R}(x)$ (see the argument below). Its order is unique if

$K[i] = K[x]/x^2+1$ is an algebraically closed field, in other words, if any algebraic extension of K does not admit an order. We call such a field a <u>real closed field</u> and we do not distinguish between it and its ordered field. We note that any ordered field has a unique real algebraic extension which preserves the order. We call it a <u>real closure</u>. Here the uniqueness means that for two real closures K_1, K_2 of $(K, \leq)$ there exists a unique K-isomorphism from K_1 to K_2, which is stronger than the algebraic closure case.

Consequently, for an integral domain A, the set of orders on A corresponds bijectively to the set of real closed fields which are algebraic over A.

§3. Real spectrum

Let A denote a commutative ring. The <u>real spectrum</u> of A, denoted by $\mathrm{Spec}_r A$, is the set of pairs $(\mathfrak{p}, \leq)$ where $\mathfrak{p}$ is a prime ideal of A and $\leq$ means an order on $A/\mathfrak{p}$. We call the krull dimension of $A/\mathfrak{p}$ the <u>dimension</u> of $(\mathfrak{p}, \leq)$.

Case of $A=\mathbb{R}[x]$. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{R}[x]$. There are three cases: (i) $\mathfrak{p} = \{0\}$, (ii) $\mathfrak{p}$ is generated by x-a for some $a\in\mathbb{R}$ and (iii) by $(x+a)^2+b^2$ for some $a, b \in\mathbb{R}$ with $b\neq 0$. Then $\mathbb{R}[x]/\mathfrak{p}$ is isomorphic to $\mathbb{R}[x]$, $\mathbb{R}$, $\mathbb{C}$, respectively. $\mathbb{R}$ is a real closed field, and $\mathbb{C}$ admits no order. Hence we need consider orders on $\mathbb{R}[x]$. It suffices to compare x with all real numbers. Assume that x is bigger than any real number. Then

$$a_n x^n + \cdots + a_0 > 0 \quad \text{for } a_i \in \mathbb{R} \text{ with } a_n \neq 0$$

if and only if $a_n > 0$. We write the element of $\mathrm{Spec}_r \mathbb{R}[x]$ with this order $+\infty$. $-\infty$ means the order where x is smaller than any real number. Next consider the case when x is bigger than a real number <u>a</u> and smaller than any real number in (a, ∞). Then

$$a_n(x-a)^n+\cdots+a_m(x-a)^m > 0 \quad \text{for } a_i\in\mathbb{R} \text{ with } n\geq m,\ a_m\neq 0,$$

if and only if $a_m>0$. Let this case be denoted by a_+. We define a_- in the same way. Consequently

$$\mathrm{Spec}_r\mathbb{R}[x]=\mathbb{R}\cup\{a_\pm:\ a\in\mathbb{R}\}\cup\{\pm\infty\}.$$

How to regard geometrically the above right set? We regard $a_\pm$ as the germ at $\underline{a}$ of $(a, +\infty)$, $(-\infty, a)$, respectively, and $\pm\infty$ as the germ at $\pm\infty$ of $\mathbb{R}$. If the closure of an open semialgebraic set U in $\mathbb{R}$ does not contain $\underline{a}$, the germ of U at $\underline{a}$ is empty. We ignore this case. The germ at $\underline{a}$ of $\mathbb{R}-a$ is regarded as the set consisting of a_+ and a_-. Hence if the closure of U contains $\underline{a}$ and if U does not contain $\underline{a}$, then the germ of U at $\underline{a}$ is a_+, a_- or $\{a_+, a_-\}$. Moreover $a_\pm$ are irreducible in the following sense. If a_+ (resp. a_-) is the union of two germs b, c at $\underline{a}$ of some open semialgebraic sets, then a_+ (resp. a_-)=b or c. Of course $\pm\infty$ are also irreducible. Thus we regard $\{a_\pm:\ a\in\mathbb{R}\}\cup\{\pm\infty\}$ as the set of irreducible germs of open semialgebraic sets in $\mathbb{R}$.

This geometric interpretation is justified as follows. If $(\mathfrak{p}, \leq)$ $=a$ in $\mathrm{Spec}_r\mathbb{R}[x]$, then $\mathfrak{p}$ consists of polynomials vanishing at $\underline{a}$ and the class $\bar{f}$ in $\mathbb{R}[x]/\mathfrak{p}$ of $f\in\mathbb{R}[x]$ is positive if $f(a)>0$. If $(\mathfrak{p}, \leq)=a_+$ in $\mathrm{Spec}_r\mathbb{R}[x]$, then $\mathfrak{p}$ consists of polynomials vanishing on some open semialgebraic set which presents a_+ (consequently $\mathfrak{p}=\{0\}$) and $\bar{f}\in\mathbb{R}[x]/\mathfrak{p}$ for $f\in\mathbb{R}[x]$ is positive if $f>0$ on the set. For a_-, $\pm\infty$, we obtain similar statements.

