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An elementary construction of canonical stratification of smooth mappings: The canonical stratification of jet space

Isao Nakai (中居功)
Department of Mathematics
Faculty of Liberal Arts
Nagasaki University
Nagasaki Japan

Let $(N,\mathscr{A}),(P,\mathscr{B})$ be analytic manifolds and subanalytic stratifications. We construct a canonical stratification for generic proper C^r smooth mappings of N to P refining the stratifications $\mathscr{A}$, $\mathscr{B}$ by a purely geometric method without the use of Thom-Mather calculus. We prove also the topologicl stabilization theorem for smooth mappings of stratified manifolds.

Canonical stratification for smooth mappings has been one of fundamental tools in the singularity theory of mappings since being initiated by Thom [17,18,19]. This was constructed by Mather [10] for generic smooth mappings by using his differential calculus with the preparation theorem for smooth functions in a series of papers. However no further advanced theory on this method has been developed since then. In this paper we introduce an elementary and geometric construction of a canonical stratification for mappings of stratified manifolds without the use of the unfolding theory and the preparation theorem. The fundamental method used is partially seen in the papers [2,3] by Fukuda and goes back three decades to the papers [17,18] written by Thom, where the singularity theory of smooth mappings was initiated with a number of brilliant ideas.

Our method uses only elementary calculus and applies to many cases in the singularity theory, for which any finite determinacy does not in general in the corresponding jet spaces. In this paper all theorems are stated for mappings of subanalytic stratified manifolds for the stratification problem of composed mappings well as projections of subanalytic sets.

The idea of our construction can be seen already in the papers [17,18], where Thom showed a program for a topological stabilization (determinacy) theorem for map germs as well as varieties , and introduced many original ideas such as the use of Malgrange's preparation theorem, Thom-Bordman symbols, regular stratifications, the isotopy theorems (Theorems 1,2 [18]) including a certain universality of the jet section now denoted ${}_sJ^k$ (Lemma B). And in the final sections he indicated an sketch to construct a canonical stratification of jet space by the stabilization procedure. We inherit the mind of those papers but reject the ideas of the use of preparation theorem, Bordman symbols, and we realize the geometric construction of canonical stratification (Steps (i) to (v) in §1.2, 1. 3).

Although the idea presented by Thom [18] is completely verified by our results, we need to follow an apparently different way to prove the various properties. We represent the program here in a slightly modified form as follows. For simplicity we discuss the canonical stratification of the infinite jet space

$J(n,p)$ of map germs of $\mathbb{R}^n,0$ to $\mathbb{R}^p,0$.

Consider the unfolding $F : \mathbb{R}^n \times J^k(n,p) \to \mathbb{R}^p \times J^k(n,p)$ defined by $F(x,z) = (f_z,(x),z)$ with the polynomial representatives f_z of z. By the stratification theory of semialgebraic sets due to Łojasiewicz

[8], we construct a natural stratification of F,

$$F : \mathbb{R}^n \times J^k(n,p) \ , \mathcal{S}_n \longrightarrow \mathbb{R}^p \times J^k(n,p) \ , \mathcal{S}_p$$

which is A_F-regular at generic points. Define the semialgebraic stratification $\mathcal{S}'^{k}$ of k-jet space by the "stratification-type" of $\mathcal{S}_n$, $\mathcal{S}_p$ at $o \times J^k$ in the source and target. A k-jet z is sufficient if there exists a neighbourhood U of z in the k-jet space such that the restrictions of $\mathcal{S}'^{\ell}$ to the preimages $\pi^{-1}_{\ell,k}(U)$, $k \leq \ell$ by the natural projection $\pi_{\ell,k}$ of jet spaces are induced from $\mathcal{S}'^{k}$. Call the smallest of such k the order of sufficiency. Denote the set of non-sufficient k-jets by Σ^k and the restriction of $\mathcal{S}'^{k}$ to the complement by $\mathcal{S}^k$. These $\mathcal{S}^{k+1}$ extend the preimages of $\mathcal{S}^k$ by the projection successively for k = 1,2,... and define the stratification $\mathcal{S}$ of infinite jet space of finite type (locally defined in finite jet space) off the projective limit Σ of Σ^k with infinite codimension, which we call the first canonical stratification. The resulting stratification $\mathcal{S}$ is locally but not globally semialgebraic since the order of sufficiency for jets in connected strata is not bounded.

In this paper we first construct the restriction $\mathcal{S}_t$ of $\mathcal{S}$ to the set of transversal jets (correct + transversality condition to $\mathcal{S}$ in [18]), of which the complement has codimension > n (§1.2-1.4). The restriction is semialgebraic defined in a finite jet space by the construction in §1.2, 1.3, which we call pre canonical stratification.

It is remarkable that the geometric construction induces the RL-invariance and the S (= contact)-invariance (Theorem 3), which were proved for the case of $\mathbb{R}^n$, $\mathbb{R}^p$ with trivial stratification by Mather [9] using a deep calculus with the preparation theorem. Consider the natural inclusions $J^k(n,p) \subset J^k(n+s,p+s)$ defined by the trivial suspension. By the S-invariance pre canonical stratifications of the target induce stratifications $\mathcal{S}(s)$, s = 1,2,... , off subsets $\Sigma(s)$ with codimension > n + s, which fill up the complement Σ_t to form the first canonical stratification $\mathcal{S}$.

As seen in the above program as well as the statements in Theorems 3, 4, the problem is closely related to construction of a canonical stratification of map germs. For example recall that in Mather's construction of the canonical stratification of infinite jet space J(N,P) in [10], the well-known universality of stable unfoldings of map germs played a key role. A stable unfolding of a map germ contains all nearby singularities spring out the original singularity as sectional map germs. This universality enables us to construct canonical stratification of those singularities simultaneously by giving a stratification for a stable unfolding.

The stabilization theorem [18] gives, roughly stating, a natural stratification of generic map germs and decomposition of infinite jet space of finite type by stratification (topological)-type of map germs. This leads to the first step toward a geometric construction of a canonical stratification of jet space. The first half of the paper on the stabilization of varieties (Theorem 3 [18]) was justified and developed to the theory of regularities by Kuo, Trotman [20]. The second half on map germs has been recently explained by Fukuda [2,3] and du

Plessis [14] independently by using Mather's canonical stratification of jet space. Fukuda constructs a canonical Whitney A-regular stratification for unstable map germs by a certain stabilization procedure (the first stabilization) of jets by using the universality of ${}_sJ^k$ and Mather's canonical stratification, and he proves the stabilization theorem with the isotopy theorem with "carpeting function" (see [18]). Although, the quest toward the canonical stratification was neglected, being familiarized to Mather's stratification.

