

R W A - 振動子に対する森のランジュバン方程式

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§1. INTRODUCTION.

We are concerned with Mori's Langevin equation for a model of a quantum harmonic oscillator coupled to infinitely many scalar bosons whose Hamiltonian H is given formally by

$$(1.1) \quad \left\{ \begin{array}{l} H = H_0 + H_I \\ H_0 = H_S + H_B \\ H_S = \hbar\omega_0 a^+ a \quad (0 < \omega_0) \\ H_B = \sum_{k=1}^{\infty} \hbar\omega_k b_k^+ b_k \quad (0 < \omega_k < \omega_{k+1}, k \in \mathbb{N}) \\ H_I = \sum_{k=1}^{\infty} (\Gamma_k a^+ b_k + \bar{\Gamma}_k b_k^+ a) \quad (\Gamma_k \in \mathbb{C}, k \in \mathbb{N}) \end{array} \right.$$

Here a and a^+ (resp. b_k and b_k^+) denote annihilation and creation operators of a quantum harmonic oscillator (resp. boson), respectively, which act in the symmetric Boson Fock space $\mathcal{F}_S(\mathbb{C}\ell^2(\mathbb{N}))$ over $\mathbb{C}\ell^2(\mathbb{N})$. Operators H_S and H_B denote a Hamiltonian of a quantum harmonic oscillator and the one of infinitely many scalar bosons, respectively. The operator

H_I which represents the interaction between the quantum harmonic oscillator and infinitely many scalar bosons is said to be the coupling Hamiltonian for the rotating wave approximation (abbr.RWA)-oscillator. We shall simply call the operator H the Hamiltonian for the RWA-oscillator.

The behavior of the Heisenberg picture $e^{itH/\hbar} a e^{-itH/\hbar} = a(t)$ has been studied in H.Haken([9]), K.Lindenberg and B.J. West([11]), and E.Braun([4]). They considered their own equations of Langevin type for $a(t)$ whose form are dependent on their consideration. In particular, K.Lindenberg and B.J.West rewrite the total Hamiltonian H given by (1.1) into

$$(1.2) \quad \left\{ \begin{array}{l} H = H_S^{(m)} + H_B + H_I^{(m)} \\ H_S^{(m)} = \hbar(\omega_0 - \Delta) a^\dagger a \\ H_B + H_I^{(m)} = \sum_{k=1}^{\infty} \hbar \omega_k B_k^\dagger B_k \\ B_k = b_k + \frac{\bar{\Gamma}_k}{\hbar \omega_k} a \end{array} \right.$$

$$\text{where } \Delta = \sum_{k=1}^{\infty} \frac{|\Gamma_k|^2}{\hbar^2 \omega_k} .$$

By solving a simultaneous system of differential equations for $a(t)$ and $B_k(t)$:

$$(1.3) \quad \begin{cases} \frac{d}{dt}a(t) = \frac{i}{\hbar}[H, a(t)] \\ \frac{d}{dt}B_k(t) = \frac{i}{\hbar}[H, B_k(t)] \end{cases} .$$

they derived an equation of Langevin type for $q(t)$. However, the physical meaning of the quantities $\omega_0^{-\Delta}$ and Δ are not clear in [11]. While, in his theory of generalized Brownian motion in statistical physics, H. Mori derived the so-called Mori's Langevin equation which consists of the Mori's frequency, Mori's memory function and Mori noise ([12], [13]). Mathematically, Mori's Langevin equation can be derived if a Mori-Okabe (abbr. an MO)-structure is given. Here an MO-structure consists of the triplet of a Hilbert space X , a self-adjoint operator L on X and a non-zero A_0 in the domain of L ([15], [16], [17]).

The purpose is to show that the Hamiltonian H in (1.1) has an MO-structure and investigate Mori's Langevin equation for the Heisenberg picture $q(t)$ in detail. In order to carry out it, we shall construct a Hilbert space $X_C(H)$ of unbounded operators on $\mathcal{F}_S(\mathbb{C} \otimes \ell^2(\mathbb{N}))$ containing the annihilation and creation operators, where the inner product of $X_C(H)$ is introduced from Bogoliubov scalar product (Kubo-

Mori scalar product, or the canonical correlation) (see [5, p96], [13]). Furthermore, we shall construct a self-adjoint operator L (Liouville operator) on $X_C(H)$ in such a way that the Heisenberg picture of q by the operator H coincides with the time evolution of q by the operator L . Since the triplet $(X_C(H), L, q)$ satisfies an MO-structure, we can develop Mori's theory of generalized Brownian motion on $X_C(H)$. The main point is to express the Mori's frequency, the complex mobility of the Mori's memory function and the canonical correlation function of $q(t)$ in terms of the parameters in the Hamiltonian H in (1.1), by obtaining the Bogoliubov transformation of H . Furthermore, we shall give a physical meaning of two constants $\omega_0 - \Delta$ and Δ in (1.2), and show that the canonical correlation function is almost periodic, and does not converge as the time tends to infinity.

§2. MO-STRUCTURES.

