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MODIFIED ANALYTIC TRIVIALIZATION OF MAPS

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It has been well-understood that finding a 'nice and natural' equivalence relation of real analytic objects (functions, maps, varieties and so on) is one of the fundamental problems in the theory of real singularities. T.-C. Kuo has introduced the notion of modified analytic trivialization and proposed us that this induces a 'nice and natural' equivalence relation. In this talk, we try to seek some condition for modified analytic trivialization of maps.

We first recall some well-known fact in singularity theory. Let $W_t(x,y) = xy(x-y)(x-ty)$, $t\geq 2$. Then W_t is C^1-equivalent to $W_{t'}$ if and only if $t=t'$. Generally speaking if $f(x_1,\ldots,x_n)$ is not simple germ, there are C^∞-moduli near f. If we want to classify all function germs, by differential equivalence, then the situation is disastrous and causes many problems. For the toplogical equivalence relation, it does not seem to cause moduli but is too weak to provide a workable theory. Thus we are interested in the following observation, due to T.-C. Kuo[4]. Let $\pi:M \longrightarrow \mathbb{R}^2$ be the blowing up at the origin. Then there is a family of analytic isomorphisms H_t of M which induces a family of homeomorphisms h_t of $\mathbb{R}^2$ with $W_t\circ h_t = W_2$. Since the proof of this observation is instructive, we repeat it briefly. Let *grad* W be the vector field defined by $\frac{\partial W_t}{\partial x}\frac{\partial}{\partial x} + \frac{\partial W_t}{\partial y}\frac{\partial}{\partial y} + \frac{\partial W_t}{\partial t}\frac{\partial}{\partial t}$, and V' be the orthogonal projection of $\frac{\partial}{\partial t}$ to the orthogonal complement to *grad* W. Let V be the multiplcation of V' such that the t-component of V is 1. Easy computation shows that $V = \left(\left(\frac{\partial W_t}{\partial x}\right)^2 + \left(\frac{\partial W_t}{\partial y}\right)^2\right)^{-1}\left(\frac{\partial W_t}{\partial t}\right)\left(\frac{\partial W_t}{\partial x}\frac{\partial}{\partial x} + \frac{\partial W_t}{\partial y}\right.$

$\left.\frac{\partial}{\partial y}\right) + \frac{\partial}{\partial t}$ and has an analytic lift on $M\times[2,\infty)$. The integration of V gives the disired isomorphisms. Thus we are lead to the definition of modified analytic equivalence.(due to essentially Kuo)

Let X, Y, $\tilde{X}$,... be real analytic manifolds and $\pi:\tilde{X} \longrightarrow X$ a proper real-analytic modification of which complexification is a proper complex-analytic modification. A continuous map $f:X \longrightarrow Y$ is said to be <u>modified analytic via</u> π if $f\circ\pi$ is real analytic. We say f is <u>modified analytic</u> if f is modified analytic via some π. A homeomorphism $\phi: X \longrightarrow Y$ of real spaces is called <u>modified analytic</u> if both ϕ and ϕ^{-1} are.

Let $A_{n,p}$ be the set of real analytic map germs of $(\mathbb{R}^n,0)$ to $(\mathbb{R}^p,0)$. We denote A_x, A_y, A_u the set of real analytic function-germs with coordinate system x, y, u, respectively.

For given f_1, $f_2 \in A_{n,p}$, we say they are <u>modified analytically $\mathcal{R}$-equivalent</u> if there exists a modified analytic local homeomorphism $\phi: (\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^n,0)$ such that $f_2\circ\phi = f_1$. We say f_1 and f_2 are <u>modified analytically $\mathcal{A}$-equivalent</u> if there exist modified analytic local homeomorphisms $\phi:(\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^n,0)$ and $\psi:(\mathbb{R}^p,0) \longrightarrow (\mathbb{R}^p,0)$ such that $f_2\circ\phi = \psi\circ f_1$. We say f_1 and f_2 are <u>modified analytically $\mathcal{K}$-equivalent</u> if there exist a modified analytic local homeomorphism $\phi:(\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^n,0)$ and a modified analytic family of local modified analytic homeomorphisms $\psi_x:(\mathbb{R}^p,0) \longrightarrow (\mathbb{R}^p,0)$ such that $f_2\circ\phi(x) = \psi_x\circ f_1(x)$. If ϕ is identity germ, we say f_1 and f_2 are <u>modified analytically $\mathcal{C}$-equivalent</u>. We say f_1 and f_2 are <u>modified analytically V-equivalent</u> if there exists a modified analytic local homeomorphism $\phi:(\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^n,0)$ which transfers $f_1^{-1}(0)$ to $f_2^{-1}(0)$. Obviously $\mathcal{R}$-equivalent implies $\mathcal{A}$-equivalent, $\mathcal{A}$-equivalent implies