Independently of this tautological link of Thom-Mather theory and the various works following after Thom's original ideas, Varchenko [21] had already proved a similar stabilization theorem for map germs by using the stratification theory of varieties. However he mentioned it was not clear that his stratification gave a natural stratification to lead the topological stability theorem for global mappings.

Holding the program in mind, we aware that the method used by Fukuda should apply to construct canonical stratifications of all map germs as being purposed by Thom [17,18]. In our construction the universality of ${}_sJ^k$ replaces allover role of the universal unfolding used by Mather. Our method unifies and generalizes the above individual results. (The preparation theorem was used by Thom in order to discuss the analytic structure of the discriminant varieties. We instead use the theory of subanalytic sets. The partition of mappings by Bordman symbols does not satisfy the regularity conditions. We have then to refine the partition by a canonical procedure. This is implicitly contained in Steps (i) to (iv) in §1.2.)

A complete proof of the isotopy theorems was later given by

Thom [19] for Whitney B-regular stratified sets and mappings in a form close to the present form seen in [4], to which we refer in this paper. To achieve the Whitney B-regularity in our geometric construction we apply a further stabilization procedure, namely <u>the second stabilization</u> (§1.3).

space for subanalytic stratified manifolds. Topological

Another significance of the geometric construction no less than the generality for stratified manifolds is the naturality with respect to the Thom-Mather theory. We introduce in §0.2 the RL and S-(contact) equivalence relations for map germs of stratified manifolds. The invariances of the stratification under those relations follows from the construction. This tells how two independent methods are correlated with each other. Actually the stratification geometrically constructed coincides with that due to Mather for manifolds with trivial stratifications.

Finally we refer to some related works. By using the subanalytic-set theory due to Hironaka [6], triangulation of subanalytic mappings (of which the graphs are subanalytic) is constructed by Hardt [5], Teissier [16] and Denkowska, Kurdyka [23]. On the other hand Verona [22] constructs triangulations of generic stratified mappings of depth one. These together with our stratification suggest that generic C^r mappings of pre-subanalytic stratified manifolds may be triangulable.

§ 0. Definitions and the main results

0.1 Stratification of global mappings

In this paper we suppose manifolds are paracompact, real analytic, possibly have corners (see §1.4). stratification are subanalytic: locally a finite union of differences of images of proper analytic mappings; and regular means the Whitney B-regularity unless otherwise mentioned. In stating global results, the supports stratifications (the unions of closures of strata with positive codimension) are compact for simplicity. For the basic notions of stratification of mappings, the readers are suggested to refer to the book [4], the paper [10], and for subanalytic sets, the paper [6].

Families of mappings f_u, $g_u:(N,\mathscr{A}) \to (P,\mathscr{B})$ of stratified manifolds with the parameter u in a manifold Q are topologically conjugate (topologically equivalent) if there exists continuous families of homeomorhpisms φ_u of $(N,\mathscr{A})$, ψ_u of $(P,\mathscr{B})$ covering a homeomorphism λ of Q such that $\varphi_u \circ f_u = g_u \circ \psi_u$ for all u. We say an $f \in C^r(N,P)$ is topologically stable if the topological equivalence class $\mathscr{O}(f)$ is a neighbourhood of f in the Whitney topology, and we say f is homotopically stable is there exists a neighbourhood $\mathscr{O}$ of f such that any family g_u , $g_{u_0} = f$ within $\mathscr{O}$ is topologically trivial: conjugate with the trivial family $f'_u = f$ of f.

A stratification of a mapping $f:N \to P$ is a pair $(\mathscr{S}_N,\mathscr{S}_P)$ of stratifications of N, P such that f restricts on each stratum X of $\mathscr{S}_N$ to a submersion into some stratum Y of $\mathscr{S}_P$. When N, P are previously stratified by $\mathscr{A}$, $\mathscr{B}$, we suppose that $\mathscr{S}_N$, $\mathscr{S}_P$ refine $\mathscr{A}$, $\mathscr{B}$ respectively. We say $\mathscr{S}_N$ as well as the pair $(\mathscr{S}_N,\mathscr{S}_P)$ is A_f-regular if the following condition is satisfied : if a sequence y_i in a stratum $Y \in \mathscr{S}_N$ is convergent to an x in another $X \in \mathscr{S}_N$ and $\ker d(f|Y)_{y_i}$ is convergent to a subspace $T \subset T_xN$, then $\ker d(f|X)_x \subset T$. This regularity condition says that roughly the fibers of f are "almost parallel" in a certain sense so that f possesses an analogy of the covering homotopy property respecting the stratifications.

The main global theorem is

Theorem 1 (Topological stability theorem). Let $(N^n,\mathcal{A})$, $(P^p,\mathcal{B})$ be subanalytic stratified manifolds and assume that the supports of $\mathcal{A}$ is compact. Then there exists a positive integer $k(N,\mathcal{A},B) \leq \infty$ such that

(1) k is finite if $\mathcal{A}$ is semialgebraic with respect to suitable coordinate open neighbourhoods of N (for the case of non compact support)

(2) There exists an open dense subset $\mathcal{F}$ in the proper C^r mapping space $C^r_{pr}(N,P)$, $k(N,\mathcal{A},P) < r$ or $r = k = \infty$ with the Whitney topology with the following properties. For any C^r smooth family f_u in $\mathcal{F}$ parametrized by u in a manifold Q, the map $F = (f_u,u):N \times Q \to P \times Q$ admits C^{r-k+2} smooth (C^{r-k+1} smooth if $p = 1$) stratifications $\mathcal{S}_{N\times Q}$, $\mathcal{S}_{P\times Q}$ refining $\mathcal{A} \times Q$, $\mathcal{B} \times Q$ such that the second projection of $P \times Q$ onto Q is a stratified submersion. And the set $\Sigma(F|\mathcal{A}\times Q)$ of those $(x,u) \in N \times Q$ where F is not a stratified submersion is a union of some strata X of $\mathcal{S}_{N\times Q}(F)$ embedded in strata A of $\mathcal{A}$ with positive codimension. In particular mappings in $\mathcal{F}$ are homotopically stable with respect to $\mathcal{A}$, $\mathcal{B}$ by Thom's second isotopy theorem.

Subanalytic subsets admit subanalytic stratifications. So we obtain

Corollary 2 (Topological stability theorem for mappings of subanalytic sets). Let $X \subset N^n$, $Y \subset P^p$ be subanalytic subsets of real analytic manifolds. If X is compact, there exists an open dense subset $\mathcal{F} \subset C^\infty_{pr}(N,P)$ such that C^∞ smooth families $f_u:N \to P$, $u \in Q$ in $\mathcal{F}$ admit C^∞ smooth A_F-regular stratifications $(\mathcal{S}_{N\times Q},\mathcal{S}_{P\times Q})$, $F = (f_u,u)$, $X \times Q$, $Y \times Q$ are unions of strata and the second projection of $\mathcal{S}_{P\times Q}$ onto Q is a submersion. In particular f in $\mathcal{F}$ are homotopically stable with respect to X and Y : any family f_u close to the trivial family $f' = f_u$ is actually conjugate with f' by continuous families of homeomorphisms respecting X and Y.

