数理解析研究所講究録
第 693 巻 1989 年 12-22

STRATIFICATION OF THE DISCRIMINANT VARIETIES OF TYPE A_ℓ and B_ℓ

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§1. **Introduction** Let R be a reduced irreducible root system in $\mathbf{R}^\ell$. Let $\mathcal{H} = \{H_\alpha\}(\alpha \in \Lambda)$ be the corresponding arrangement of the hyperplanes. The Weyl group W is the group generated by the reflections along $\{H_\alpha \ ; \ \alpha \in \Lambda\ \}$. It acts on $\mathbf{C}^\ell$ so that the quotient space $\mathbf{C}^\ell/W$ is isomorphic to the affine space $\mathbf{C}^\ell$ whose affine coordinate ring is the ring of the invariant polynomial $\mathbf{C}[\xi_1,\dots,\xi_\ell]^W$ (Chapter 6, [1]). Let $|\mathcal{H}| = \bigcup_{\alpha\in\Lambda} H_\alpha$. The action on the complement $\mathbf{C}^\ell - |\mathcal{H}|$ is free and $|\mathcal{H}|$ is W-invariant. We call the quotient space $|\mathcal{H}|/W$ the discriminant variety of the root system and we denote it by $\mathcal{D}$. The discriminant variety is a hypersurface in the quotient space $\mathbf{C}^\ell/W$. There are many interesting results by many authors about the topology of the arrangement $|\mathcal{H}|$ or $\mathbf{C}^{\ell+1} - |\mathcal{H}|$. See Orlik [6] and its references. The complement $\mathbf{C}^\ell - \mathcal{D}$ is known to be a $K(\pi,1)$-space by [2] and [3]. Let $\mathcal{S}$ be a stratification of $|\mathcal{H}|$ which is compatible with the W–action. For instance, we can take the minimal stratification $\mathcal{S}_{min} = \{\ H_\Xi^* \ ; \ \Xi \subset \Lambda\ \}$ where $H_\Xi^* = \bigcap_{\alpha\in\Xi} H_\alpha - \bigcup_{\alpha\notin\Xi} H_\alpha$. For a given $\mathcal{S}$, $\mathcal{D}$ inherits a canonical stratification $\overline{\mathcal{S}}$ which is defined by the images of the strata of $\mathcal{S}$. The purpose of this paper is to show that the discriminant variety for the arrangements of type A_ℓ and B_ℓ has canonical regular stratifications which are constructed in the above way. Here the regularity means the b-regularity in the sense of Whitney [7]. It is known that the b-regularity implies the a-regularity ([5]). For $A_{\ell+1}$ and $B_{\ell+1}$, we can simply take $\mathcal{S} = \mathcal{S}_{min}$.

Let $\mathcal{T}$ be an analytic stratification of an analytic variety V in an open set U of $\mathbf{C}^n$. Let (M,N) be a pair of strata of $\mathcal{T}$ with $\overline{M} \supset N$ and let $q \in N$. Let $p(u)$ $(0 \le u < 1)$ be a real analytic curve such that $p(0) = q$ and $p(u) \in M$ for $u > 0$. Let $T = \lim_{u\to 0} T_{p(u)}M$. We say that the pair (M,N) has a unique tangential limit at q if this limit T depends only on q and M. If $\mathcal{T}$ enjoys this property at any point q of N for any pair (M,N), we say that $\mathcal{T}$ has *the unique tangential limits property.* Of course, the existence of a stratification with the unique tangential limits property poses a strong geometric restriction on V.

We will show that the stratifications $\overline{\mathcal{S}}$ for $A_{\ell+1}$ and $B_{\ell+1}$-discriminants have the unique tangential limits property.

§2. **A_ℓ-arrangement.** We first consider the A_ℓ-arrangement. As a root system, A_ℓ is the restriction of $B_{\ell+1}$ to the following hyperplane

(2.1) $$L : \xi_1 + \cdots + \xi_{\ell+1} = 0.$$

The corresponding arrangement $\mathcal{H}$ consists of $\binom{\ell+1}{2}$ hyperplanes $\{\xi_i - \xi_j = 0\}$ $(i < j)$ and the Weyl group W is the symmetric group $S_{\ell+1}$. The invariant ring is generated by

$$s_i = \sum_{\tau \in S_{\ell+1}} \xi_{\tau(1)} \cdots \xi_{\tau(i)} \quad (i = 1, \ldots, \ell+1). \tag{2.2}$$

We refer to Chapter 6 of [1] for the basic results about the irreducible root systems. We use the following symmetric polynomials for the calculation's sake.

$$\tau_i = \xi_1^i + \cdots + \xi_{\ell+1}^i \quad (i = 1, \ldots, \ell+1). \tag{2.3}$$