$\mathcal{K}$-equivalent, and $\mathcal{K}$-equivalent implies V-equivalent. These definition ia an analogy of J.Mather[5]. As the usual way we can define the notion of modified analytically $\mathcal{G}$-trivial unfolding, via π, of a map germ where $\mathcal{G} = \mathcal{R}, \mathcal{A}, \mathcal{K}, \mathcal{C}$, and V.

§1. The blowing up.

To describe the first result for the modified analytic trivialization we prepare some language.

A polynomial germ $f_0:(\mathbb{R}^n,0) \longrightarrow (\mathbb{R}^p,0)$ is <u>weighted homogeneous of type</u> $(a_1, \ldots, a_n; d_1, \ldots, d_p)$ if each coordinate germ $f_{0j} = y_j \circ f_0$ is a weighted homogeneous polynomial germ of type $(a_1, \ldots, a_n; d_j)$. We call f_0 is <u>homogeneous</u> if $a_1 = \ldots = a_n = 1$.

A polynomial unfolding f of a weighted homogeneous polynomial germ f_0 is an <u>unfolding of non-negative weight</u> if writing $f(x,u)=(\bar{f}(x,u),u)$ and $\bar{f}(x,u) = f_0(x) + \sum u_{\alpha,j} x^{\alpha} \varepsilon_j$ with $u_{\alpha,j} \in A_u$, then $\mathrm{wt}(x^{\alpha}) \geq d_j$ and $> \max\{-a_1, \ldots, -a_n\}$ when $u_{\alpha,j} \neq 0$.

Theorem. ($\mathcal{G} = \mathcal{R}$, $\mathcal{K}$ or $\mathcal{C}$)

<u>Let f be a homogeneous polynomial map germ. If f is $\mathcal{G}$-finite, then any unfolding of f of non-negative weight is a modified analytically $\mathcal{G}$-trivial unfolding via the blowing up at the origin.</u>

This theorem is originally due to Kuo when $\mathcal{G}=\mathcal{R}$, and p=1.
To show this theorem we need the following lemma proved by direct computation.

Lemma. (A generalization of Cramer's rule)

<u>Let p and n be integers with $p \leq n$. Let $A = (a_{ji})$ be a $p \times n$ matrix and $b = {}^t(b_1, \ldots, b_p)$ be a column vector. Let</u>

$$A(i_1, \ldots, i_p) := \det \begin{pmatrix} a_{1i_1} \ldots a_{1i_p} \\ a_{pi_1} \ldots a_{pi_p} \end{pmatrix},$$

$$B(i_1,\dots,i_{p-1}) := \det\begin{pmatrix} a_{1i_1}\dots a_{1i_{p-1}} & b_1 \\ a_{pi_1}\dots a_{pi_{p-1}} & b_p \end{pmatrix},$$ and

$$\xi'_i := \sum_{i_1<\dots<i_{p-1}} A(i_1,\dots,i_{p-1},i)B(i_1,\dots,i_{p-1}).$$

Then $A\xi' = \det({}^tA A)b$, where $\xi' = {}^t(\xi'_1,\dots,\xi'_n)$.

Remark. When n=p, this is the usual Cramer's rule.

Remark. $\det({}^tAA) = \sum_{i_1<\dots<i_p} A(i_1,\dots,i_p)^2$.

Proof for $\mathcal{G} = \mathcal{R}$.

Assume that f_0 be a $\mathcal{R}$-finite homogeneous map germ and f be a polynomial unfolding of f_0 of non-negative weight. Let $a_{ji} = \frac{\partial}{\partial x_i}\left(y_j \circ \bar{f}\right)$ and $b_j = -\frac{\partial \bar{f}}{\partial u_{\alpha,j}}$. Let $\xi_{\alpha,j} = \det({}^tAA)^{-1}\sum \xi'_i \frac{\partial}{\partial x_i}$. Direct computation shows $\xi_{\alpha,j}$ has an analytic lift on the blowing up at the origin. By lemma $(d\bar{f})\xi_{\alpha,j} = \frac{\partial \bar{f}}{\partial u_{\alpha,j}}$, thus the trajectory of $\xi_{\alpha,j}$ gives the desired trivialization.