By an MO-structure, we mean a triplet (X, L, A_0) such that X is a Hilbert space with an inner product $(\cdot, \cdot)_X$, L a self-adjoint operator on X with domain $D(L)$, and A_0 a non-zero element in $D(L)$, where the inner product $(\cdot, \cdot)_X$ is linear

in the right vector . For any MO-structure , we consider a stationary curve $A=\{A(t);t\in\mathbb{R}\}$ defined by

$$(2.1) \quad A(t) := e^{itL}A_0 \quad (t \in \mathbb{R})$$

and the correlation function R_A of A is given by

$$(2.2) \quad R_A(t) := (A(0),A(t))_X .$$

Let X_0 be the closed subspace generated by A_0 , and P_0 and X_1 the orthogonal projection operator on X_0 and the complementary subspace of X_0 in X , respectively. Then we define a linear operator L_1 on the Hilbert space X_1 by

$$(2.3) \quad \begin{cases} D(L_1) := X_1 \cap D(L) \\ L_1x := (1 - P_0)Lx \quad (x \in D(L_1)) . \end{cases}$$

Lemma 2.1. ([12],[15],[16]) L_1 is self-adjoint on the Hilbert space X_1 .

We define a real number $\omega=\omega(A_0)$, a stationary curve $I_M=\{I_M(t);t\in\mathbb{R}\}$ in X_1 and a non-negative definite function ϕ on \mathbb{R} by

$$(2.4) \quad \omega = \omega(A_0) := (A(0),LA(0))_X \cdot (A(0),A(0))_X^{-1} .$$

$$(2.5) \quad I_M(t) := i \cdot e^{itL_1} \cdot (1 - P_0)LA_0 ,$$

$$(2.6) \quad \phi(t) := (I_M(0),I_M(t))_X \cdot (A(0),A(0))_X^{-1} .$$

Concerning the correlation function R_A , we have

Theorem 2.2.([7,§6.2],[12],[15],[16]) (a) For all $t\in\mathbb{R}$,

$$\frac{d}{dt}R_A(t) = i\omega \cdot R_A(t) - \int_0^t ds \phi(t-s)R_A(s) .$$

(b) For all $z \in \mathbb{C}^+ := \{z \in \mathbb{C}; \text{Im}z > 0\}$,

$$\int_0^\infty dt e^{itz} R_A(t) = R_A(0) \cdot \frac{1}{i\omega - iz + \int_0^\infty dt e^{itz} \phi(t)}$$

Furthermore, the equation of motion describing the time evolution of stationary curve $A = \{A(t); t \in \mathbb{R}\}$ is given in the following

Theorem 2.3. ([7, §6.2], [12], [15], [16]) For all $t \in \mathbb{R}$,

$$(2.7) \quad \frac{d}{dt}A(t) = i\omega \cdot A(t) - \int_0^t ds \phi(t-s)A(s) + I_M(t) .$$

The quantities $\omega = \omega(A_0)$, ϕ and I_M are called the Mori's frequency, Mori's memory function and Mori noise , respectively .

The equation (2.7) is said to be the Mori's Langevin equation associated with the MO-structure (X, L, A_0) .

§3. CONSTRUCTION OF A HILBERT SPACE ASSOCIATED WITH A CLASS OF CLOSABLE OPERATORS.

Let \mathcal{F} be a separable Hilbert space with an inner product (\cdot, \cdot) , which is linear in the right vector. Let H be a non-

negative self-adjoint operator acting in \mathcal{F} with the following properties (H.1) and (H.2) :

(H.1) There exists a complete orthonormal basis $\{\varphi_n; n \in \mathbb{N}^* = \{0\} \cup \mathbb{N}\}$ in $D(H)$ such that $H\varphi_n = \lambda_n \varphi_n$ with $0 \leq \lambda_n \leq \lambda_{n+1}$ ($n \in \mathbb{N}^*$).

(H.2) $e^{-\tau H} \in \mathcal{F}_1$ for all $\tau \in (0, \beta]$, where $\beta > 0$ is the inverse temperature and \mathcal{F}_1 denotes the family of all trace class operators on \mathcal{F} .

Let

$$(3.1) \quad D := \left\{ \sum_{k=0}^n \alpha_k \varphi_k; n \in \mathbb{N}^*, \alpha_k \in \mathbb{C}, k=0, 1, \dots, n \right\}.$$

Obviously D is dense in \mathcal{F} . We denote by $\mathcal{L}(D, \mathcal{F})$ the space of bounded linear operators from D into \mathcal{F} . Every element A in $\mathcal{L}(D, \mathcal{F})$ has a unique extension to an element in $\mathcal{L}(\mathcal{F})$, the space of bounded linear operators on \mathcal{F} . We denote the extension of A by \bar{A} or A^- .

Let $\mathcal{F}(D, H)$ be the set of linear operators A acting in \mathcal{F} with the following properties (F.1) - (F.3) :

$$(F.1) \quad D \subset D(A) \cap D(A^*) .$$

$$(F.2) \quad \text{For all } x \in D, Ax \text{ and } A^* x \text{ are in } D .$$

$$(F.3) \quad \text{For all } \tau \in (0, \beta], e^{-\tau H} A \text{ (resp. } A e^{-\tau H}) \text{ is in } \mathcal{L}(D, \mathcal{F}) \\ \text{and } (e^{-\tau H} A)^- \text{ (resp. } (A e^{-\tau H})^-) \text{ is in } \mathcal{F}_2, \text{ where } \mathcal{F}_2 \\ \text{denotes the family of all Hilbert-Schmidt operators}$$

with the Hilbert-Schmidt norm $\| \cdot \|_2$.