Note that $\{\tau_1, \ldots, \tau_{\ell+1}\}$ is also a basis of the ring of invariant polynomials and that $s_1 = \tau_1 = 0$ on L. We define the mapping $\Phi : \mathbf{C}^{\ell+1} \to \mathbf{C}^{\ell+1}$ by $\Phi(\xi_1, \ldots, \xi_{\ell+1}) = (\tau_1, \ldots, \tau_{\ell+1})$. Let $\bar{L}$ be the hyperplane in the quotient space defined by $\tau_1 = 0$. Let $\phi_L : L \to \bar{L}$ and $\phi : |\mathcal{H}| \to \mathcal{D}$ be the respective restriction of Φ to L and $|\mathcal{H}|$. We have the following commutative diagrams.

$$\begin{array}{ccccc} \mathbf{C}^{\ell+1} & \hookleftarrow & L & \hookleftarrow & |\mathcal{H}| \\ \downarrow \Phi & & \downarrow \phi_L & & \downarrow \phi \\ \mathbf{C}^{\ell+1} & \hookleftarrow & \bar{L} & \hookleftarrow & \mathcal{D} \end{array} \tag{2.4}$$

Here the horizontal maps are the respective inclusion maps. It is well-known that $\mathcal{D}$ is defined by $\prod_{i<j}(\xi_i - \xi_j)^2 = 0$ which can be written in a weighted homogeneous polynomial of $\{s_1, \ldots, s_{\ell+1}\}$ or equivalently of $\{\tau_1, \ldots, \tau_{\ell+1}\}$. This is equal to the discriminant polynomial of $x^{\ell+1} - s_1 x^\ell + \cdots + (-1)^{\ell+1} s_{\ell+1} = 0$ in the usual sense ([4]).

Now we consider the stratification $\mathcal{S} = \mathcal{S}_{min}$ of $|\mathcal{H}|$. Let $\mathcal{C}_1$ be the set of the non-maximal subdivisions of the set $\{1, \ldots, \ell+1\}$. Namely an element $\mathcal{F}$ of $\mathcal{C}_1$ can be written as $\{I_1, \ldots, I_k\}$ where $I_i \cap I_j = \emptyset$ for $i \neq j$ and $\bigcup_{j=1}^k I_j = \{1, \ldots, \ell+1\}$. The maximal element $\mathcal{M} = \{\{1\}, \ldots, \{\ell+1\}\}$ is excluded as $M(\mathcal{M}) = \mathbf{C}^{\ell+1} - |\mathcal{H}|$. Note that the Weyl group W acts canonically on $\mathcal{C}_1$. Let $\mathcal{C}_2$ be the set of the non-maximal partitions of the integer $\ell+1$. An element $\mathcal{K}$ of $\mathcal{C}_2$ is written as $\{m_1, \ldots, m_k\}$ such that $\sum_{j=1}^k m_j = \ell+1$ with $m_j > 0$. For a subset I of $\{1, \ldots, \ell+1\}$, we denote its cardinality by $|I|$. Then there is a canonical surjection from $\mathcal{C}_1$ to $\mathcal{C}_2$ by $\mathcal{F} \mapsto |\mathcal{F}|$ where $|\mathcal{F}| = \{|I_1|, \ldots, |I_k|\}$. For each $\mathcal{F} = \{I_1, \ldots, I_k\}$ of $\mathcal{C}_1$, we define

$$M(\mathcal{F}) = \{\ \xi = (\xi_i) \in \mathbf{C}^{\ell+1}\ ;\ \xi_i = \xi_j \Leftrightarrow \exists a\ ;\ \{i,j\} \subset I_a\}.$$

It is clear that $\{M(\mathcal{F})\}_{\mathcal{F} \in \mathcal{C}_1}$ is equal to $\mathcal{S} = \mathcal{S}_{min}$ which is a regular stratification of $|\mathcal{H}|$. Let $\mathcal{F} = \{I_1, \ldots, I_k\}$ and $\mathcal{G} = \{J_1, \ldots, J_m\}$ be elements of $\mathcal{C}_1$. $\mathcal{F}$ is called a *subdivision* of $\mathcal{G}$ if for each i, there exists a j such that $I_i \subset J_j$. We define a partial ordering in $\mathcal{C}_1$ (respectively in $\mathcal{C}_2$) by $\mathcal{F} \succeq \mathcal{G}$ if and only if $\mathcal{F}$ is a subdivision of $\mathcal{G}$. (Respectively $|\mathcal{F}| \succeq |\mathcal{G}| \Leftrightarrow |\mathcal{F}|$ is a subpartition of $|\mathcal{G}|$.) The canonical map $\mathcal{F} \mapsto |\mathcal{F}|$ is obviously order-preserving.

PROPOSITION (2.5). *Let $\mathcal{F}, \mathcal{F}' \in \mathcal{C}_1$. The following conditions are equivalent.*
(i) $\overline{M(\mathcal{F})} \supseteq M(\mathcal{F}')$. (ii) $\overline{M(\mathcal{F})} \cap M(\mathcal{F}') \neq \emptyset$. (iii) $\mathcal{F} \succeq \mathcal{F}'$.