The above theorem is proved by using the local properties of pre canonical stratification of jet space in Theorems 3,4 together with an elementary argument of transversality of multi jet sections.

0.2 Pre canonical and the first canonical stratifications

Let $\mathcal{A}$, $\mathcal{B}$ be stratifications of manifolds N, P. Map germs $f : N,x \to P,y$ and $g : N,x' \to P,y'$ are <u>RL-equivalent with respect to</u> $\mathcal{A}$, $\mathcal{B}$ if there exists germs of real analytic diffeomorphisms $\varphi : N,x \to N,x'$, $\psi : P,y \to P,y'$ such that $\varphi(\mathcal{A}) = \mathcal{A}$, $\psi(\mathcal{B}) = \mathcal{B}$ and $\varphi \circ f = g \circ \psi$. The <u>S-equivalence relation</u> of map germs of $(N\times\mathbb{R}^s,\mathcal{A}\times\mathbb{R}^s)$ to $(P\times\mathbb{R}^s,\mathcal{B}\times\mathbb{R}^s)$ for s = 0,1,... is generated by the RL-equivalence relations with respect to $\mathcal{A} \times \mathbb{R}^s$, $\mathcal{B} \times \mathbb{R}^s$ and the relation $(f_{u,o},u) \sim (f_{u,v},u,v)$ of unfoldings

$$(f_{u,o},u) : N \times \mathbb{R}^s \to P \times \mathbb{R}^s , \qquad u \in \mathbb{R}^s ,$$

$$(f_{u,v},u,v) : N \times \mathbb{R}^{s+t} \to P \times \mathbb{R}^{s+t} , \qquad u \in \mathbb{R}^s , v \in \mathbb{R}^t$$

of map germs $f_{0,0}$, $0 \le s$, t. By definition the S-equivalence class of an unfolding $(f_u,u) : N \times \mathbb{R}^s \to P \times \mathbb{R}^s$ is determined by that of $f_0 : N \to P$. The S-equivalence relation coincides with the contact ($\mathcal{K}$-) equivalence defined by Mather when the stratifications are trivial at least for $\mathcal{K}$-finite germs.

Germs of stratifications $(\mathcal{A},x)$, $(\mathcal{A}',x')$ of N, N' are <u>S-equivalent</u> if the germs of $\mathcal{A} \times \mathbb{R}^s$, $\mathcal{B} \times \mathbb{R}^s$ at (x,o), (x',o) are real analytically diffeomorphic for some s and s'.

Two r-jets z, z' $\in J^r(N,P)$ are <u>RL-equivalent</u> (respectively S-equivalent) if they are represented by C^r smooth germs which are RL-equivalent (S-equivalent) in the above sense with real analytic diffeomorphisms.

These equivalence relations are defined with real analytic diffeomorphisms in order to make the argument in this paper simple. Note that we may replace them by C^r diffeomorphisms with a sufficiently large r when $\mathcal{A}$ is semialgebraic and has compact support (see also §1.5).

For a stratification $\mathcal{A}$ of N and a mapping $f : N \to P$, we define the singular point set $\Sigma(f|\mathcal{A})$ to be the set of those $x \in N$ where f is not a stratified submersion. If $\mathcal{A}$ is Whitney A-regular then $\Sigma(f|\mathcal{A})$ is closed. In this paper we always assume that the restriction of f to the singular point set is finite-to-one for map germs $f : N,x \to P,y$ as well as global mappings. When f is multi transversal with respect to a stratification of N,x, for example the trivial stratification or the canonical stratification $\mathcal{S}_f = Jf^{-1}(\mathcal{S}_t)$ in Theorems 3, 4, the restriction satisfies this condition. Such a map germ admits a representative $\tilde{f} : U \to V$ defined on open neighbourhoods U, V of x, y such that the restriction $\tilde{f}|\Sigma(\tilde{f}|\mathcal{A})$ is proper, finite-to one and $\Sigma(\tilde{f}|\mathcal{A}) \cap \tilde{f}^{-1}(o) = x$ (or empty). The image $\tilde{f}(\Sigma(\tilde{f}|\mathcal{A}))$ is then closed and its germ at y is independent of the choice of the representative, so denoted $f(\Sigma(f|\mathcal{A}))$. By Theorems 3,4, if Jf is transversal to the canonical stratification $\mathcal{S}$ of jet space, then $\Sigma(f|\mathcal{A})$ is a union of some strata of $\mathcal{S}_f$ on each stratum of which f restricts to a local submersion. The direct image $f_*\mathcal{S}_f$ is the germ of partition of P at y characterized similar to those of constructible sheaves as follows. The germ $Y_{y'}$ at y' of $Y \in f_*\mathcal{S}_f$ containing y' is the intersection $Y_{y'} = \bigcap_{f(x')=y'} f(X_{x'})$. Here $X_{x'}$ stand for the germs at x' of the stratum X containing x'.

In this paper J(N,P) denotes the infinite jet space, on which the topology is induced from the projections onto finite jet spaces.

A stratification $\mathcal{S}$ of complement of a subset $\Sigma \subset J^\infty(N,P)$ is of <u>finite type</u> if there exists upper semicontinuous functions $0 \leq \bar{k} < k \leq \infty$ (k is finite and $\bar{k}$ is defined off Σ) called the <u>local order</u>, the <u>order of</u> $\mathcal{S}$ with the following properties.

(1) $\mathcal{S}$, Σ are <u>k-sufficient</u> : for any $z \in J(N,P)$ there exist a stratification $\mathcal{S}'$, a subset Σ' and functions k',k" on a neighbourhood U of z^k in $J^{k(z)}(N,P)$ from which the restrictions of $\mathcal{S}$, Σ, k, $\bar{k}$ to $\pi^{-1}_{k(z)}(U)$ are induced via $\pi_{k(z)}$. Here π_k is the projection of the infinite jet space onto the k(z)-jet space and z^k the image of z.

(2) $\mathcal{S}$ is <u>locally $\bar{k}$-sufficient</u> : for any $z \in \mathcal{S}$ there exist a neighbourhood $U' \subset U$ of z^k, a stratification $\mathcal{S}''$ and a function k" on a neighbourhood U" of $z^{\bar{k}}$ from which the restrictions of $\mathcal{S}$, $\bar{k}$ to U' are induced via the projection $\pi_{\bar{k}(z)}$.

