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$L^2(\Omega)$-ERROR ESTIMATE OF GALERKIN BOUNDARY ELEMENT METHOD WITH SINGLE LAYER POTENTIAL

Michio SAKAKIHARA
Department of Applied Mathematics
Okayama University of Science
Ridai-cho 1-1, Okayama 700,JAPAN

1. *INTRODUCTION.*

As a numerical method for solving a Dirichlet boundary value problem such as

$$-\Delta u + u = 0 \quad in \quad \Omega, \tag{1.1}$$

$$u = g \quad on \quad \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain in R^2 with the C^2-boundary $\partial\Omega$ the boundary element method is convenient to obtain the discretized equation and to solve. When we formulate an integral equation on the boundary with the single layer potential representation of the function which satisfies the equation (1.1), we have to deal with the first kind Fredoholm integral equation. In this case it is important to prove that the integral equation has a unique solution in an appropriate Sobolev space. The discussion of the integral equation about Laplace equation was presented by Nedelec and Planchard[6]. He proved that a bilinear form arising from a Dirichlet problem for Laplace equation in R^3 is $H^{-1/2}(\partial\Omega)$-elliptic. Then a variational problem on the boundary corresponding to the problem has a unique solution. For the case in R^2 Le Roux[5] presented same results to Nedelec and Planchard. The same results for Laplace equation was also presented by Okamoto[7] with a different method from Nedelec and Planchard's method. Applications of the boundary element method to the equation such as (1.1) appear in formulations of numerical methods for parabolic partial differential equations, for example, steady convective diffusion problems[d], Laplace transformed equations of transient diffusion equations, semi-discrete equation in time for transient diffusion equations[1]0, and convective diffusion problems with first order reaction[8]. Furthermore in some linearizations with quasi-Newton methods for mildly non-linear partial differential equations[9], we can find some examples.

So we are interested in the boundary element method for the problem (1.1-2). It are shown that the integral equation on the boundary corresponding to the problem (1.1-2) has a unique solution in $H^{-1/2}(\partial\Omega)$, that when we discretize the integral equation by Galerkin method the Galerkin solution converge to the exact solution and that we obtain $H^1(\Omega)$ and $L^2(\Omega)$-error estimates. To this end the author gives a different way from Nedelec and Planchard. The present results are based on the results presented by Babuška[1] and Blair[2].

2. *INTEGRAL EQUATION.*

The single layer potential representation of solution to the equation (1.1) is expressed as

$$U(x) = \frac{1}{2\pi} \oint_{\partial\Omega} K_0(|x-y|)\rho(y)ds(y), \tag{2.1}$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $|x-y|$ is the distance between the points x and y, K_0 denotes the second kind modified Bessel function which is a fundamental solution for the equation (1.1), ρ is a density function defined on the boundary and s denotes the arc length of the boundary. Here we denote x the coordinate of the point in Ω. It is obvious that

$$-\Delta U(x) + U(x) = 0. \tag{2.2}$$

The integral equation on the boundary for the problem (1.1-2) :

$$\frac{1}{2\pi}\oint_{\partial\Omega} K_0(|z-y|)\rho(y)ds(y) = g(z), \tag{2.3}$$

is given as tending the internal point x to the point z on the boundary. To discuss the problem in the weak sense it is natural that we consider the integral equation (2.3) in the Sobolev space $H^{-1/2}(\partial\Omega)$. The reason is as follow. Here the ρ is the gap cross the boundary such as

$$\rho(z) = q(z)_{in} - q(z)_{ex}, \tag{2.4}$$

in which $q(z)_{in}$ and $q(z)_{ex}$ denote the outer normal derivatives defined by the limiting processes from the internal region and external region, respectively. When we consider the weak solution for the problem its flux $q \in H^{-1/2}(\partial\Omega)$. Then ρ is too. From the integral equation (2.3) we obtain the variational problem on the boundary in the form (P)

find $\rho \in H^{-1/2}(\partial\Omega)$ *such as*

$$\langle K\rho, r\rangle = \langle g, r\rangle, \tag{2.5}$$

for all $r \in H^{-1/2}(\partial\Omega)$, *in which* $g \in H^{1/2}(\partial\Omega)$.

Here

$$\langle u, v\rangle = \oint_{\partial\Omega} uvds,$$

and

$$K\rho = \frac{1}{2\pi}\oint_{\partial\Omega} K_0(|x-y|)\rho(y)ds(y).$$

In the next section we shall prove that the bilinear form $\langle K\rho, \rho\rangle$ is $H^{-1/2}$ – *elliptic.*

3. *EXISTENCE OF SOLUTION FOR (P).*

The main result in this section is as follow:

THEOREM 1. *There exists uniqe solution for the problem* (P).