PROPOSITION (2.6). *Let $\mathcal{F}, \mathcal{F}' \in \mathcal{C}_1$.* (I) *The following conditions are equivalent.*
(i) $\phi(M(\mathcal{F})) = \phi(M(\mathcal{F}'))$. (ii) $\phi(M(\mathcal{F})) \cap \phi(M(\mathcal{F}')) \neq \emptyset$.
(iii) *There exists an element $g \in W$ such that $g(M(\mathcal{F})) = M(\mathcal{F}')$.* (iv) $|\mathcal{F}| = |\mathcal{F}'|$ *in $\mathcal{C}_2$.*
(II) $\overline{\phi(M(\mathcal{F}))} \supseteq \phi(M(\mathcal{F}'))$ *if and only if $|\mathcal{F}| \succeq |\mathcal{F}'|$.*

PROOF: Proposition (2.5) is immediate from the definition of $M(\mathcal{F})$. We prove Proposition (2.6). The equivalence (iii) $\Leftrightarrow$ (iv) is obvious. The implications (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) are also trivial. Assume that $\phi(\xi) = \phi(\xi')$ for some $\xi \in M(\mathcal{F})$ and $\xi' \in M(\mathcal{F}')$. This implies that there exists a $g \in W$ such that $g(\xi) = \xi'$. As $\mathcal{H}$ is invariant by the action of W, we can write $g(M(\mathcal{F})) = M(\mathcal{G})$ for some $\mathcal{G} \in \mathcal{C}_1$. As $\{M(\mathcal{F})\}_{\mathcal{F} \in \mathcal{C}_1}$ are disjoint, this implies $\mathcal{F}' = \mathcal{G}$. Thus (ii) $\Rightarrow$ (iii). As $\overline{\phi(M(\mathcal{F})} = \phi(\overline{M(\mathcal{F})})$, the assertion (II) is an immediate consequence of (I) and Proposition (2.5).

DEFINITION (2.7). *For $\mathcal{K} \in \mathcal{C}_2$, we define $V(\mathcal{K}) = \phi(M(\mathcal{F}))$ where $|\mathcal{F}| = \mathcal{K}$.*

We define an important vector-valued function $X(x)$ by

$$X(x) = (x, x^2, \ldots, x^{\ell+1}). \tag{2.8}$$

Let $X'(x) = (1, 2x, \ldots, (\ell+1)x^{\ell})$ be the derivative of $X(x)$. Then $\Phi(\xi) = \sum_{i=1}^{\ell+1} X(\xi_i)$ and the tangential map $d\Phi_\xi : T_\xi \mathbf{C}^{\ell+1} \to T_{\Phi(\xi)}\mathbf{C}^{\ell+1}$ satisfies $d\Phi_\xi(\frac{\partial}{\partial \xi_i}) = \sum_{j=1}^{\ell+1} j\xi^{j-1}\frac{\partial}{\partial \tau_j}$. We identify the tangent space $T_{\Phi(\xi)}\mathbf{C}^{\ell+1}$ with $\mathbf{C}^{\ell+1}$ in a canonical way. Then the above equality says

$$d\Phi_\xi(\frac{\partial}{\partial \xi_i}) = X'(\xi_i), \quad i = 1, \ldots, \ell+1. \tag{2.9}$$

For any subset I of $\{1, \ldots, \ell+1\}$, we define

$$\frac{\partial}{\partial \xi_I} = \frac{1}{|I|}\sum_{i \in I}\frac{\partial}{\partial \xi_i}, \qquad \xi_I = \frac{1}{|I|}\sum_{i \in I}\xi_i. \tag{2.10}$$

Let $\mathcal{F} = \{I_1, \ldots, I_k\}$ and let $\xi \in M(\mathcal{F})$. As ξ_j does not depend on $j \in I_i$ for i being fixed, we have $\xi_j = \xi_{I_i}$ for any $j \in I_i$.

PROPOSITION (2.11). *Let $\mathcal{F} = \{I_1, \ldots, I_k\}$ and let $\xi \in M(\mathcal{F})$.*
(i) *$T_\xi M(\mathcal{F})$ is the (k-1)-dimensional vector space which is equal to*

$$T_\xi M(\mathcal{F}) = \left\{ \sum_{t=1}^{k} \lambda_t \frac{\partial}{\partial \xi_{I_t}} \; ; \; \sum_{t=1}^{k} \lambda_t = 0 \right\}.$$

(ii) *The restriction* $\phi : M(\mathcal{F}) \to V(|\mathcal{F}|)$ *is a finite covering.*

(iii) $V(|\mathcal{F}|)$ *is non-singular and*

$$T_{\phi(\xi)}V(|\mathcal{F}|) = \left\{ \sum_{t=1}^{k} \lambda_t X'(\xi_{I_t}) \; ; \; \sum_{t=1}^{k} \lambda_t = 0 \right\}.$$