We say $\mathcal{S}$ is of <u>weakly finite type</u> if Condition (2) and the following condition holds

(1)' $\mathcal{S}$ is induced from $\mathcal{S}'$ on $\pi_k^{-1}(U)$ We call the r-jet term z^r of an infinite jet z with $k(z) \le r$ a <u>sufficient r-jet</u> and define $k(z^r) = k(z)$, and we denote the project of the germs of $\mathcal{S}$ to r-jet space simply by the same $\mathcal{S}$. By the above local property (2) the transversality of jet sections Jf to a stratum X at a jet z makes a sense for $C^{\bar{k}(z)}$ smooth mappings as their $\bar{k}(z)$-jet sections are C^1 smooth. We then say simply, Jf is <u>transversal to</u> X in this paper without reference.

Theorem 3 (Pre canonical stratification) Let $(N,\mathcal{A})$, $(P,\mathcal{B})$ be subanalytic stratified manifolds. Then there exist a subanalytic subset $\Sigma_t \subset J^\infty(N,P)$ of codimension $\ge n + 1$ and a subanalytic stratification $\mathcal{S}_t$ of the complement of finite type with finite order $(k < \infty)$ which possess the following properties.

(1) $\mathcal{S}_t$, Σ_t are <u>RL-invariant</u> : if sufficient r-jets z, z' are RL-equivalent, then $k(z) = k'(z)$ and the equivalence induces a real analytic diffeomorphism of r-jet space which preserves the germs of $\mathcal{S}_t$, Σ_t at z, z'.

(2) $\mathcal{S}_t$ is <u>partially S-invariant</u> : if sufficient r-jets z, z' $\in \mathcal{S}_t$ are S-equivalent, then the germs of $\mathcal{S}_t$ at z, z', $\mathcal{A}$ at the sources and $\mathcal{B}$ at the targets are respectively S-equivalent.

(3) Let $f_u : N,x \to P,y$, $u \in \mathbb{R}^s$ be a C^r smooth unfolding with jet $z = J^r f_0(x)$ in a stratum X of $\mathcal{S}_t$, $k(z) \le r$ and let $F = (f_u,u) : N \times \mathbb{R}^s \to P \times \mathbb{R}^s$. Then the family of jet section $\bar{J}F = (Jf_u) : N \times$

$\mathbb{R}^s \to J(N,P)$ is transversal to $\mathcal{S}_t$ at (x,o) and the induced stratification $\mathcal{S}_F = \bar{J}F^{-1}(\mathcal{S}_t)$ is a Whitney regular $C^{r-\bar{k}(z)}$ smooth stratification refining $\mathcal{A} \times \mathbb{R}^s$. The singular point set $\Sigma(F|\mathcal{A}\times\mathbb{R}^s)$ is the union of the strata X of $\mathcal{S}_f$ embedded in some $A \times \mathbb{R}^s$, $A \in \mathcal{A}$ with positive codimension. On those strata F restricts to immersions and the images are in general position (F is multi transversal with respect to $\mathcal{S}_F$). The direct image and the intersection of its preimage give the $C^{r-\bar{k}(z)}$ smooth A_F-regular stratification of F

(i) $\quad F : \mathcal{S}_F \cap F^* F_* \mathcal{S}_F \longrightarrow F_* \mathcal{S}_F$.

The direct image $F_*\mathcal{S}_F$ is transversal to $\mathcal{B} \times \mathbb{R}^s$ and the transversal intersection

(ii) $\quad F : \mathcal{S}_F \cap F^*(F_*\mathcal{S}_F \cap \mathcal{B}\times\mathbb{R}^s) \longrightarrow F_*\mathcal{S}_F \cap \mathcal{S}\times\mathbb{R}^s$

is A_F-regular. The restriction $F|\ \bar{J}F^{-1}(X)$ is an immersion, $\Sigma(F|\mathcal{A}\times\mathbb{R}^s) \cap F^{-1}F(\bar{J}F^{-1}(X)) = \bar{J}F^{-1}(X)$ and $\bar{J}F^{-1}(X)$, $F(\bar{J}F^{-1}(X))$ are the strata containing (x,o), (x',o). The second projections of the stratifications in (ii) onto $\mathbb{R}^s$ are stratified submersions and induce the A_{f_u}-regular stratification

$$f_u : \mathcal{S}_{f_u} \cap f_u^*(f_{u*}\mathcal{S}_{f_u} \cap \mathcal{B}) \to f_{u*}\mathcal{S}_{f_u} \cap \mathcal{B}$$

of f_u on each fibers $N \times u$, $P \times u$.

In §1 we give the construction of pre canonical stratification. In the proof of Theorems 3, 4, Lemmas A, B play allover role. An S-equivalence class of an unfolding $F = (f_u,u) : N \times \mathbb{R}^s \to P \times \mathbb{R}^s$ at (x,o) is determined by that of f_o at x. By the S-invariance the germ of $\mathcal{S}_t(N \times \mathbb{R}^s, P \times \mathbb{R}^s)$ at $JF(x,o)$ induces a germ of stratification of $J(N,P)$ at $Jf_o(x)$ denoted $\pi_*\mathcal{S}_t(N \times \mathbb{R}^s,$

$P \times \mathbb{R}^s$). These germs for $s = 1,2,\ldots$ glue up to extend the $\mathcal{S}_t(N,P)$ to the stratifications $\mathcal{S}(s)$ and the first canonical stratification $\mathcal{S}$ in the following theorem. The full detail is available in §3.

Theorem 4 (the first canonical stratification of jet space) Let $(N,\mathcal{A})$, $(P,\mathcal{B})$ be subanalytic stratified manifolds. Then there exist subanalytic stratifications $\mathcal{S}(s)$ of weakly finite type with finite order k of complements of subanalytic subsets $\Sigma(s) \subset J^\infty(N,P)$ of codimension $\geq n + s + 1$ for $s = 0,1,\ldots$, which possess the following properties.

(1) $(\mathcal{S}(0),\Sigma(0))$ is pre canonical stratification. Let k_s be the orders for the stratifications $\mathcal{S}(s)$. Then k_s is increasing on s,

$$\pi_{k_{s+1}(z)k_s(z)}(\Sigma(s+1)) \subset \Sigma(s) \quad ,$$

as germs at z^{k_s}, and $\pi^{-1}_{k_{s+1}k_s}(\mathcal{S}(s))$ is a restriction of $\mathcal{S}(s+1)$ on which the order k_{s+1} is induced from k_s as germs at z^{k_s} for all z and $s = 0,1,\ldots$.