By the above explanation it is natural to give an order to Spec_r $\mathbb{R}[x]$ as follows. For $a<b\in\mathbb{R}$,

$$-\infty<a<a_+<b_-<b<+\infty.$$

Case of $A=\mathbb{R}[x_1,\ldots,x_n]$. We shall regard $\mathrm{Spec}_r\mathbb{R}[x_1,\ldots,x_n]$ as the set of a sort of irreducible germs of Nash manifolds (=semialgebraic C^ω manifolds) in $\mathbb{R}^n$. We need to explain the germ. Let M_i, $i=0,\dots,m\le n$, be a sequence of Nash manifolds in $\mathbb{R}^n$ for $i>0$ or $i=m=0$ and in $\mathbb{R}^n\cup\{\infty\}$ for $i=0$ and $m>0$ of dimension=i with $M_i\subset\overline{M_{i+1}}$ and $M_i\cap M_{i+1}=\phi$. Let M'_i, $i=0,\dots,m'$, be another one. Then we regard $\{M_i\}$ and $\{M'_i\}$ as the same one if $m=m'$, $M_0=M'_0$, $M_1\cap U_0= M'_1\cap U_0$ for some open semialgebraic neighborhood U_0 of M_0, $M_2\cap U_1= M'_2\cap U_1$ for some open semialgebraic neighborhood U_1 of $M_1\cap U_0$, and so on. Let S denote the family of these sequence classes, and let S_1 denote the subfamily of S consisting of α such that for any representative sequence $\{M_i\}$ there is a smaller representative sequence $\{M'_i\}$ of α than $\{M_i\}$ such that each M'_i is connected. Then

$$\mathrm{Spec}_r\mathbb{R}[x_1,\dots,x_n]=S_1.$$

This equality follows from the following correspondence. For an element α of S_1, $(\mathfrak{p}, \le)$ is defined as follows. Let $\{M_i\}_{i=0,\dots,m}$ be a representative sequence of α such that M_m is connected. Set

$$\mathfrak{p}=\{f\in\mathbb{R}[x_1,\dots,x_n]:\ f|_{M_m}=0\},$$

and let the class $\overline{f}$ in $\mathbb{R}[x_1,\dots,x_n]/\mathfrak{p}$ for a polynomial f be called positive if $f>0$ on M'_m for some representative sequence $\{M'_i\}$ of α. We note that the dimension of α equals m.

In the same way as the case of $\mathbb{R}[x]$, we can prove that each element of S is a finite union of some elements of S_1 and that elements of S_1 are irreducible. Consider a sequence X_i, $i=0,\dots,m$, of semialgebraic sets in $\mathbb{R}^n$ everywhere of dimension=i with $X_i\subset\overline{X_{i+1}}$ and $X_i\cap X_{i+1}=\phi$, and define an equivalence relation in such sequences in the same way as above. Let S_2 denote the family of these sequence classes. Then $S_2=S$. Hence we can regard $\mathrm{Spec}_r\mathbb{R}[x_1,\dots,x_n]$ also as the set of a sort of irreducible germs of semialgebraic sets in $\mathbb{R}^n$.

Why semialgebraic germs? An element of Spec $\mathbb{C}[x_1,\dots,x_n]$ means an irreducible algebraic set in $\mathbb{C}^n$. However elements of Spec $\mathbb{R}[x_1,\dots,x_n]$ have not such a geometric meaning in $\mathbb{R}^n$. Hence the usual spectrum is not very useful for real algebraic geometry. Conversely consider the family of all irreducible algebraic sets in $\mathbb{R}^n$. This family is hard to deal with because the image of an algebraic set under a proper polynomial map is not necessarily algebraic. The family of semialgebraic sets is easier to deal with. However this family also has a fault. This is not Noetherian, namely, there is an infinite decreasing sequence of semialgebraic sets, and hence we can not decompose a semialgebraic set into irreducible ones. The family of semialgebraic set germs in the above sense solves these faults. Of course we are interested mostly in the global problems on real algebraic geometry. Hence, when we use real spectra in real algebraic geometry, the problem is to extend the local properties to the global. Later I shall explain such a problem, which seems most important at present in real algebraic geometry.

In all the statements above we can replace $\mathbb{R}$ by any real closed field. However the generalization to an ordered field is false. For example, the ideal $\mathfrak{p}$ of $\mathbb{Q}[x]$ generated by x^2-2 is prime, $\mathbb{Q}[x]/\mathfrak{p} = \mathbb{Q}[2^{1/2}]$ admits an order, and the zero set of x^2-2 is empty in $\mathbb{Q}$.

§4. Topology on real spectra

Let A denote a commutative ring. We give $\mathrm{Spec}_r A$ a topology by adopting the following type of sets as an open subbase:

$$\tilde{U}(f)=\{(\mathfrak{p}, \leq)\in \mathrm{Spec}_r A:\ \bar{f} \text{ in } A/\mathfrak{p}>0\}$$

for an element f of A. It is easy to see that $\mathrm{Spec}_r A$ is compact but not a Hausdorff space except for some special cases. We call a subset of $Spec_r A$ <u>constructible</u> if it is a finite union of finite intersections of some sets of form $\tilde{U}(f)-\tilde{U}(g)$ for f,g of A. For an open subset X of $Spec_r A$, X is constructible if and only if X is compact.