PROOF: (i) is obvious by the definition of $M(\mathcal{F})$. Thus

$$d\Phi_\xi(T_\xi M(\mathcal{F})) = \left\{ \sum_{t=1}^{k} \lambda_t X'(\xi_{I_t}) \; ; \; \sum_{t=1}^{k} \lambda_t = 0 \right\}.$$

By the Vandermonde determinant formula, this image has dimension $(k-1)$. Thus the restriction $\phi|M(\mathcal{F})$ is a submersion and the local image by ϕ is smooth. Now assume that $\phi(\xi) = \phi(\eta)$ for $\xi, \eta \in M(\mathcal{F})$ with $\xi \neq \eta$. Then there exists a permutation $g \in S_{\ell+1}$ so that $g(\xi) = \eta$. Then $g(M(\mathcal{F})) = M(\mathcal{F})$. Thus the local images near ξ and η by ϕ coincide. This proves that $V(|\mathcal{F}|)$ is smooth and the assertions (ii) and (iii) follow immediately.

Let us examine the order of the covering $\phi : M(\mathcal{F}) \to V(|\mathcal{F}|)$ more explicitly. Let $\{\alpha_1, \dots, \alpha_m\} = \{n \; ; \; \exists i, \; n = |I_i| \; \}$. Clearly we have $m \leq k$ and $\{\alpha_i\}$ are mutually distinct. Let ρ_i be the number of j's such that $|I_j| = \alpha_i$ $(i = 1, \dots, k)$. We consider the subgroups

$$W(\mathcal{F}) = \{ \; g \in W \; ; \; g(M(\mathcal{F})) = M(\mathcal{F}) \; \}, \quad I(\mathcal{F}) = \{ \; g \in W \; ; \; g|M(\mathcal{F}) = id \; \}.$$

Then $I(\mathcal{F})$ is a normal subgroup of $W(\mathcal{F})$ and the quotient group $W(\mathcal{F})/I(\mathcal{F})$ acts freely on $M(\mathcal{F})$ with the quotient space $V(|\mathcal{F}|)$. More precisely let $\bar{g} \in W(\mathcal{F})/I(\mathcal{F})$. Then for each $s = 1, \dots, m$, $\bar{g}$ induces a permutation of $\{\xi_{I_j} \; ; \; |I_j| = \alpha_s\}$. Thus we have

PROPOSITION (2.12). *There is a canonical isomorphism* $W(\mathcal{F})/I(\mathcal{F}) \cong S_{\rho_1} \times \cdots \times S_{\rho_m}$. *Thus the order of the above covering is* $\rho_1! \cdots \rho_m!$.

Let $f(x)$ be a vector valued rational function of one variable. We define the rational functions $f_k(x_1, \dots, x_k)$ $(k = 1, \dots, \ell+1)$ inductively by $f_1(x_1) = f(x_1)$ and

(2.13) $$f_k(x_1, \dots, x_k) = \{f_{k-1}(x_1, \dots, x_{k-2}, x_{k-1}) - f_{k-1}(x_1, \dots, x_{k-2}, x_k)\} / (x_{k-1} - x_k)$$

We call $f_k(x_1, \dots, x_k)$ the k-fold derived function of $f(x)$.

PROPOSITION (2.14). *We have the following formulae.*

(i) $$f(x_k) = f(x_1) + \sum_{j=2}^{k} \left(\prod_{h=1}^{j-1} (x_k - x_h) \right) f_j(x_1, \ldots, x_j)$$

(ii)

$$f_{s+1}(x_1, \ldots, x_s, x_{s+k}) = f_{s+1}(x_1, \ldots, x_{s+1}) + \sum_{j=2}^{k} \left(\prod_{h=1}^{j-1} (x_{s+k} - x_{s+h}) \right) f_{s+j}(x_1, \ldots, x_{s+j}).$$

PROOF: As (i) is a special case of (ii), we prove (ii) by the induction on k. The assertion on $k = 1$ is trivial. We assume the assertion for $k - 1$. By the definition of the derived function, we have

$$\begin{aligned} f_{s+1}(x_1, \ldots, x_s, x_{s+k}) - f_{s+1}(x_1, \ldots, x_s, x_{s+1}) &= (x_{s+k} - x_{s+1}) f_{s+2}(x_1, \ldots, x_{s+1}, x_{s+k}) \\ &= (x_{s+k} - x_{s+1}) f_{s+2}(x_1, \ldots, x_{s+2}) \\ &+ (x_{s+k} - x_{s+1}) \sum_{j=2}^{k} \left(\prod_{h=1}^{j-1} (x_{s+k} - x_{s+1+h}) \right) f_{s+1+j}(x_1, \ldots, x_{s+1+j}) \\ &= \sum_{j=2}^{k} \left(\prod_{h=1}^{j-1} (x_{s+k} - x_{s+h}) \right) f_{s+j}(x_1, \ldots, x_{s+j}). \end{aligned}$$

This completes the proof.