(2) (i) $\mathcal{S}(s)$, $\Sigma(s)$ are <u>RL-invariant</u> : if sufficient r-jets z, z' are RL-equivalent, then $k_s(z) = k_s(z')$ and the germs of $\mathcal{S}(s)$, $\Sigma(s)$ at z, z', $\mathcal{A}$ at the sources, $\mathcal{B}$ at the targets of z, z' are respectively real analytically diffeomorphic.

(ii) $\mathcal{S}(s)$ are <u>partially S-invariant</u> : if two r-jets z, z' are S-equivalent, $z \in \mathcal{S}(s)$ and z is sufficient, then z' is a sufficient r-jet in an $\mathcal{S}(s+t)$ and the germs of $\mathcal{S}(s+t)$ at z, z', $\mathcal{A}$ at the sources and $\mathcal{B}$ at the targets of z, z' are S-equivalent respectively.

(3) If a family of jet sections $\bar{J}F = (Jf_u) : N \times \mathbb{R}^s \to J(N,P)$ of a

C^r smooth family $f_u : N,x \to P,y_u$; $u \in \mathbb{R}^t$ is transversal to $\mathcal{S}(s)$ at z for (x,o) and k(z) < r. Then the statements in Theorem 3, (3) holds except for the final paragraph.

(4) The projective limits of $\Sigma(s)$, $\mathcal{S}(s)$ as s tends to infinity give a (pro-locally subanalytic) subset Σ of J(N,P) with infinite codimension and a (locally) subanalytic stratification $\mathcal{S}$ of the complement of weakly finite type by Property (2).

We call the stratification $\mathcal{S}$ in (4) <u>the first canonical stratification</u>. From the partial S-invariance (2)(ii) it follows that

(5) $\mathcal{S}$ is <u>S-invariant</u> : if infinite jets z, z' are S-equivalent, then the germs of $\mathcal{S}$, Σ at z, z', the germs of $\mathcal{A}$ at the sources, $\mathcal{B}$ at the targets of z, z' are respectively S-equivalent. In particular $\mathcal{S}$, Σ
are locally analytically trivial over P is $\mathcal{B}$ is trivial, and trivial over $N \times P$ if $\mathcal{A}$ is also trivial.

(6) Let $\mathcal{S}^n$ denote the union of the strata of $\mathcal{S}$ with codimension $\leq$ n. (Then $\mathcal{S}^n$ is of finite type and the set of those $z \in \mathcal{S}^n$ for any representative f of which the jet section Jf is transversal to $\mathcal{S}^n$ (hence to $\mathcal{S}$) is an open dense subset of J(N,P), on which $\mathcal{S}^n$ restricts to give pre canonical stratification $\mathcal{S}_t(N,\mathcal{A},P,\mathcal{B})$.

§1. Geometric construction of pre canonical stratification.

1.1 Preliminaries and Lemmas A, B.

For a smooth map germ $f : \mathbb{R}^n, x \to \mathbb{R}^p, f(x)$ let $\tau_x^\ell f$ denote the Taylor polynomial mapping of f at x of order ℓ.

Let m, q be positive integers. The m-<u>universal</u> q-<u>jet section</u>

$${}_mJ^q : (\mathbb{R}^n)^m \times J^\ell(\mathbb{R}^n,\mathbb{R}^p) \longrightarrow J^q(\mathbb{R}^n,\mathbb{R}^p)^m$$

is defined by ${}_mJ^q((x_i), J^\ell f(x)) = (J^q\ \tau_x^\ell f(x_i))_{i=1,\dots,m}$. For a subset Σ of $J^k(\mathbb{R}^n,\mathbb{R}^p)$, $k \le \ell$ let $(\mathbb{R}^n-*)^m-\Delta(\mathbb{R}^n)^m \times_* \pi_{\ell k}^{-1}(\Sigma)$ denote the fibre product of the bundle over $\mathbb{R}^n$ with fibre $(\mathbb{R}^n-x)^m \times \Delta(\mathbb{R}^n)^m$ on $x \in \mathbb{R}^n$ and the projection of $\pi_{\ell k}^{-1}(\Sigma)$ onto the sources, $\Delta(\mathbb{R}^n)^m$ being the generalized diagonal set of $(\mathbb{R}^n)^m$. We denote the restriction of ${}_mJ^q$ to this fibre product by ${}_m\bar{J}^q$.

Lemma A (due to Fukuda [2,3]) If $k + m(q+1) \le \ell$, then ${}_m\bar{J}^q$ is a submersion for any submanifold $\Sigma \subset J^k(\mathbb{R}^n,\mathbb{R}^p)$.

Lemma B (The universality of ${}_mJ^q$ due to Fukuda [1,2] and Thom [18]) Let $f_u : \mathbb{R}^n, x_u \to \mathbb{R}^p, y_u$, $u \in \mathbb{R}^s$ be a C^r smooth family of map germs and $\ell = k + m(q+1) \le 1$. Then there exists a $C^{r-\ell}$ smooth family of sections

$$\Omega_{f_u} : (\mathbb{R}^n)^m \to (\mathbb{R}^n)^m \times \pi_{\ell k}^{-1}(J^k f_u(x_u)) ,$$

$\Omega_{f_u}(x_u,\dots,x_u) = (J^\ell f_u(x_u), x_u,\dots,x_u)$ of which the first factor is invariant under the premutations of the sources $(x_1,\dots,x_m) \in (\mathbb{R}^n)^m$, such that the following diagram commutes,

$$
\begin{array}{ccccc}
{}_mJ^q : & (\mathbb{R}^n)^m & \times & J^\ell(\mathbb{R}^n,\mathbb{R}^p) & \longrightarrow J^q(\mathbb{R}^n,\mathbb{R}^p)^m \\
 & \swarrow & & \uparrow & \nearrow \\
 & (\mathbb{R}^n)^m & & \Omega = (\Omega_{f_u}) & \\
 & \nwarrow & & & \\
({}_mJ^q f_u) : & (\mathbb{R}^n)^m & \times & \mathbb{R}^s &
\end{array}
$$

Proof of Lemmas A, B. We assume $s = 0$ for simplicity. It is enough to prove for the case $p = 1$. So let $f : \mathbb{R}^n, x_0 \to \mathbb{R}, y_0$ be a C^r smooth function defined on a convex neighbourhood U of $x_0 \in \mathbb{R}^n$ and $x_1, \ldots, x_m \in U$. We begin writing f as

$$(*)_0 \qquad f = \tau^k_{x_0} f + \sum_{|\lambda_0|=k+1} f_{\lambda_0} \cdot (x-x_0)^{\lambda_0} ,$$

where $\lambda_0 = (\lambda_0^1, \ldots, \lambda_0^n)$ runs over the set of multi indices of order $k+1$. Write similarly as

$$(*)_1 \qquad f_{\lambda_0} = \tau^q_{x_1} f_{\lambda_0} + \sum_{|\lambda_1|=q+1} f_{\lambda_0\lambda_1} \cdot (x-x_0)^{\lambda_0} \cdot (x-x_1)^{\lambda_1}$$

and substitute $(*)_1$ for f_{λ_0} in $(*)_0$. Then we obtain

$$f = \tau^k_{x_0} f + \sum_{\lambda_0} \tau^q_{x_1} f_{\lambda_0} \cdot (x-x_0)^{\lambda_0} + \sum_{\lambda_0,\lambda_1} f_{\lambda_0\lambda_1} \cdot (x-x_0)^{\lambda_0} \cdot (x-x_1)^{\lambda_1} .$$