The following lemma presented by Babuška is necessary in order to prove theorem 1.

LEMMA 1. *Let $h \in H^{-1/2}(\partial\Omega)$ and u be a solution of the Neumann problem for the equation $-\Delta u + u = 0$ on Ω, $\partial u/\partial n = h$ on $\partial\Omega$ in $H^1(\Omega)$. There exists constants $0 < C_1 < C_2 < \infty$ such that*

$$C_1 \oint_{\partial\Omega} huds \leq \| h \|^2_{-1/2,\partial\Omega} \leq C_2 \oint_{\partial\Omega} huds, \tag{3.1}$$

and

$$\| u \|^2_{1,\Omega} = \oint_{\partial\Omega} huds. \tag{3.2}$$

Proof. See [1]

Throughout this paper $\| u \|_{k,\Omega}$ and $\| v \|_{k,\partial\Omega}$ denote the norms of the Sobolev spaces $H^k(\Omega)$ and $H^k(\partial\Omega)$, respectively. n is the outer normal on the boundary. We denote n' the outer normal on the boundary with respect to the exterior region $\Omega^c = R^2 - \bar{\Omega}$ in which $\bar{\Omega}$ is the closure of Ω. In order to discuss fluently it is necessary to define sub-spaces $G(\Omega)$ and $\tilde{G}(\Omega)$ in $H^1(\Omega)$.

$$G(\Omega) = \{u \in H^1(\Omega) | -\Delta u + u = 0 in \Omega \quad in \quad the \quad weak \quad sense\}.$$

$$\tilde{G}(\Omega) = \{u | u = K\rho, \rho \in H^{-1/2}(\Omega)\}.$$

We have the following lemma which is similar to the lemma 1.

LEMMA 2. *Let $h \in H^{-1/2}(\partial\Omega)$ and u be a solution of the Neumann problem for the equation $-\Delta u + u = 0$ on Ω^c, $\partial u/\partial n' = h$ on $\partial\Omega$ in $G(\Omega^c)$. There exists constants $0 < C_1 < C_2 < \infty$ such that*

$$C_1 \oint_{\partial\Omega} huds \leq \| h \|_{-1/2,\partial\Omega} \leq C_2 \oint_{\partial\Omega} huds, \tag{3.3}$$

and

$$\| u \|^2_{1,\Omega^c} = \oint_{\partial\Omega} huds. \tag{3.4}$$

Proof. The Neumann problem has a solution in $G(\Omega)^c$. The statement (3.4) follows immediately from the definition of a weak solution on Ω^c. Then the proof of the lemma is done with same way to the proof of the lemma 1.

LEMMA 3. *Let the operator $Q : H^r(\partial\Omega) \to H^r(\partial\Omega)$ be defined as*

$$Qp \equiv \frac{1}{2}p + p.v. \oint_{\partial\Omega} \frac{\partial}{\partial n_y} K_0(|x-y|)p(y)ds(y). \qquad (4.2)$$

Then the operator is bounded in $H^{-1/2}(\partial\Omega)$, that is, there exists, respectively, a positive constant such that

$$\|Qp\|_{-1/2,\partial\Omega} \le C\|p\|_{-1/2,\partial\Omega}.$$

Proof. In order to prove this lemma we have to prove that

$$\langle Qp, \psi\rangle \le C\|p\|_{-1/2,\partial\Omega}\|\psi\|_{1/2,\partial\Omega}.$$

Let Q^* be the adjoint of Q. We have

$$\langle Qp, \psi\rangle = \langle p, Q^*\psi\rangle \le \|p\|_{-1/2,\partial\Omega}\|Q^*\psi\|_{1/2,\partial\Omega}.$$

If we prove that

$$\|p\|_{-1/2,\partial\Omega}\|Q^*\psi\|_{1/2,\partial\Omega} \le C\|p\|_{-1/2,\partial\Omega}\|\psi\|_{1/2,\partial\Omega},$$

the proof of this lemma is complete. To this end, by using the trace theorem we have that there exists a function w, $\tilde{w} \in H^2(\Omega)$ such as

$$\|Q^*\psi\|_{1/2,\partial\Omega} \le C\|w\|_{2,\Omega},$$

$$\|\psi\|_{1/2,\partial\Omega} \le C\|\tilde{w}\|_{2,\Omega}.$$

Since there exists a positive constant C' such as

$$\|w\|_{2,\Omega} \le C'\|\tilde{w}\|_{2,\Omega},$$

we have the inequality we need by using the invers trace theorem.