In this section, we shall construct a Hilbert space $X_C(H)$ and a Liouville operator L associated with the Hamiltonian H and show that the triplet $(X_C(H), L, A_0)$ satisfies an MO-structure for any non-zero $A_0 \in D(L)$.

If two operators A and B in $\mathcal{F}(D, H)$ satisfy $Ax=Bx$ for all $x \in D$, then we write as $A \sim B$, which gives an equivalence relation in $\mathcal{F}(D, H)$. We denote by $[A]$ the equivalence class of $A \in \mathcal{F}(D, H)$ and by $\mathcal{F}(D, H)/\sim$ the set of all the equivalence classes.

We can introduce in $\mathcal{F}(D, H)/\sim$ the operation of addition, scalar multiplication and involution $*$ as follows:

$$(3.2) \quad [A] + [B] := [A+B].$$

$$(3.3) \quad \alpha[A] := [\alpha A] \quad (\alpha \in \mathbb{C}).$$

$$(3.4) \quad [A]^* := [A^*].$$

We can define a correlation; Bogoliubov scalar product (Kubo-Mori scalar product, or the canonical correlation) $\langle \cdot ; \cdot \rangle_H$ on $\mathcal{F}(D, H)/\sim$ as follows: For any $[A], [B] \in \mathcal{F}(D, H)/\sim$,

$$(3.5) \quad \langle [A]; [B] \rangle_H := \frac{1}{\beta} \frac{\int_0^\beta d\lambda \sum_n (A e^{-(\beta-\lambda)H} \varphi_n, e^{-\lambda H} B \varphi_n)}{\text{tr}(e^{-\beta H})}$$

It can be easily seen to prove that $\langle ; \rangle_H$ is inner products on $\mathcal{F}(D, H)/\sim$. We denote the norm of $[A]$ by

$$(3.6) \quad \|[A]\|_H := \langle [A]; [A] \rangle_H^{1/2} .$$

We can define an element $A(t)$ in $\mathcal{F}(D, H)$ and its equivalence class $[A](t)$ by

$$(3.7) \quad A(t) := e^{itH/\hbar} A e^{-itH/\hbar} \quad (A \in \mathcal{F}(D, H) , t \in \mathbb{R}) ,$$

$$(3.8) \quad [A](t) := [A(t)] \quad (A \in \mathcal{F}(D, H) , t \in \mathbb{R}) ,$$

where $\hbar > 0$ is a parameter denoting the Planck constant divided by 2π .

Lemma 3.1. For all $A \in \mathcal{F}(D, H)$, $\|[A]\|_H = \|[A]^*\|_H$.

We obtain a Hilbert space $X_C(H)$ as the completion of $\mathcal{F}(D, H)/\sim$ by the norm $\| \cdot \|_H$.

Remark 3.1. We can define an involution $[A] \rightarrow [A]^+$ on $X_C(H)$ such that $[A]^+ = [A]^*$ for all $[A] \in \mathcal{F}(D, H)/\sim$.

Let

$$(3.9) \quad \mathcal{F}_\delta := \{A \in \mathcal{F}(D, H) ; AH , HA \in \mathcal{F}(D, H)\} ,$$

$$(3.10) \quad D(\delta) := \{[A] \in X_C(H) ; A \in \mathcal{F}_\delta\} .$$

$D(\delta)$ is a dense subspace in $X_C(H)$.

We define a linear operator $\delta: D(\delta) \rightarrow X_C(H)$ by

$$(3.11) \quad \delta[A] := \left[\frac{1}{\hbar} [H, A] \right] \quad ([A] \in D(\delta)) .$$

where $[H, A] := HA - AH$. Then, δ is a symmetric operator.

We define for each $t \in \mathbb{R}$ the operator $U(t) : \mathcal{F}(D, H) / \sim \rightarrow X_C(H)$ by

$$(3.12) \quad U(t)[A] := [A](t) \quad (A \in \mathcal{F}(D, H)) .$$

Proposition 3.2. For any $[A] \in \mathcal{F}(D, H) / \sim$, and $t, s \in \mathbb{R}$,

(a) $U(t)$ is unitary on $\mathcal{F}(D, H) / \sim$,

(b) $U(0)[A] = [A]$,

(c) $U(t + s)[A] = U(t)U(s)[A]$,

(d) $s\text{-}\lim_{t \rightarrow 0} U(t)[A] = [A]$.

Since it follows from Proposition 3.2 that $\{U(t); t \in \mathbb{R}\}$ is uniquely extended to a strongly continuous unitary group on $X_C(H)$, we denote its extension by the same symbol. By Stone's theorem, there exists a unique self-adjoint operator L on $X_C(H)$ such that

$$(3.13) \quad U(t) = e^{itL} .$$

Proposition 3.3. $L \supset \delta$.