Consider the case of $A=\mathbb{R}[x_1,...,x_n]$. For a polynomial function f

$$\tilde{U}(f) \cap \mathbb{R}^n = \{x \in \mathbb{R}^n: f(x)>0\}$$

(Set U(f)=this set). Hence the induced topology on $\mathbb{R}^n$ is the usual topology. Conversely

$$\tilde{U}(f) = \{\alpha \in Spec_r \mathbb{R}[x_1,...,x_n]: M_m \subset U(f) \text{ for some representative } \{M_i\}_{i=0,..,m} \text{ of } \alpha\}.$$

If n=1,

$$\tilde{U}(f) = \bigcup_{\text{finite}} [x_1, x_2] \quad \text{for } x_1=a_+ \text{ or } -\infty,\ x_2=b_- \text{ or } +\infty.$$

For a constructible subset X of $Spec_r \mathbb{R}[x_1,...,x_n]$, $X \cap \mathbb{R}^n$ is semialgebraic. Moreover the correspondence $X \to X \cap \mathbb{R}^n$ is a bijective map from the family of constructible sets to the family of semialgebraic sets. Let $\tilde{}$ denote the inverse map, namely, $\tilde{U} \cap \mathbb{R}^n = U$. Note that the closed point set of $Spec_r \mathbb{R}[x_1,...,x_n]$ consists of $\mathbb{R}^n$ and curve germs at infinity, namely, classes of sequences of form $\{M_0=\infty, M_1\}$ in S_1.

By the term of real spectrum, many classical results of real algebraic geometry are stated and proved shortly. For example, the semialgebraic curve selection lemma is restated as follows. For an infinite constructible set X in $Spec_r \mathbb{R}[x_1,...,x_n]$ and a closed point $\underline{a}$ of the closure $\overline{X}$ there exists an element α in X of dimension 1 with $a \in \overline{\alpha}$.

§5. Structure sheaf on real spectra

Let A, B be commutative rings, and let $f:A \to B$ be a homomorphism. Then we have a natural continuous map $\mathrm{Spec}_r B \to \mathrm{Spec}_r A$. We want to define a structure sheaf $\mathcal{N}$ on $\mathrm{Spec}_r A$. Let U be a constructible open set in $\mathrm{Spec}_r A$. Let I_U denote the direct system consisting of couples (A', s) where A' is an étale A-algebra, and s is a continuous section of the map $\mathrm{Spec}_r A' \to \mathrm{Spec}_r A$ on U. Set

$$\mathcal{N}'(U) = \varinjlim_{(A',s)\in I_U} A' ,$$

and let $\mathcal{N}$ denote the sheaf associated with the presheaf defined by this equality. For $\alpha=(\mathfrak{p}, \leq)$ of $\mathrm{Spec}_r A$, let $k(\alpha)$ denote the ordered quotient field of $(A/\mathfrak{p}, \leq)$. Then the stalk $\mathcal{N}_\alpha$ is a Henselian local ring, and its residue field is the real closure of $k(\alpha)$.

Case of $A=\mathbb{R}[x_1,\ldots,x_n]$. For an open semialgebraic set U of $\mathbb{R}^n$, we can regard $\mathcal{N}(\tilde{U})$ as the ring of Nash ($=C^\omega$ semialgebraic) functions on U. Hence the stalk $\mathcal{N}_0$ of $\mathcal{N}$ at 0 is the Henselization of the quotient ring of $\mathbb{R}[x_1,\ldots,x_n]$ with respect to the ideal generated by $x_1,\ldots,x_n$, in other words, the ring of formal power series algebraic over $\mathbb{R}[x_1,\ldots,x_n]$. If n=1, $\mathcal{N}_{0_+}$ is the ring of Puiseux series $\sum_{i=k}^{\infty} a_i x^{i/q}$, $k\in\mathbb{Z}$, $q\in\mathbb{N}-\{0\}$, $a_i\in\mathbb{R}$, which are algebraic over $\mathbb{R}[x]$.

Conjecture 1. Let $\mathcal{J}$ be a coherent sheaf of $\mathcal{N}$-ideals on $\mathrm{Spec}_r \mathbb{R}[x_1,\ldots,x_n]$. Then $\mathcal{J}$ is generated by the global cross-sections $\Gamma(\mathcal{J})$ of $\mathcal{J}$.

Conjecture 2. $\Gamma(\mathcal{N}) \to \Gamma(\mathcal{N}/\mathcal{J})$ is surjective.

If n=1, these are trivial. The case of n=2 follows from [S_2]. For n>0,

$$H^1(\mathrm{Spec}_r \mathbb{R}[x_1,\ldots,x_n], \mathcal{N}) \neq \{0\}.$$

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