Repeating this for the $x_2, \ldots, x_m$ rest, we obtain

$$f = \tau^k_{x_0} f + \sum_{\lambda_0} \tau^q_{x_1} f_{\lambda_1} \cdot (x-x_0)^{\lambda_0} + \ldots$$
$$+ \sum_{\lambda_0,\ldots,\lambda_m} \tau^q_{x_m} f_{\lambda_0\ldots\lambda_m} \cdot (x-x_0)^{\lambda_0} \cdot \ldots \cdot (x-x_m)^{\lambda_m}$$

We define the section Ω_f by

$$\Omega_f(x_1, \ldots, x_m) =$$
$$(J^\ell(f - \sum_{\lambda_0,\ldots,\lambda_m} f_{\lambda_0\ldots\lambda_m} \cdot (x-x_0)^{\lambda_0} \cdot \ldots \cdot (x-x_m)^{\lambda_m})(x_0), x_1, \ldots, x_m) .$$

Clearly the diagram in Lemma B commutes as the above second term is q-flat at $x_1, \ldots, x_m$. To show the $C^{r-\ell}$ smoothness we look at the argument closer. The coefficients $f_{\lambda_0 \ldots \lambda_i}$ can be represented as

$$\int_0^1 \ldots \int_0^1 \partial^q / \partial x^{\lambda_i} f_{\lambda_0 \ldots \lambda_{i-1}}(x + t(x-x_i))\, dt^q \quad .$$

This shows the smoothness by induction.

Conversely the commutativity of the diagram tells that ${}_m\bar{J}^q$ for $\Sigma = \{J^k f(x_0)\}$ is submersive as $\Omega_f(x_1,\ldots,x_m)$ since there exists a family f_u , $f_0 = f$ with a constant k-jet at x_0 but the multi jet ${}_mJ^q f_u$ is submersive at $(o,x_1,\ldots,x_m)$.

Taking the average of the first factor of Ω_f with respect the permutations of $x_1,\ldots,x_m$, we may assume the first factor is invariant. This completes the proof of Lemmas A, B.

1.2 The first stabilization

We seek to construct the stratification $\mathscr{S}_t$ locally at a jet z. So we first assume that $N = \mathbb{R}^n$, $P = \mathbb{R}^p$ and the stratifications are trivial. The construction falls into the steps of the induction on the hierarchy of the strata. Namely we assume that there exist a germ of subanalytic subset $\Sigma_{a+1} \subset J^k(\mathbb{R}^n,\mathbb{R}^p)$, $0 \le a + 1 < p$ at z with codimension $\ge \max\{0,n-p\} + a + 1$ and a germ of subanalytic stratification $\mathscr{S}_a$ of the complement by strata with codimension 0 and
$\max\{0,n-p\} + i$, $i = 1,\ldots,a$ which possess the properties in Theorem 3. We then seek to construct a stratum X_A (possiblly a union of disconnected components) open dense in each connected component of Σ_{a+1} - sing Σ_{a+1} of codimension $\max\{0,n-p\} + a + 1$.
We define $\mathscr{S}_{a+1}$, Σ_{a+2} to the the union of $\mathscr{S}_a$ and those strata, and

its complement (for the detail see §1.3). Here sing Y denotes the set of those $y \in Y$ where Y is not an analytic submanifold of maximal dimension. By a result due to Tamm [15] sing Y is subanalytic for subanalytic sets Y. The induction begins with the set of singular (non-full rank) jets $\Sigma_{\max\{1,p-n+1\}} \subset J^1(\mathbb{R}^n,\mathbb{R}^p)$ and the complement $\mathcal{S}_{\max\{0,p-n\}}$, and stops when $\mathcal{S}_p$ is constructed. We then define $\mathcal{S}_t = \mathcal{S}_p$.

We begin with a connected component X_{a+1} of Σ_{a+1} - sing Σ_{a+1} of codimension $\max\{0,n-p\} + a + 1$. Let k(z) be the order of jet space in which $\mathcal{S}_a$ and Σ_{a+1} are defined, and let $\bar{k}(z)$, $z \in \mathcal{S}_a$, be the smallest order in which $\mathcal{S}_a$ is locally defined at z. (For the definition of these orders, see §0.2. Note that $\bar{k}(z)$ is constant on connected components by analyticity.)

Let ${}^1X_{a+1} \subset X_{a+1}$ be the set of those jets $z' \in X_{a+1}$ such that

(i) $\mathcal{S}_a$ is Whitney regular over X_{a+1} on a neighbourhood of z'
(This can be omitted. The theorems are proved by the other conditions. The regularity follows from Theorem 3.)

(ii) for any representative f of which the jet section $J^{k(z)}f$ is transversal to X_{a+1} and

(iii) $f \mid X_{a+1}(f)$ is an immersion

Here $X_{a+1}(f) = (J^{k(z)}f)^{-1}(X_b)$. By the transversality theorem due to Mather (Theorem 6.1 [9]), we see ${}^1X_{a+1}$ is locally defined in the jet space of the order $\bar{k}(z) + 1$ at z and codim X_{a+1} < codim $X_{a+1} - {}^1X_{a+1}$.