Remark 3.2. Proposition 3.3 means that the time evolution by Liouville operator L coincides with the Heisenberg picture on $D(\delta)$, i.e.,

$$(3.14) \quad e^{itL}[A] = [e^{itH/\hbar} A e^{-itH/\hbar}] \quad ([A] \in D(\delta)) .$$

Definition 3.4. We say that $A \in \mathcal{F}(D, H)$ is in $M(D, H)$ if

it satisfies the following condition (C.1) :

(C.1) For all τ, τ' with $0 < \tau' \leq \tau \leq \beta$, there exist non-negative functions $f_A(\tau, \tau'), g_A(\tau, \tau'), f_A^*(\tau, \tau')$ and $g_A^*(\tau, \tau') \geq 0$ such that , for all $x \in D$,

$$\begin{cases} \|e^{-\tau H} A x\| \leq f_A(\tau, \tau') \|e^{-(\tau-\tau')H} A e^{-\tau' H} x\| + g_A(\tau, \tau') \|e^{-\tau' H} x\| \\ \|e^{-\tau H} A^* x\| \leq f_A^*(\tau, \tau') \|e^{-(\tau-\tau')H} A^* e^{-\tau' H} x\| \\ \quad + g_A^*(\tau, \tau') \|e^{-\tau' H} x\| . \end{cases}$$

If A and $B \in \mathcal{F}(D, H)$ are in $M(D, H)$, we have $AB \in \mathcal{F}(D, H)$. Then , for any $A, B \in M(D, H)$, we define the product of $[A]$ and $[B] \in \mathcal{F}(D, H) / \sim$ by

$$(3.15) \quad [A][B] := [AB] .$$

This definition is independent of the choice of the representatives of $[A]$ and $[B]$.

Proposition 3.5. Suppose that $\{B_0, B_1, \dots, B_N\} \subset \mathcal{F}(D, H)$ ($N \in \mathbb{N}^*$) satisfy the following conditions :

$$B_k B_\ell^*, B_\ell^* B_k \in \mathcal{F}(D, H) \text{ and } [B_k, B_\ell] x \delta_{k\ell} x \quad (k, \ell = 0, 1, \dots, N, x \in D)$$

Then , $\{[B_0], [B_1], \dots, [B_N]\}$ is linearly independent .

Definition 3.6. We say that $A \in \mathcal{F}(D, H)$ is H -diagonal if there exists $\sigma_A \in \mathbb{R}$ such that

$$(3.16) \quad [H, A]x = -\hbar \sigma_A A x \quad , \quad x \in D .$$

Remark 3.3. If A is H -diagonal with $[A] \neq 0$, then σ_A is

uniquely determined .

Lemma 3.7. For any H-diagonal $A \in \mathcal{F}(D, H)$, $x \in D$, $\tau > 0$ and $t \in \mathbb{R}$.

$$(a) A e^{-\tau H} x = \exp[-\tau \hbar \sigma_A] e^{-\tau H} A x ,$$

$$(b) A e^{itH/\hbar} x = \exp[it\sigma_A] e^{itH/\hbar} A x .$$

Proposition 3.8. If $A, B \in \mathcal{F}(D, H)$ are H-diagonal and $\sigma_A \neq \sigma_B$, then $\langle [A]; [B] \rangle_H = 0$.

Lemma 3.9. If $A \in \mathcal{F}(D, H)$ is H-diagonal, then A is in $M(D, H)$.

Proposition 3.10. (a) If $A \in \mathcal{F}(D, H)$ is H-diagonal, then $A^* \in \mathcal{F}(D, H)$ is also H-diagonal. Moreover, if $[A] \neq 0$, then

$$\sigma_{A^*} = -\sigma_A .$$

(b) If $A, B \in \mathcal{F}(D, H)$ are H-diagonal , then $AB \in \mathcal{F}(D, H)$ is also H-diagonal . Moreover, if $[AB] \neq 0$, then $\sigma_{AB} = \sigma_A + \sigma_B$.

Lemma 3.11. If $A \in \mathcal{F}(D, H)$ is H-diagonal , then $A \in \mathcal{F}_\delta$ and so $[A] \in D(\delta)$.

We define a subset $\mathfrak{A}_f(H)$ of $X_C(H)$ as follows :

$$(3.17) \quad \mathfrak{A}_f(H) := \{u_0[A_0] + u_1[A_1] + \dots + u_N[A_N] \in X_C(H); N \in \mathbb{N}^*, u_0, u_1, \dots, u_N \in \mathbb{C}, \text{ and } A_0, A_1, \dots, A_N \in \mathcal{F}(D, H) \text{ are H-diagonal}\} .$$

Proposition 3.12. $\mathfrak{A}_f(H)$ is a *-algebra and $\mathfrak{A}_f(H) \subset D(\delta)$.

Remark 3.4. $X_C(H)$ is a partial *-algebra with a unity (see , [1, Definition 2.1. and Definition 2.2.])

Lemma 3.13. For all $[A]$, $[B] \in \mathfrak{A}_f(H)$ and $t \in \mathbb{R}$.

- (a) $L[A]$ and $e^{itL}[A] \in \mathfrak{L}_f(H)$,
 (b) $L([A][B]) = (L[A])[B] + [A](L[B])$,
 (c) $L[A]^* = - (L[A])^*$.

Lemma 3.14. Suppose that $A \in \mathfrak{F}(D, H)$ is H -diagonal with $\sigma_A > 0$ and $[A, A^*]x = x$ for all $x \in D$. Then ,

- (a)
$$\frac{\text{tr}((e^{-\beta H} A^* A)^-)}{\text{tr}(e^{-\beta H})} = \frac{1}{\exp[\beta \hbar \sigma_A] - 1}$$

 (b)
$$\| [A] \|_H^2 = \| [A]^* \|_H^2 = \frac{1}{\beta \hbar \sigma_A}$$

Definition 3.15. We say that the linear operator A is in $\mathfrak{C}_\eta(H)$ ($\eta \in (0, \beta)$) if A satisfies the following conditions

(C.1) and (C.2) _{η} :

(C.1) (= (F.1)) $D(A)$, $D(A^*) \supset D$.