Now we consider the derived functions $X_k(x_1, \ldots, x_k)$ and $X'_k(x_1, \ldots, x_k)$ of $X(x)$ and $X'(x)$ respectively. The following Lemma plays an important role throughout this paper.

LEMMA (2.15). *Let $a_{k,j}$ and $b_{k,j}$ be the j-th coordinate of $X_k(x_1, \ldots, x_k)$ and $X'_k(x_1, \ldots, x_k)$ respectively. Then $a_{k,j}$, $b_{k,j}$ are symmetric polynomials of $x_1, \ldots, x_k$ defined by*

(i) $$a_{k,k+j} = \sum_{\nu_1 + \cdots + \nu_k = j+1} x_1^{\nu_1} \cdots x_k^{\nu_k}, \qquad b_{k,k+j} = (k+j) \sum_{\nu_1 + \cdots + \nu_k = j} x_1^{\nu_1} \cdots x_k^{\nu_k}$$

(ii) $$X_k(x, \cdots, x) = X^{(k-1)}(x)/(k-1)!, \qquad X'_k(x, \ldots, x) = X^{(k)}(x)/(k-1)!$$

where $X^{(j)}(x) = (\frac{d}{dx})^j X(x)$.

PROOF: (i) is immediate from the inductive calculation and the equality: $(x^a - y^a)/(x - y) = x^{a-1} + x^{a-2}y + \cdots + y^{a-1}$. The assertion (ii) follows immediately from (i).

LEMMA (2.16). *Let $\xi \in M(\mathcal{F})$ and let $\mathcal{F} = \{ I_1, \ldots, I_k \}$. Then*

$$X'_t(\xi_{I_{\sigma(1)}}, \ldots, \xi_{I_{\sigma(t)}}) \in T_{\phi(\xi)} V(|\mathcal{F}|) \quad \text{for any } t = 2, \ldots, k \text{ and } \sigma \in S_t$$

PROOF: By Proposition (2.11), we have that

$$X'(\xi_{I_i}) - X'(\xi_{I_j}) = (\xi_{I_i} - \xi_{I_j})X_2'(\xi_{I_i}, \xi_{I_j}) \in T_{\phi(\xi)}V(|\mathcal{F}|) \ (i \neq j).$$

This implies that $X_2'(\xi_{I_i}, \xi_{I_j}) \in T_{\phi(\xi)}V(|\mathcal{F}|)$ for $i \neq j$. Now the assertion follows by an easy inductive argument.

The following is a generalization of the Vandermonde determinant formula and it plays a key role to show the linear independence of certain vectors in the later arguments.

LEMMA (2.17). *(Generalized Vandermonde formula) Let $\lambda_1, \ldots, \lambda_k$ be mutually distinct complex numbers and let $\mathcal{N} = \{\nu_1, \ldots, \nu_k\}$ be an element of $\mathcal{C}_2$. Then we have the formula:*

$$\det\left({}^tX'(\lambda_1), \ldots, {}^tX^{(\nu_1)}(\lambda_1), \ldots, {}^tX'(\lambda_k), \ldots, {}^tX^{(\nu_k)}(\lambda_k)\right) = (\ell+1)! \prod_{j>i} (\lambda_j - \lambda_i)^{\nu_i \nu_j}.$$

In particular, $\{ X^{(j)}(\lambda_i) \}$ $(j = 1, \ldots, \nu_i,\ i = 1, \ldots, k)$ are linearly independent.

PROOF: Let $\Psi(x_1, \ldots, x_{\ell+1}) = \det({}^tX'(x_1), \ldots, {}^tX'(x_{\ell+1}))$. Then it is easy to see that

(2.18) $$\Psi(x_1, \ldots, x_{\ell+1}) = (\ell+1)! \prod_{j>i} (x_j - x_i)$$

by the Vandermonde determinant formula. We consider the differential operators:

$$D_i = \left(\frac{\partial}{\partial x_{\nu_1+\cdots+\nu_{i-1}+2}}\right)^1 \cdots \left(\frac{\partial}{\partial x_{\nu_1+\cdots+\nu_i}}\right)^{\nu_i - 1} \text{ and } D = D_1 \cdots D_k.$$

Let $E = \{ (j,h) \ ; \ \nu_1 + \cdots + \nu_{i-1} + 1 \leq h < j \leq \nu_1 + \cdots + \nu_i,\ i = 1, \ldots, k \}$ and let $\mathcal{E}$ be the ideal generated by $\{ x_j - x_h \ ; \ (j,h) \in E \}$. As $\sum_{j=1}^{\nu_i - 1} j = \binom{\nu_i}{2}$, it is easy to see that