Proof for $\mathcal{G} = \mathcal{C}$.

Let $g_j = \sum_i x_i^{2d_j}$. Define a meromorphic vector field $\eta_{\alpha,j}$ on $\mathbb{R}^n \times \mathbb{R}^p$ by

$$\sum_{j=1}^{p} \frac{(y_j \circ \frac{\partial \bar{f}}{\partial u_{\alpha,j}}) \sum_{k=1}^{p} (y_k \circ \bar{f}) \prod_{m\neq k} g_m y_k}{\sum_{k=1}^{p} (y_k \circ \bar{f})^2 \prod_{m\neq k} g_m} \frac{\partial}{\partial y_j}.$$

If f_0 is $\mathcal{C}$-finite, $\eta_{\alpha,j}$ has an analytic lift via $\pi \times id_{\mathbb{R}^p}$, where π be the blowing up at the origin in $\mathbb{R}^n$. Thus the trajectory of $\eta_{\alpha,j}$ gives the desired homeomorphisms.

Proof for $\mathcal{G} = \mathcal{K}$.

Let $\bar{\xi}_{\alpha,j} = \left((\det {}^tAA) + \sum_{j=1}^{p} (y_j \circ \bar{f})^2 \prod_{k\neq j} g_k^2\right)^{-1} \sum \xi'_i \frac{\partial}{\partial x_i}$, and

$$\bar{\eta}_{\alpha,j} = \left((\det A^tA) + \sum_{j=1}^{p}(y_j\circ\bar{f})^2 \prod_{k\neq j} g_k^2\right)^{-1}\left(\sum_j y_j\circ\left(\frac{\partial\bar{f}}{\partial u_{\alpha,j}}\right)\sum_{k=1}^{p} y_k\circ\bar{f} \prod_{m\neq k} g_m y_k\right)\frac{\partial}{\partial y_i}.$$

Then, by the lemma, $(d\bar{f})\bar{\xi}_{\alpha,j} = \frac{\partial\bar{f}}{\partial u_{\alpha,j}} + \bar{\eta}_{\alpha,j}\bar{f}$.
Direct computation shows that $\bar{\xi}_{\alpha,j}$, $\bar{\eta}_{\alpha,j}$ have analytic lifts via $\pi\times id_{\mathbb{R}^p}$. Thus the trajectories of $\bar{\xi}_{\alpha,j}$ and $\bar{\eta}_{\alpha,j}$ gives the desired homeomorphisms.

§2. Newton polygons.

First we recall Newton polygons. Let $f=(f_1,\ldots,f_p):\mathbb{R}^n \longrightarrow \mathbb{R}^p$ be a real analytic mapping of n variables $x = (x_1,\ldots,x_n)$. Let $\sum a_{i,\nu}x^\nu$ be the Taylor expansion of f_i (i=1,...,p). Let us set $\Gamma_+(f_i)$ = the convex hull of $\{\nu+\mathbb{R}^n_+ \mid a_{i,\nu}\neq 0\}$, $\Gamma_f = (\Gamma_+(f_1),\ldots,\Gamma_+(f_p))$, and $\Gamma_+(f) = \Gamma_+(f_1) + \ldots + \Gamma_+(f_p)$. Let f_γ be $\left(\sum_{\nu\in\gamma_1} a_{1,\nu}x^\nu,\ldots,\sum_{\nu\in\gamma_p} a_{p,\nu}x^\nu\right)$ for any subset $\gamma = (\gamma_1,\ldots,\gamma_p)$ in Γ_f. We say a face γ of Γ_f is <u>compact</u> if $\gamma_1+\ldots+\gamma_p$ is a compact face of $\Gamma_+(f)$. We say γ is <u>coordinate</u> if $\gamma_1+\ldots+\gamma_p$ is a coordinate face of $\Gamma_+(f)$, up to a parallel transformation τ with $\tau(\Gamma_+(f)) \subset \mathbb{R}^n_+$.