(C.2) _{η} $(e^{-\eta H/2} A)^-$, $(Ae^{-(\beta - \eta)H/2})^- \in \mathfrak{F}_2$.

Lemma 3.16. For all $\eta \in (0, \beta)$,

- (a) If A is in $\mathfrak{C}_\eta(H)$, then A^* is in $\mathfrak{C}_{(\beta - \eta)}(H)$,
 (b) $\mathfrak{C}_\eta(H)$ is a complex vector space ,
 (c) $\mathfrak{F}(D, H) \subset \mathfrak{C}_\eta(H)$.

Lemma 3.17. For each $\eta \in (0, \beta)$ and $A \in \mathfrak{C}_\eta(H)$, there exists a Cauchy sequence $\{[A_N]; N \in \mathbb{N}^*\} \subset \mathfrak{F}(D, H)/\sim$ such that

- (a) $A_N \in \mathfrak{F}_\delta$,
 (b) $\lim_{N \rightarrow \infty} A_N x = Ax$ in \mathfrak{F} for all $x \in D$,

(c) As a function of $\lambda \in [0, \beta]$,

$\sum_n ((A - A_N) e^{-(\beta - \lambda)H} \varphi_n, e^{-\lambda H} (A - A_N) \varphi_n)$ converges uniformly to 0 as $N \rightarrow \infty$.

For every $A \in \mathfrak{C}_\eta(H)$ ($\eta \in (0, \beta)$) , we can define an element of $X_C(H)$ $[A]$ by

$$(3.18) \quad [A] := \lim_{N \rightarrow \infty} [A_N] .$$

Remark 3.5. If we can take another convergent sequence $\{[B_N]; N \in \mathbb{N}^*\} \subset \mathfrak{S}(D, H) / \sim$ to the operator A in the sense of (b) and (c) of Lemma 3.17 , we can show that $\lim_{N \rightarrow \infty} [B_N] = [A] \in X_C(H)$. Furthermore , we can show that there exists an injective mapping $\iota : \mathfrak{C}_\eta(H) \upharpoonright D \rightarrow X_C(H)$ defined by

$$\iota(A \upharpoonright D) := [A] ,$$

where $\mathfrak{C}_\eta(H) \upharpoonright D := \{A \upharpoonright D ; A \in \mathfrak{C}_\eta(H)\}$.

Definition 3.18. For each $\eta \in (0, \beta)$,

$$\mathfrak{C}_\eta(H) / \sim := \iota(\mathfrak{C}_\eta(H) \upharpoonright D) .$$

Lemma 3.19. For each $A \in \mathfrak{C}_\eta(H)$, $\eta \in (0, \beta)$ and $t \in \mathbb{R}$,

(a) $e^{itH/\hbar} A e^{-itH/\hbar}$ is in $\mathfrak{C}_\eta(H)$,

(b) $[e^{itH/\hbar} A e^{-itH/\hbar}] = e^{itL} [A]$,

(c) $[A]^+ = [A^*]$, and $[A]^+ \in \mathfrak{C}_{(\beta - \eta)}(H)$.

Remark 3.6. For $[A] \in \mathfrak{C}_\eta(H) / \sim$ ($\eta \neq \beta/2$) , it does not always hold that $[A]^+$ is in $\mathfrak{C}_\eta(H) / \sim$.

§4. MORI'S LANGEVIN EQUATION FOR THE RWA-OSCILLATOR

Let a complex Hilbert space $\ell^2(\mathbb{N})$ be given by

$$(4.1) \quad \ell^2(\mathbb{N}) := \{(c_k; k \in \mathbb{N}) \in \mathbb{C}^{\mathbb{N}}; \sum_{k=1}^{\infty} |c_k|^2 < \infty\} .$$

For each $f \in \mathbb{C} \oplus \ell^2(\mathbb{N})$, we denote f by

$$(4.2) \quad f = (f_0, f_1, f_2, \dots) ,$$

where $f_0 \in \mathbb{C}$ and $(f_1, f_2, \dots) \in \ell^2(\mathbb{N})$.

An inner product $\langle | \rangle$ of $\mathbb{C} \oplus \ell^2(\mathbb{N})$ is given by

$$(4.3) \quad \langle f | g \rangle := \sum_{k=0}^{\infty} \bar{f}_k g_k \quad (f, g \in \mathbb{C} \oplus \ell^2(\mathbb{N})) .$$

Let $\mathcal{F}_S(\mathbb{C} \oplus \ell^2(\mathbb{N}))$ be the symmetric Boson Fock space over $\mathbb{C} \oplus \ell^2(\mathbb{N})$, i.e.,

$$(4.4) \quad \mathcal{F}_S(\mathbb{C} \oplus \ell^2(\mathbb{N})) = \bigoplus_{n=0}^{\infty} S_n(\mathbb{C} \oplus \ell^2(\mathbb{N}))^n ,$$

where, for all $n \in \mathbb{N}$, $S_n(\mathbb{C} \oplus \ell^2(\mathbb{N}))^n$ is the n -fold symmetric tensor product of $\mathbb{C} \oplus \ell^2(\mathbb{N})$, $S_0(\mathbb{C} \oplus \ell^2(\mathbb{N}))^0 := \mathbb{C}$ (see, e.g., [20, p.53]).