(2.19) $$D\Psi \equiv (\ell+1)! \prod_{(j,h) \notin E} (x_j - x_h) \quad \text{modulo } \mathcal{E}.$$

Thus the assertion follows immediately from

$$\begin{aligned} &\det({}^tX'(\lambda_1), \ldots, {}^tX^{(\nu_1)}(\lambda_1), \ldots, {}^tX'(\lambda_k), \ldots, {}^tX^{(\nu_k)}(\lambda_k)) \\ &\quad = (D\Psi)(\underbrace{\lambda_1, \ldots, \lambda_1}_{\nu_1}, \ldots, \underbrace{\lambda_k, \ldots, \lambda_k}_{\nu_k}) = (\ell+1)! \prod_{j>i} (\lambda_j - \lambda_i)^{\nu_i \nu_j}. \end{aligned}$$

Here the last equality is due to (2.19).

§3. **Regularity and the limit of the tangent space.** Now we are ready to show the regularity of the stratification $\overline{\mathcal{S}}$ of the discriminant variety of $A_{\ell+1}$-arrangement and the unique tangential limits property. Let $M(\mathcal{F})$ and $M(\mathcal{G})$ be stratum of $\mathcal{S}$ such that $\overline{M(\mathcal{F})} \supset M(\mathcal{G})$. Let q be an arbitrary point of the stratum $V(|\mathcal{G}|)$ and let $\bar{p}(u)$ and $\bar{q}(u)$ be real analytic curves defined on the interval $[0,1]$ such that (i) $\bar{p}(0) = \bar{q}(0) = q$ and $\bar{q}(u) \in V(|\mathcal{G}|)$ for any $u \in [0,1]$. (ii) $\bar{p}(u) \in V(|\mathcal{F}|)$ for $u > 0$. We also assume that

$$\lim_{u\to 0} T_{\bar{p}(u)}V(|\mathcal{F}|) = T, \qquad \lim_{u\to 0}[\bar{p}(u), \bar{q}(u)] = \gamma. \tag{3.1}$$

Here $[\bar{p}(u), \bar{q}(u)]$ is the line spanned by $\bar{p}(u) - \bar{q}(u)$. Changing the parameter u by $u^{1/m}$ for some integer m if necessary, we may assume that there are lifting real analytic curves $p(u)$ and $q(u)$ in $\overline{M(\mathcal{F})}$ and $M(\mathcal{G})$ respectively so that $\bar{p}(u) = \phi(p(u))$ and $\bar{q}(u) = \phi(q(u))$ respectively. We may assume that $p(0) = q(0)$ and let $\eta = p(0) \in M(\mathcal{G})$. Let $\mathcal{G} = \{J_1, \dots, J_m\}$. By Proposition (2.5), we can write $\mathcal{F} = \{\ J_{i,j}\ ;\ i = 1, \dots, m,\ i = 1, \dots, \nu_i\ \}$ where $J_{i,j} \subset J_i$ for $j = 1, \dots, \nu_i$.

THEOREM (3.2). *$\overline{\mathcal{S}}$ is a regular stratification with the unique tangential limits property. Namely* (i) *T is generated by*

$$\left\{\sum_{i=1}^{m} \lambda_i X'(\eta_{J_i})\ ;\ \sum_{i=1}^{m} \lambda_i = 0\ \right\} \bigcup \left\{X^{(j)}(\eta_{J_i}),\ 1 \le i \le m,\ 2 \le j \le \nu_i\ \right\}.$$

(ii) *(Regularity)* $\gamma \in T$.

PROOF: By Proposition (2.11), the vectors $\lambda_1 X'(p(u)_{J_{1,1}}) + \cdots + \lambda_m X'(p(u)_{J_{m,1}})$ with $\sum_{i=1}^m \lambda_i = 0$ are contained in $T_{\bar{p}(u)}V(|\mathcal{F}|)$. Thus by taking the limit as $u \to 0$, we see that $\sum_{i=1}^m \lambda_i X'(\eta_{J_i}) \in T$. This gives only a subspace of T of dimension $m - 1$. We still need $\nu_1 + \cdots + \nu_m - m$ independent vectors to generate T. For this purpose, we apply Lemma (2.15). We know that $X'_k(p(u)_{J_{i,1}}, \dots, p(u)_{J_{i,k}}) \in T_{\bar{p}(u)}V(|\mathcal{F}|)$ $(2 \le k \le \nu_i,\ 1 \le i \le m)$. We take the limits of these vectors as $u \to 0$ and we apply Lemma (2.15) to obtain that $X^{(j)}(\eta_{J_i}) \in T$ $(2 \le j \le \nu_i,\ 1 \le i \le m)$. Now we apply Lemma (2.17) to see that the vectors $\{X^{(j)}(\eta_{J_i})\ ;\ 1 \le i \le m,\ 1 \le j \le \nu_i\}$ are linearly independent. This completes the proof of (i).