Definition. We say f is <u>$\mathcal{R}$-nondegenerate</u> if the matrix $Jf_\gamma := \left(\left(x_i\frac{\partial f_j}{\partial x_i}\right)_{\gamma_j}\right)_{i=1,\ldots,n;j=1,\ldots,p}$ has the maximal rank on the set $\{x_1\cdots x_n \neq 0\}$, for any compact face γ of Γ_f. We say f is <u>$\mathcal{C}$-nondegenerate</u> if the set $\{f_\gamma = 0\}\cap\{x_1\cdots x_n\neq 0\}$ is empty for any compact face γ of Γ_f. We say f is <u>$\mathcal{K}$-nondegenerate</u> if the matrix Jf_γ has the maximal rank on the set $\{f_\gamma = 0,\ x_1\cdots x_n \neq 0\}$, for any compact face γ of Γ_f.

We can construct a proper analytic modification $\pi:X\longrightarrow\mathbb{R}^n$ corresponding to $\Gamma_+(f)$.(See [2,6], for example.)

Theorem.($\mathcal{G} = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$) <u>Let $(f_u(x),u)$ be an unfolding of f_0. Suppose</u>

<u>that Γ_{f_u} is independent of u, f_u is $\mathcal{G}$-nondegenerate and $f_{u,\gamma}$ is independent for any coordinate face γ of Γ_{f_u}. Then $(f_u(x),u)$ is a modified analytically $\mathcal{G}$-trivial unfolding via π.</u>

Proof. The proof is similar to the case for the blowing up at the origin, and the detailed proof will appear in [3] for the case of $\mathcal{G}$ = $\mathcal{K}$. The idea of proof is to consider the gradient vector with respect to the singular metric defined by $\left(x_i\frac{\partial}{\partial x_i},\ x_j\frac{\partial}{\partial x_j}\right) = \delta_{ij}$. The construction of vector fields is similar to the case for the blowing up.

§3 $\mathcal{A}$**-triviality.**

It seems to be hard to obtain conditions for modified analytic $\mathcal{A}$-triviality. The only statement, I can state now, is the following

Proposition.

<u>Let f_0 be a homogeneous polynomial map germ with $d_1 = \ldots = d_p = d$. If f_0 is $\mathcal{A}$-finite and $n \le p$, then any polynomial unfolding f of f_0 of non-negative weight is a modified analytically $\mathcal{A}$-trivial unfolding.</u>

Unfortunately, I do not have non-trivial examples which satisfies the suppositions of this proposition.

Proof. Under the supposition of the proposition, there is an $\ell > 0$ such that if $g \in A_{x,u}\{\frac{\partial}{\partial x_i}\}$ and $\mathrm{wt}(g) \ge \ell$, then

$$g \in A_{x,u}\{\frac{\partial \bar{f}}{\partial x_i}\} + A_{y,u}\{\frac{\partial}{\partial y_j}\}.$$ (This is due to [1],p.312, Lemma5.8)

Thus there is an N so that, for $\ell > N$,

$$-y_j^{\ell}\frac{\partial \bar{f}}{\partial u} \in A_{x,u}\{\frac{\partial \bar{f}}{\partial x_i}\} + A_{y,u}\{\frac{\partial}{\partial y_j}\}.$$

Let $\ell > N$ and $\rho = \sum y_j^{2\ell}$, then

$$-\rho \frac{\partial \bar{f}}{\partial u} \in A_{x,u}\{\frac{\partial \bar{f}}{\partial x_i}\} + A_{y,u}\{\frac{\partial}{\partial y_j}\}.$$

Thus we can write $-\rho \frac{\partial \bar{f}}{\partial u} \in \left(\sum \alpha_i(x)\frac{\partial}{\partial x_i}\right) \bar{f} - \left(\sum \beta_j(y)\frac{\partial}{\partial y_j}\right)$. Since ρ and $\frac{\partial \bar{f}}{\partial u}$ are homogeneous, we may assume α_i and β_j are homogeneous. Then $wt(\alpha_i) - wt(\rho) = wt\left(\frac{\partial \bar{f}}{\partial u}\right) + 1 > 0$, and $wt(\beta_j) - wt(\rho) = wt\left(\frac{\partial \bar{f}}{\partial u}\right) + d > 0$. Direct computation shows that $\rho^{-1} \sum \beta_j(y)\frac{\partial}{\partial y_j}$ has an analytic lift on the blowing up at the origin. Since f_0 is $\mathcal{A}$-finite, thus $\mathcal{K}$-finite, and $n \le p$, direct computation shows that $\rho^{-1} \sum \alpha_i\frac{\partial}{\partial x_i}$ has an analytic lift via the blowing up at the origin. Therefore the trajectories of these vector fields give the desired homeomorphisms.

Problem. Find conditions for modified analytical $\mathcal{A}$-triviality.

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