For any $f \in \mathbb{C} \oplus \ell^2(\mathbb{N})$, we define $B^+(f) : S_n(\mathbb{C} \oplus \ell^2(\mathbb{N}))^n \rightarrow S_{n+1}(\mathbb{C} \oplus \ell^2(\mathbb{N}))^{n+1}$ by

$$(4.5) \quad B^+(f)\psi := \sqrt{n+1} S_{n+1}(f \otimes \psi) \quad (\psi \in S_n(\mathbb{C} \oplus \ell^2(\mathbb{N}))^n) .$$

Let

$$(4.6) \quad \mathcal{F}_F(\mathbb{C} \oplus \ell^2(\mathbb{N})) := \{\psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_S(\mathbb{C} \oplus \ell^2(\mathbb{N})) ; \text{ there exists } n_0 \in \mathbb{N}^* \text{ such that, for all } n \geq n_0, \psi^{(n)} = 0\} .$$

The subspace $\mathcal{F}_F(\mathbb{C}\otimes\ell^2(\mathbb{N}))$ is dense in $\mathcal{F}_S(\mathbb{C}\otimes\ell^2(\mathbb{N}))$ (see, e.g., [6, p.68]). We denote the Fock vacuum Ω_0 by

$$(4.7) \quad \Omega_0 := \{1, 0, 0, \dots\} .$$

For each $f \in \mathbb{C}\otimes\ell^2(\mathbb{N})$, we define a linear operator $A^+(f)$ on $\mathcal{F}_F(\mathbb{C}\otimes\ell^2(\mathbb{N}))$ by

$$(4.8) \quad \begin{cases} (A^+(f)\psi)^{(n)} := B^+(f)\psi^{(n-1)} & (n \in \mathbb{N}) \\ (A^+(f)\psi)^{(0)} := 0 \end{cases} .$$

Then, $A^+(f)$ is densely defined, and we put

$$(4.9) \quad A(f) := \overline{A^+(f)^* \upharpoonright \mathcal{F}_F(\mathbb{C}\otimes\ell^2(\mathbb{N}))} .$$

$A(f)$ and $A^+(f)$ are called the annihilation and creation operator, respectively (see, e.g., [6, §.3.1]). Operators $A^+(f)$ and $A(f)$ are closable and the closure of them are denoted by the same notation.

Let $N = d\Gamma(I)$ be the second quantization of the identity I (the number operator). It is well-known that

$$(4.10) \quad D(A^+(f)), D(A(f)) \supset D(N^{1/2})$$

with estimates

$$(4.11) \quad \begin{cases} \|A^+(f)\psi\| \leq \|f\| \cdot \|(N+1)^{1/2}\psi\| \\ \|A(f)\psi\| \leq \|f\| \cdot \|N^{1/2}\psi\| \end{cases} \quad (\psi \in D(N^{1/2}))$$

Futhermore, $A^+(f)$ and $A(f)$ leave $\mathcal{F}_F(\mathbb{C}\otimes\ell^2(\mathbb{N}))$ invariant which satisfy the canonical commutation relations

$$(4.12) \quad \begin{cases} [A(f), A(g)] = 0, [A^+(f), A^+(g)] = 0 \\ [A(f), A^+(g)] = \langle f | g \rangle, \end{cases} \quad (f, g \in \mathbb{C} \otimes \ell^2(\mathbb{N}))$$

on $\mathcal{F}_F(\mathbb{C} \otimes \ell^2(\mathbb{N}))$, where $[A, B] := AB - BA$.

Let $\{e_k; k \in \mathbb{N}^*\}$ be a complete orthonormal system of $\mathbb{C} \otimes \ell^2(\mathbb{N})$ given by

$$(4.13) \quad e_k := \underset{\substack{\text{(k+1)-th} \\ \cup}}{(0, 0, \dots, 0, 1, 0, 0, \dots)} .$$

We put

$$(4.14) \quad \begin{cases} a := A(e_0) , \\ a^+ := A^+(e_0) , \\ b_k := A(e_k) , \\ b_k^+ := A^+(e_k) . \end{cases} \quad (k \in \mathbb{N})$$

Let $D^{(0)}$ denote the algebraic span of a complete orthonormal basis of $\mathcal{F}_S(\mathbb{C} \otimes \ell^2(\mathbb{N}))$ (see [3]),

$$\frac{1}{\sqrt{n_0! n_1! \dots n_M!}} (a^+)^{n_0} (b_1^+)^{n_1} \dots (b_M^+)^{n_M} \Omega_0, \quad M \in \mathbb{N}^*, n_0, n_1, \dots, n_M \in \mathbb{N}^* .$$

Operators a and a^+ physically denote the annihilation and creation operators of a quantum harmonic oscillator, respectively. On the other hand, the operators b_k and b_k^+ denote the annihilation and creation operators of a heat bath, respectively.