LEMMA 4. *For all p and $r \in H^{-1/2}(\partial\Omega)$*

$$\langle Kp, r\rangle \leq C \parallel p \parallel_{-1/2,\partial\Omega} \parallel r \parallel_{-1/2,\partial\Omega} .$$

Proof. Let v be a solution for the Dirichlet problem $-\Delta v + v = 0$ in Ω, $v = Kp$ on $\partial\Omega$. Note that $p = q_{in} - q_{ex}$ same as (2.4). Applying Schwarz inequality and trace theorem we have

$$\begin{aligned}\langle Kp, r\rangle &\leq \|Kp\|_{1/2,\partial\Omega}\|r\|_{-1/2,\partial\Omega}\\ &\leq C\|v\|_{1,\Omega}\|r\|_{-1/2,\partial\Omega}.\\ &\leq C\|q_{in}\|_{-1/2,\partial\Omega}\|r\|_{-1/2,\partial\Omega}\end{aligned}$$

since $v \in \tilde{G}(\Omega)$ and we have

$$\|v\|_{1,\Omega} = \frac{|\oint_{\partial\Omega} q_{in} v ds|}{\|v\|_{1,\Omega}} \leq C\frac{|\oint_{\partial\Omega} q_{in} v ds|}{\|v\|_{1/2,\partial\Omega}}.$$

From lemma 3,

$$\|q_{in}\|_{-1/2,\partial\Omega} \leq C\|p\|_{-1/2,\partial\Omega}.$$

Hence we have this lemma.

Finally we prove theorem 1.

Proof of theorem 1. From lemma 1 the bilinear form we can realize that $\langle K\rho, \rho\rangle$ is $H^{-1/2}$-elliptic. Lemma 3 implies that the bilinear form is bounded in $H^{-1/2}(\partial\Omega)$. Then according to Lax-Milgram lemma we have that the problem (P) has a unique solution in $H^{-1/2}(\partial\Omega)$.

4. $H^1(\Omega)$-*ERROR ESTIMATE*

The convergence of the Galerkin solution ,with an appropriate subspace which is constracted to obtain an internal approximation of the solution, for the integral equation (2.3) is easy to prove since we have Cea's lemma[3]:

LEMMA 5. *Suppose that the bilinear form $a(.,.)$ and the linear form f satisfy the Lax-Milgram lemma, u satisfies that*

$$a(u,v) = f(v) \qquad for \qquad all \qquad v \in V,$$

and V_h is a finite-dimensional subspace of the Banach space V. Then There exists a constant C independent of the subspace $V_h \subset V$ such that

$$\|u - u_h\|_V \leq C \inf_{v_h \in V_h} \|u - v_h\|_V.$$

From lemma 5 we have the following corollary:

COROLLARY 1. *Suppose that $V_h \subset H^{-1/2}(\partial\Omega)$. Then ρ_h, which satisfies that*

$$b\langle \rho_h, r\rangle = \langle g, r\rangle \quad for \quad all \quad r \in V_h,$$

in which $b\langle p, r\rangle \equiv \langle Kp, r\rangle$, converges to the solution for the problem (P). Moreover there exists apositive constant such that

$$\|\rho - \rho_h\|_{-1/2,\partial\Omega} \leq C \inf_{R_h \in V_h} \|\rho - R_h\|_{-1/2,\partial\Omega}.$$

Furthermore we have the following $H^{-1/2}(\partial\Omega)$-error estimate about the approximation of ρ.

THEOREM 1. *Suppose that ρ_h is constracted by set of functions χ_i on the boundary such as*

$$\chi_i = 1 \quad on \quad S_i, \quad \chi_i = 0 \quad on \quad \partial\Omega - S_i.$$

where $\cup S_i = \partial\Omega$. Then we have

$$\|\rho - \rho_h\|_{-1/2,\partial\Omega} \leq h\|\rho\|_{1/2,\partial\Omega}. \tag{4.1}$$

Proof. Suppose that $e = \rho - \rho_h$. From the definition of the norm of $H^{-1/2}(\partial\Omega)$ we have to prove that

$$|\langle e, f\rangle| \leq Ch\|\rho\|_{1/2,\partial\Omega}\|f\|_{1/2,\partial\Omega}.$$