Now we consider the regularity (ii). Using the equality $\sum_{j=1}^{\nu_i} |J_{i,j}| = |J_i|$, we have

$$\bar{p}(u) - \bar{q}(u) = \sum_{i=1}^{m}\sum_{j=1}^{\nu_i} |J_{i,j}| \left(X(p(u)_{J_{i,j}}) - X(q(u)_{J_i})\right). \tag{3.3}$$

Using Proposition (2.14), we can write

$$X(p(u)_{J_{i,j}}) - X(q(u)_{J_i}) = \sum_{h=1}^{j} \alpha_{i,j,h}(u) X_{h+1}(q(u)_{J_i}, p(u)_{J_{i,1}}, \dots, p(u)_{J_{i,h}}) \tag{3.4}$$

where $\alpha_{i,j,h}(u)$ is defined by

(3.5) $$\alpha_{i,j,h}(u) = (p(u)_{J_{i,j}} - q(u)_{J_i}) \prod_{k=1}^{h-1} (p(u)_{J_{i,j}} - p(u)_{J_{i,k}}), \quad h = 1, \ldots, \nu_i.$$

Substituting (3.4) in (3.3), we obtain

(3.6) $$\bar{p}(u) - \bar{q}(u) = \sum_{i=1}^{m} \sum_{h=1}^{\nu_i} \alpha_{i,h}(u) X_{h+1}(q(u)_{J_i}, p(u)_{J_{i,1}}, \ldots, p(u)_{J_{i,h}}).$$

where $\alpha_{i,h}(u) = \sum_{j=h}^{\nu_i} |J_{i,j}| \alpha_{i,j,h}(u)$. In particular, we have

(3.7) $$\alpha_{i,1}(u) = \sum_{j=1}^{\nu_i} |J_{i,j}| \left(p(u)_{J_{i,j}} - q(u)_{J_i}\right).$$

We define a non-negative integer β by

(3.8) $$\beta = \min \{\text{order } (\alpha_{i,h}(u)) \; ; \; i = 1, \ldots, m, \; h = 1, \ldots, \nu_i\}$$

and let $\alpha_{i,h}(u) = \alpha_{i,h} u^\beta + (\text{higher terms})$. Then (3.6) and Lemma (2.15) imply that

(3.9) $$\bar{p}(u) - \bar{q}(u) = \left(\sum_{i=1}^{m} \sum_{h=1}^{\nu_i} \alpha_{i,h} X^{(h)}(\eta_{J_i})/h! \right) u^\beta + (\text{higher terms}).$$

By the Generalized Vandermonde formula (Lemma (2.17)), we can see easily that

(3.10) $$\sum_{i=1}^{m} \sum_{h=1}^{\nu_i} \alpha_{i,h} X^{(h)}(\eta_{J_i})/h! \neq 0 \text{ and } \gamma = \left[\sum_{i=1}^{m} \sum_{h=1}^{\nu_i} \alpha_{i,h} X^{(h)}(\eta_{J_i})/h! \right].$$

Here $[v]$ denotes the line generated by the vector v. Thus the assertion (ii) of Theorem (3.2) follows immediately from (i) and (3.10) and the following.

Assertion (3.11). $\sum_{i=1}^{m} \alpha_{i,1} = 0$.

Proof: By (3.7) we have

$$\sum_{i=1}^{m} \alpha_{i,1}(u) = \sum_{i=1}^{m} \alpha_{i,1} t^\beta + (\text{higher terms}) = \sum_{i=1}^{m} \sum_{j=1}^{\nu_j} |J_{i,j}| p(u)_{J_{i,j}} - \sum_{i=1}^{m} |J_i| q(u)_{J_i} \equiv 0.$$

The last equality is derived from the fact that $p(u)$ and $q(u)$ are in the hyperplane L. Now the assertion is immediate from the above equality.

§4. **$B_{\ell+1}$-arrangement.** Let R be the root system of type $B_{\ell+1}$ in $\mathbf{R}^{\ell+1}$. The corresponding arrangement $\mathcal{H}$ consists of $2\binom{\ell+1}{2} + \ell + 1$ hyperplanes: $\{\xi_i \pm \xi_j = 0\}$ and $\{\xi_i = 0\}$. The Weyl group

W is isomorphic to a semi-direct product of the symmetric group $S_{\ell+1}$ and the abelian group $(\mathbf{Z}/2\mathbf{Z})^{\ell+1}$(Chapter 6, [1]). The invariant polynomial ring is generated by

$$t_i = \sum_{\tau \in S_{\ell+1}} \xi^2_{\tau(1)} \cdots \xi^2_{\tau(i)}, \quad i = 1, \ldots, \ell+1. \tag{4.1}$$

We will use the following generators.

$$\zeta_i = \xi_1^{2i} + \cdots + \xi_{\ell+1}^{2i} \quad i = 1, \ldots, \ell+1. \tag{4.2}$$