Let $\{\omega_k; k \in \mathbb{N}^*\}$ and $\{\Gamma_k; k \in \mathbb{N}\}$ be sequences satisfying the following conditions (A.1), (A.2) and (A.3) :

$$(A.1) \quad \left\{ \begin{array}{l} 0 < \omega_0 ; \quad 0 < \omega_1 < \omega_2 < \dots , \\ \sum_{k=1}^{\infty} \frac{1}{\omega_k} < \infty . \end{array} \right.$$

$$(A.2) \quad \sum_{k=1}^{\infty} \omega_k^2 |\Gamma_k|^2 < \infty .$$

$$(A.3) \quad \hbar \omega_0 > \sum_{k=1}^{\infty} \frac{|\Gamma_k|^2}{\hbar \omega_k} .$$

Example: We have two examples , as the frequency $\{\omega_k; k \in \mathbb{N}\}$ satisfying (A.1) :

$$\omega_k = (k^2 + m^2)^{1/2} , \quad m > 0 \quad (\text{the relativistic case}) ,$$

$$\omega_k = k^2 (2M)^{-1} , \quad M > 0 \quad (\text{the non-relativistic case}) .$$

We can define a positive self-adjoint operator ω on $\mathbb{C} \otimes \ell^2(\mathbb{N})$ by

$$(4.15) \quad \left\{ \begin{array}{l} \omega e_0 := \hbar \omega_0 e_0 , \\ \omega e_k := \hbar \omega_k e_k \quad (k \in \mathbb{N}) . \end{array} \right.$$

Then we get the free Hamiltonian H_0 associated with ω defined by

$$(4.16) \quad H_0 := d\Gamma(\omega) ,$$

where $d\Gamma(\omega)$ is the second quantization of ω .

Let

$$(4.17) \quad \gamma := (0, \overline{\Gamma_1}, \overline{\Gamma_2}, \dots) \in \mathbb{C} \otimes \ell^2(\mathbb{N}) .$$

and let

$$(4.18) \quad H_I := A^+(\gamma)A(e_0) + A^+(e_0)A(\gamma) .$$

The operator H_I describes the Hamiltonian of the oscillator interacting with infinitely many scalar bosons . We note that

$$(4.19) \quad H_I = \sum_{k=1}^{\infty} (\overline{\Gamma_k} b_k^+ a + \Gamma_k a^+ b_k) \quad \text{on } D^{(0)} .$$

Since the operator H_I is well-defined on $\mathcal{F}_F(\mathbb{C} \otimes \ell^2(\mathbb{N}))$ and symmetric , we denote the closure of $H_I \upharpoonright \mathcal{F}_F(\mathbb{C} \otimes \ell^2(\mathbb{N}))$ by the same symbol . Then , the total Hamiltonian H is given by

$$(4.20) \quad H := H_0 + H_I .$$

We have $D(H_0) \subset D(H)$, and the closure of $H \upharpoonright D(H_0)$ is essentially self-adjoint on any core for H_0 . We shall denote the closure of $H \upharpoonright D(H_0)$ by the same symbol H .

We define a linear operator \mathbb{L} on $\mathbb{C} \otimes \ell^2(\mathbb{N})$ as follows :

$$(4.21) \quad D(\mathbb{L}) := \text{E=L.h.} [\{e_k; k \in \mathbb{N}^*\}]$$

$$(4.22) \quad \mathbb{L} e_k := \begin{cases} \omega_0 e_0 + \sum_{\ell=1}^{\infty} \frac{\overline{\Gamma_\ell}}{\hbar} e_\ell & (k=0) \\ \frac{\Gamma_k}{\hbar} e_0 + \omega_k e_k & (k=1, 2, \dots) . \end{cases}$$

It is easy to see that the operator \mathbb{L} can be extended to

a closed symmetric operator on $\mathbb{C}\otimes\ell^2(\mathbb{N})$, and E is a core for its extension.

Lemma 4.1. (a) L is self-adjoint, and there exists a unitary operator U on $\mathbb{C}\otimes\ell^2(\mathbb{N})$ such that $U^*LUe_p = \epsilon_p e_p$, where $\epsilon_p > 0$ for any $p \in \mathbb{N}^*$ and $\{\epsilon_p; p \in \mathbb{N}^*\}$ is the all zeros

$$\text{of } D(z) := z - \omega_0 + \sum_{k=1}^{\infty} \frac{|\Gamma_k|^2}{\hbar^2(\omega_k - z)}.$$

(b) Put $\langle e_k | Ue_p \rangle =: u_{kp}$ ($k, p \in \mathbb{N}^*$), then

$$u_{kp} = - \frac{\overline{\Gamma_k}}{\hbar(\omega_k - \epsilon_p)} u_{0p} \quad (k \in \mathbb{N}; p \in \mathbb{N}^*),$$

$$\epsilon_p - \omega_0 + \sum_{k=1}^{\infty} \frac{|\Gamma_k|^2}{\hbar^2(\omega_k - \epsilon_p)} = 0 \quad (p \in \mathbb{N}^*),$$

$$|u_{0p}|^2 = \left\{ 1 + \sum_{k=1}^{\infty} \frac{|\Gamma_k|^2}{\hbar^2(\omega_k - \epsilon_p)^2} \right\}^{-1} \quad (p \in \mathbb{N}^*).$$