Let $Y = Y_1 \times_{\mathbb{R}^p} \dots \times_{\mathbb{R}^p} Y_m$ be an m-fold fibre product of strata of $\mathcal{S}_a$ (including possiblly X_{a+1} itself). By the RL-invariance Y_i are locally trivial over $\mathbb{R}^p$, so the Y is smooth. By Lemma A the universal jet section

$$ {}_m j^{\bar{k}(z)} : (\mathbb{R}^n - *)^m - \Delta \ \times_* \ {}^1X^{\ell}_{a+1} \longrightarrow J^{\bar{k}(z)}(\mathbb{R}^n, \mathbb{R}^p)^m $$

is submersive hence transversal to Y for a sufficiently large $\ell \geq \ell(Y) = k(z) + m(\bar{k}(z)+1)$, where ${}^1X^{\ell}_{a+1} = \pi^{-1}_{\ell k(z)}({}^1X_{a+1})$. We define the project X^{ℓ}_A of the first stratum X_A of $\mathcal{S}_{a+1}$ open dense in ${}^1X^{\ell}_{a+1}$ by

$$ \text{(iv)} \quad X^{\ell}_A = {}^1X^{\ell}_{a+1,m} - \bigcup_Y \bar{B}\left({}^1X^{\ell}_{a+1,m} , ({}_m j^{\bar{k}(z)})^{-1}(Y) \right) . $$

Here $\bar{B}(V,W)$ denotes the closure of the set $B(V,W)$ of those $x \in V$ where W is not Whitney B-regular over V, ${}^1X^{\ell}_{a+1,m}$ denotes the set of those $(z',x',\ldots,x') \in {}^1X^{\ell}_{a+1}$, x' being the source of z', which is naturally identified with ${}^1X^{\ell}_{a+1}$, and Y runs over the finite set of fibre products $Y_1 \times_{\mathbb{R}^p} \cdots \times_{\mathbb{R}^p} Y_m$ satisfying one of the following conditions :

(1) (<u>accessible</u>) $\operatorname{codim} Y_i = \max(0,n-p) + c_i$, $\Sigma c_i < b$, $X_b \subset \bar{Y}_i$

(2) (<u>weakly accessible</u>) $2 \leq m$, $b \leq \Sigma c_i$, $X_{a+1} \subset \bar{Y}_i$ and any partial sum $\Sigma' c_i$ is strictly smaller than $a + 1$

(3) (<u>Weakly accessible</u>) $m = 2$, $c_1 = a + 1$, $c_2 < a + 1$, $X_{a+1} \subset \bar{Y}_i$

The resulting stratum X_A is defined in the jet space of the order $\ell = \ell(z)$ the maximum of $\ell(Y)$ for those Y.

1.3 The second stabilization and the global strata

We can show that a map germ f with a jet z in X_A^{ℓ} admits the canonical A-regular stratification $(\mathcal{S}_f \cap f^*f_*\mathcal{S}_f\ ,\ f_*\mathcal{S}_f)$. We now impose the B-regularity of the stratification by stabilizing the jet z further in a jet space of a sufficiently high order $\ell' > \ell$ in the same manner as the first stabilization.

We begin the second stabilization with a ℓ'-jet $z \in X_A^{\ell'}$ and define the germ of the stratum $X^{\ell'} \subset X_A^{\ell'}$ at z by deleting non-stabilized jets.

The polynomial representatives $f_{z'}$ of ℓ'-jets z' restricts to proper and finite-to-one mappings on the singular point sets $\Sigma(f_{z'}|\mathcal{A})$ singular point sets $\Sigma(f_{z'}|\mathcal{A})$, $f_{z'}$ being restricted to suitable neighbourhoods of the source and the target of z'. So the above canonical stratifications for $f_{z'}$ are subanalytic and the totality of those form a subanalytic canonical stratification $(\mathcal{S}_F \cap F^*F_*\mathcal{S}_F\ ,\ F_*\mathcal{S}_F)$ of the unfolding $F = (f_{z'}, z') : \mathbb{R}^n \times X_A^{\ell'} \to \mathbb{R}^p \times X_a^{\ell'}$ (for definition, see §3). The diagonal sets X_{As} , X_{At} respectively in the source and the target of F are the sets of those (x',z'), (y',z') with $z' \in X_A^{\ell'}$, x', y' being the source and the target of z'. These sets are naturally identified with $X_A^{\ell'}$. We define the germ of the project of the stratum X to ℓ'-jet space at z by

$$\text{(v)} \qquad X^{\ell'} = X_A^{\ell'} - \bar{B}(\ X_{As}\ ,\ \mathcal{S}_F \cap F^*F_*\mathcal{S}_F\ ) - \bar{B}(\ X_{At}\ ,\ F^*\mathcal{S}_F\ )$$

and the orders k(z) and $\bar{k}(z)$ by ℓ' and the order for the $\mathcal{S}_a$.

To make the construction explicit at this stage, we give another definition below. The equivalence of these two definitions is discussed in the following remark.

let $\tilde{Y} = ({}_m\bar{J}^{\bar{k}_a(z)})^{-1}(Y) \subset (\mathbb{R}^n)^m \times X_A^{\ell'}$ be as in the first stabilization ($\bar{k}_a(z)$ is the local order for $\mathcal{S}_a$) and consider the strata $Y(F) = P(\tilde{Y}) \subset \mathbb{R}^n \times X_A^{\ell'}$, $FY(F) \subset \mathbb{R}^p \times X_A^{\ell'}$ of the above stratification of F, where $P : (\mathbb{R}^n)^m \times X_A^{\ell'} \to \mathbb{R}^n \times X_A^{\ell'}$ is given with the projection onto the first $\mathbb{R}^n$. The germ of the stratum $X^{\ell'}$ at z is the set of those $z' \in X_A^{\ell'}$ for which Y(F) is regular over X_{As}, FY(F) is regular over X_{At} respectively on neighbourhoods of z for the strata Y accessible or weakly accessible to X (see Conditions (i) - (iii) in §1.2).

We define the order $k_{a+1}(z)$ for $\mathcal{S}_{a+1}$ by the ℓ' above used and the local order $\bar{k}_{a+1}(z)$ by the order k_a for $\mathcal{S}_a$ for infinite jets z in the stratum $X = \pi_{\ell'}^{-1}(X^{\ell'})$.

It is a routine exercise to show that the resulting germ of strata are subanalytic and semialgebraic for semialgebraic $\mathcal{A}$. If $\mathcal{A}$ is semialgebraic, the order ℓ' is bounded and the stratification is globally constructed in a jet space of a finite order.

Remark that the order ℓ' is an upper semicontinuous function of z. An explicit estimate of the order is given by the geometry of the canonical stratification of F. The above construction uses the indefinite ℓ', but the resulting stratum is independent of the ℓ'. We discuss this in the following remark.

Remark Consider the strata $X_A(F) = (\bar{J}^{\ell}F)^{-1}(X_A^{\ell})$, $FX_A(F)$ ($\bar{J}^{\ell}F$ denotes the family of jet sections $J^{\ell}f_{z'}$) of the above canonical stratification of F. Theorem 3 applies to the germs of F at $(x',z') \in X_A(F)$ to say that the the stratification is regular over $X_A(F)$, $FX_A(F)$ at (x',z'), $F(x',z')$. Clearly this implies the regularities used in the second presentation of the stratum.