When we assume that V_h denotes the finite dimensional subspace of $H^{-1/2}(\partial\Omega)$, R_h and $\tilde{R}_h$ are defined by

$$R_h = \inf_{\psi\in V_h} \|\rho - \psi\|_{-1/2,\partial\Omega},$$

$$\tilde{R}_h = \inf_{\psi\in V_h} \|\rho - \psi\|_{0,\partial\Omega},$$

we have

$$\|\rho - R_h\|_{-1/2,\partial\Omega} \leq \|\rho - \tilde{R}_h\|_{-1/2,\partial\Omega}.$$

Then we have

$$\begin{aligned}\langle E, f\rangle = \langle E, f-\theta\rangle &\leq \|E\|_{0,\partial\Omega}\|f-\theta\|_{0,\partial\Omega}\\ &\leq Ch^{1/2}\|\rho\|_{1/2,\partial\Omega}h^{1/2}\|f\|_{1/2,\partial\Omega}\end{aligned}$$

where $\theta \in V_h$ and $E = \rho - \tilde{R}_h$. Hence the theorem is valid from above result and Cea's lemma.

The above theorem was also presented in Nedelec and Planchard. By using theorem 1 and lemma 4 we obtain the following theorem.

THEOREM 2. *Suppose that*

$$U_h(x) = \frac{1}{2\pi}\oint K_0(x,y)\rho_h(y)ds(y). \tag{4.3}$$

Then we have

$$\|U - U_h\|_{1,\Omega} \leq h\|U\|_{2,\Omega}. \tag{4.4}$$

Proof. Here $\gamma : H^r(\Omega) \to H^{r-1/2}(\partial\Omega)$ and $\delta : H^r(\Omega) \to H^{r-3/2}(\partial\Omega)$ are trace operators. Then $\gamma e_\Omega = g - \tilde{g}$ in which $\tilde{g} = \gamma U_h$. We have

$$\begin{aligned}\|U - U_h\|_{1,\Omega} &\leq C\|\gamma e_\Omega\|_{1/2,\partial\Omega}\\ &\leq Ch\|e\|_{-1/2,\partial\Omega} \leq Ch\|\rho\|_{1/2,\partial\Omega}\\ &\leq Ch\|U\|_{1,\Omega},\end{aligned} \tag{4.5}$$

since

$$\begin{aligned}\|\rho\|_{1/2,\partial\Omega} &= \|q_{in} - q_{ex}\|_{1/2,\partial\Omega} \\ &\leq \|q_{in}\|_{1/2,\partial\Omega} + \|q_{ex}\|_{1/2,\partial\Omega} \\ &\leq C\|q_{in}\|_{1/2,\partial\Omega} \leq C\|U\|_{1,\Omega}\end{aligned}$$

Therefor we obtain theorem 2.

5. $L^2(\Omega)$-ERROR ESTIMATE.

In this section $L^2(\Omega)$-error estimate is given from results in the previus section. The following lemma which was given by Blair[2], play fundamental role in giving $L^2(\Omega)$-error estimate.

LEMMA 6. *Let $v \in H^1(\Omega)$ satisfy $-\Delta v + v = 0$; then $\|v\|_{0,\Omega} \leq C\|v\|_{-1/2,\partial\Omega}$.*

Then we have the error estimate as follow.

THEOREM 3.

$$\|U - U_h\|_{0,\Omega} \leq Ch^2\|U\|_{2,\Omega}$$

Proof. At first we prove the inequality

$$\|\gamma e_\Omega\|_{-1/2,\partial\Omega} \leq Ch\|\gamma e_\Omega\|_{1/2,\partial\Omega}.$$

From definition of the norm for $H^{-1/2}(\partial\Omega)$ we have to show the inequality

$$< \gamma e_\Omega, \theta > \leq Ch\|\gamma e_\Omega\|_{1/2,\partial\Omega}\|theta\|_{1/2,\partial\Omega}$$

for all $\theta \in H^{1/2}(\partial\Omega)$. If $\eta \in V_h$ then

$$\begin{aligned}< \gamma e_\Omega, \theta > &= < \gamma e_\Omega, \theta - \eta > \\ &\leq \|\gamma e_\Omega\|_{1/2,\partial\Omega}\|\theta - \eta\|_{-1/2,\partial\Omega} \\ &\leq \|\gamma e_\Omega\|_{1/2,\partial\Omega}Ch\|\theta\|_{1/2,\partial\Omega}.\end{aligned}$$

Therefor from theorem 2, lemma 6 and the above result we obtain this theorem.

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