Let $\Gamma(U)$ be a unitary operator on $\mathcal{F}_S(\mathbb{C}\otimes\ell^2(\mathbb{N}))$ defined by

$$(4.23) \quad \Gamma(U) \upharpoonright S_n(\mathbb{C}\otimes\ell^2(\mathbb{N}))^n = \otimes_{k=1}^n U \quad (n \in \mathbb{N}),$$

$$(4.24) \quad \Gamma(U) \upharpoonright S_0(\mathbb{C}\otimes\ell^2(\mathbb{N}))^0 = I \quad (\text{the identity}),$$

and define for each $f \in \mathbb{C}\otimes\ell^2(\mathbb{N})$

$$(4.25) \quad \begin{cases} \beta^+(f) := \Gamma(U)A^+(f)\Gamma(U)^{-1}, \\ \beta(f) := \Gamma(U)A(f)\Gamma(U)^{-1}. \end{cases}$$

We put

$$(4.26) \quad \begin{cases} \beta_k^+ := \beta^+(e_k) , \\ \beta_k := \beta(e_k) . \end{cases} \quad (k \in \mathbb{N}^*)$$

Since U is unitary on $\mathbb{C} \otimes \ell^2(\mathbb{N})$, the following commutation relations hold on $\mathcal{F}_F(\mathbb{C} \otimes \ell^2(\mathbb{N}))$, for all $f, g \in \mathbb{C} \otimes \ell^2(\mathbb{N})$.

$$(4.27) \quad [\beta(f), \beta^+(g)] = \langle f | g \rangle ,$$

$$(4.28) \quad [\beta(f), \beta(g)] = 0 , \quad [\beta^+(f), \beta^+(g)] = 0 .$$

Lemma 4.2. $\sigma(H) = \{ \hbar \epsilon_0 n_0 + \dots + \hbar \epsilon_N n_N ; N \in \mathbb{N}^*, n_0, \dots, n_N \in \mathbb{N}^* \}$.

Corollary: H satisfies conditions (H.1) and (H.2).

Lemma 4.3. (a) $\beta_p, \beta_p^+ \in \mathcal{S}(D, H)$ are H -diagonal.

$$(b) \quad \sigma_{\beta_p} = -\epsilon_p = -\sigma_{\beta_p^+} \quad (p \in \mathbb{N}^*) .$$

Lemma 4.4. a and $a^+ \in \mathcal{C}_\eta(H)$ ($\eta \in (0, \beta)$).

Lemma 4.5. (a) $\omega(a) = -(\mathbb{L}^{-1})_{00}^{-1}$

where $\omega(a)$ denotes the Mori's frequency of a .

$$(b) \quad \text{For } z \in \mathbb{C}^+, \quad \int_0^\infty dt \phi(t) e^{itz} = i\omega(a) \cdot \frac{F(z)}{1+F(z)} ,$$

where ϕ is the Mori's memory function of a .

$$F(z) := \sum_{p=0}^{\infty} \frac{\overline{(\mathbb{L}^{-1})_{kp}} ((z-\mathbb{L})^{-1})_{kp}}{p \neq k \quad (\mathbb{L}^{-1})_{kk} ((z-\mathbb{L})^{-1})_{kk}} ,$$

and, for any operator T on $\mathbb{C} \otimes \ell^2(\mathbb{N})$, $T_{kp} = \langle e_k | T e_p \rangle$.

Theorem 4.6. (a) Mori's Langevin equation for $[a](t)$ is

$$(4.29) \quad \frac{d}{dt}[a](t) = iD(0)[a](t) - \int_0^t ds \phi(t-s)[a](s) + [I_M](t) .$$

Here the Mori's memory function ϕ is characterized by

$$(4.30) \quad \frac{1}{2\pi} \int_0^\infty ds e^{itz} \phi(t) = \frac{iD(0)F(z)}{2\pi(1+F(z))} \quad (z \in \mathbb{C}^+) .$$

and $\phi(0) = -D(0)\Delta = -\omega(a)\Delta$, where

$$(4.31) \quad F(z) := \sum_{p=1}^{\infty} \frac{|\Gamma_p|^2}{\hbar^2 \omega_p (\omega_p - z)} \quad \text{and} \quad \Delta = \sum_{k=1}^{\infty} \frac{|\Gamma_k|^2}{\hbar^2 \omega_k} .$$

and Mori noise $[I_M](t)$ satisfies

$$(4.32) \quad \langle [I_M](0); [I_M](0) \rangle_H = - \frac{\phi(0)}{\beta \hbar \omega(a)} = \frac{1}{\beta \hbar} \Delta .$$

$$(b) \quad R_a(t) = \langle [a]; [a](t) \rangle_H \\ = \frac{1}{\beta \hbar} \sum_{p=0}^{\infty} \frac{1}{|D'(\epsilon_p)|} \cdot \frac{e^{-it\epsilon_p}}{\epsilon_p}$$

is an almost periodic function .

Remark 4.1. The equality $\omega(a) = D(0)$ physically means that Mori's frequency is equal to the difference between the frequency ω_0 of a quantum harmonic oscillator and the initial value of the canonical correlation function of Mori noise multiplied by $\beta \hbar$. In [11], K.Lindenberg and B.J.West

gave attention to the quantity Δ . (4.32) gives such a physical meaning that Δ is the initial value of the canonical correlation function of Mori noise multiplied by $\beta\hbar$.

We note that $R_{\alpha}(t)$ does not converge as $t \rightarrow \infty$.

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