To show the independence of the stratum from the order ℓ' above used, let $\ell_2 > \ell_1 = \ell'$ and consider two unfolding F_i with the parameter spaces $X_A^{\ell_i}$ and the stratifications $(\mathcal{S}_{F_i} \cap F_i^* F_{i*}\mathcal{S}_{F_i}, F_{i*}\mathcal{S}_{F_i})$. Theorem 3 says that if the stratification is regular for i = 1 then it is so for i = 2. Conversely assume the regularity for i = 2. The natural inclusions $\mathbb{R}^n \times X^{\ell_1} \subset \mathbb{R}^n \times X^{\ell_2}$, $\mathbb{R}^p \times X^{\ell_1} \subset \mathbb{R} \times X^{\ell_2}$ are respectively transversal to the stratifications of the source and the target for F_2 and induce the stratification of F_1. Thus the regularity holds for i = 1. This completes the proof of the independence.

By the above independence and the upper semicontinuity of the order ℓ , the germs of the stratum above constructed glue up together to give a global stratum X in the infinite jet space. The orders k, $\bar{k}$ are then defined by those for germs. The orders are upper semicontinuous, and if $\mathcal{A}$ is semialgebraic, bounded and X is semialgebraic.

Let N, P be analytic manifolds and $\{U_i\}$, $\{V_j\}$ be coordinate open neighbourhoods. The first and the second stabilization procedures are compatible with analytic coordinate change of the source and the target, so the stratifications of $J(U_i, V_j)$ glue up and give a global stratification $\mathcal{S}_t$ of J(N,P) of finite type.

1.4 Non trivial $\mathcal{A}$, $\mathcal{B}$

First we show the reduction to the case of trivial $\mathcal{B}$. Let $\mathcal{A}$, $\mathcal{B}$ be subanalytic stratifications of N, P. Assume that there exists the stratification $\mathcal{S}_t(N,\mathcal{A},P)$ with the properties in Theorem 3 for trivial $\mathcal{B}$. Then by the RL-invariance (Theorem 3 (2)), $\mathcal{S}_t$ is trivial over P, so the set $\mathcal{F}$ of mappings f of N to P for which $J^k f$ is transversal to $\mathcal{S}_t \cap \mathcal{B}$, in other words $f \mid J^k f^{-1}(\mathcal{S}_t)$ is transversal to $\mathcal{B}$ is open dense in the mappings space. Since the strata of $\mathcal{S}_t$ are locally defined in $\bar{k}$-jet space, this transversality at $x \in N$ depends only on the k(Jf(x))-jets of f at x. We denote the set of those transversal k-jets by $\mathcal{O}$, which is subanalytic and open dense in $\mathcal{S}_t$ by the density of $\mathcal{F}$. We define $\mathcal{S}_t(N,\mathcal{A},P,\mathcal{B})$ by the restriction of $\mathcal{S}_t$ to $\mathcal{O}$, and $\Sigma_t(N,\mathcal{A},P,\mathcal{B})$ by its complement. By the density of $\mathcal{F}$, codim Σ_t exceeds n.

By the RL-invariance (Theorem 3 (2)), it is enough to construct for $\mathbb{N} = \mathbb{R}^n$, $P = \mathbb{R}^p$ and subanalytic stratification $\mathcal{A}$ of $\mathbb{R}^n$. The construction uses the same induction as in §1.2, 1.3. So assuming that there exists the stratification $\mathcal{S}_a$ of the complement of Σ_{a+1}, we construct strata X open dense in connected components of smooth parts of $\Sigma_{a+1} \cap A$, $A \in \mathcal{A}$ with codimension $\max(0,n-p) + a + 1$. The first stabilization remains the same form as for trivial $\mathcal{A}$. In the second stabilization and the proof of Theorem 3, We use the various Łojasiewicz exponents of the tangent spaces of the stratification of F. Those orders are similarly defined for semialgebraic $\mathcal{A}$ by the well-known result due to Łojasiewicz [8], which are finite, and the stratification $\mathcal{S}_t$ is constructed in a finite jet space. For subanalytic $\mathcal{A}$, those orders and exponents are given by the generalization of the works of Łojasiewicz for subanalytic sets by Bochnak and Risler [1]. The resulting stratification $\mathcal{S}_t$ is in general of finite type but no longer bounded for subanalytic case.

1.5 A generalization

The essential conditions used in our construction in the proceeding sections is the subanalyticity of $\mathcal{S}_a$, Σ_{a+1} and the independence of the singular point set sing Σ_{a+1} from the choice of local coordinates of N, P. Now define $\text{sing}^i\ \Sigma_{a+1}$ to be the set of those $z \in \Sigma_{a+1}$ where Σ_{a+1} is not a C^i smooth submanifold of codimension max(0,n-p) + a + 1. Clearly this set is invariant under C^i smooth coordinate transformations of jet space hence under $C^{\bar{k}(z)+i}$ smooth coordinate transformations of N, P. By Tamm [15] it is known that $\text{sing}^i\ \Sigma_{a+1}$ is subanalytic when Σ_{a+1} is so, and then we can define the stratum X open dense in $\Sigma_{a+1} - \text{sing}^i\ \Sigma_{a+1}$ by the same stabilization procedures in §1.2, 1.3. Denote the resulting pre canonical stratification by $\mathcal{S}_t^i$. By the above invariance of $\text{sing}^i\ \Sigma_{a+1}$, the stratification $\mathcal{S}_t^i$ is invariant under $C^{\bar{k}+i}$ smooth local coordinate transformations of $\mathcal{A}$, $\mathcal{B}$, where $\bar{k}$ is the local order for $\mathcal{S}_t$. This invariance suggests to generalize the notion of subanalytic stratified manifolds as follows.

We call a subset $X \subset M$ of a C^r smooth manifold is <u>pre subanalytic</u> if there exists a coordinate open covering $\{U_i\}$ of M such that each intersection $X \cap U_i$ is subanalytic. And the we suppose that the coordinate system $\{U_i\}$ is fixed and that of $J^k(N,P)$ is induced from those of N, P. Let $(N,\mathcal{A})$, $(P,\mathcal{B})$ be pre subanalytic stratified C^r manifolds, and $\{U_i\}$, $\{V_j\}$ the coordinate systems. By the above invariance the stratification $\mathcal{S}_t(U_i,\mathcal{A}|U_i,V_j,\mathcal{B}|V_j)$ glue together by C^r transition function if $k(z) + i \le r$ for all jet $z \in J(U_i,V_j)$ and all i, j. (The order k depends only on $\mathscr{A}$.) If $\mathscr{A}$ has compact support and $\mathscr{A}|U_i$ are semialgebraic, the order k is bounded and we allow finite r. For general $\mathscr{A}$, we require $r = \infty$.

The resulting stratification $\mathscr{S}_t$ of $J^r(N,P)$ possesses properties similar to those in Theorems 3, 4. The important difference arising here is that the strata are subanalytic and C^i smooth but not real analytic.

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