Let $\Phi : \mathbf{C}^{\ell+1} \to \mathbf{C}^{\ell+1}/W \cong \mathbf{C}^{\ell+1}$ be the map defined by $\xi \mapsto (\zeta_1(\xi), \ldots, \zeta_{\ell+1}(\xi))$. We take $\mathcal{S} = \mathcal{S}_{min}$. The stratification $\mathcal{S}$ can be described as follows. Let $\mathcal{E}_1$ be the set of the subdivisions of the non-empty subsets of $\{1, \ldots, \ell+1\}$. Namely an element $\mathcal{F} \in \mathcal{E}_1$ can be written as $\mathcal{F} = \{I_1, \ldots, I_k\}$ where each I_i is non-empty and $I_i \cap I_j = \emptyset$ for $i \neq j$. Let $S(\mathcal{F}) = \bigcup_{i=1}^k I_i$ and $\mathcal{F}^c = \{1, \ldots, \ell+1\} - S(\mathcal{F})$. Let $\mathcal{E}_2$ be the set of the partitions of the integer m for $m = 1, \ldots, \ell+1$. There is a canonical surjective mapping from $\mathcal{E}_1$ to $\mathcal{E}_2$ by $\mathcal{F} \mapsto |\mathcal{F}| = \{|I_1|, \ldots, |I_k|\}$. Let

$$M(\mathcal{F}) = \{\ \xi \in \mathbf{C}^{\ell+1}\ ;\ \text{(i)}\ \xi_i = 0 \Leftrightarrow i \in \mathcal{F}^c, \text{(ii)}\ \xi_i^2 = \xi_j^2\ \Leftrightarrow \{i,j\} \subseteq \exists I_s\}$$

We omit $\mathcal{M} = \{\{1\}, \ldots, \{\ell+1\}\}$ and $|\mathcal{M}|$ from $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively as $M(\mathcal{M})$ and $V(|\mathcal{M}|)$ are nothing but the complement $\mathbf{C}^{\ell+1} - |\mathcal{H}|$ and $\mathbf{C}^{\ell+1} - \mathcal{D}$. Let $\alpha = \sum_{i=1}^k |I_i| - k$. Then $M(\mathcal{F})$ is a disjoint union of 2^α connected components corresponding th sign of $\xi_i = \pm\xi_j$ in the definition of $M(\mathcal{F})$. But they are in the same W-orbit. (Recall that the reflection along $\{\xi_i = 0\}$ is the multiplication by -1 in the i-th coordinate.) Thus each connected component is mapped by ϕ onto the same stratum of $\overline{\mathcal{S}}$. We define partial orderings in $\mathcal{E}_1$ and $\mathcal{E}_2$ as follows. Let $\mathcal{F} = \{I_1, \ldots, I_k\}$ and $\mathcal{G} = \{J_1, \ldots, J_n\}$. $\mathcal{F} \succeq \mathcal{G}$ if and only if (i) $\mathcal{F}^c \subseteq \mathcal{G}^c$, (ii) $\tilde{\mathcal{F}} \succeq \tilde{\mathcal{G}}$ in $\mathcal{C}_1$. Here $\tilde{\mathcal{F}}$ is defined by $\{\mathcal{F}^c, I_1, \ldots, I_k\} \in \mathcal{C}_1$. Similarly we define $|\mathcal{F}| \succeq |\mathcal{G}|$ if and only if (i) $|\mathcal{F}^c| \leq |\mathcal{G}^c|$, (ii) $|\tilde{\mathcal{F}}| \succeq |\tilde{\mathcal{G}}|$ in $\mathcal{C}_2$. Now the following propositions are completely parallel to Proposition (2.5) and Proposition (2.6).

Proposition (4.3). *Let $\mathcal{F}, \mathcal{G} \in \mathcal{E}_1$. The following conditions are equivalent.*
(i) $\overline{M(\mathcal{F})} \supseteq M(\mathcal{G})$. (ii) $\overline{M(\mathcal{F})} \cap M(\mathcal{G}) \neq \emptyset$. (iii) $\mathcal{F} \succeq \mathcal{G}$.

Proposition (4.4). *Let $\mathcal{F}, \mathcal{G} \in \mathcal{E}_1$. The following conditions are equivalent.*
(i) $\phi(M(\mathcal{F})) = \phi(M(\mathcal{G}))$. (ii)There exists a $g \in W$ such that $g(M(\mathcal{F})) = M(\mathcal{G})$. (iii) $|\mathcal{F}| = |\mathcal{G}|$.

Thus for a $\mathcal{K} \in \mathcal{E}_2$ we can define $V(\mathcal{K}) = \phi(M(\mathcal{F}))$ for any $\mathcal{F} \in \mathcal{E}_1$ such that $|\mathcal{F}| = \mathcal{K}$. Now we study the tangential map. Note that

$$d\Phi_\xi(\frac{\partial}{\partial \xi_i}) = 2\xi_i X'(\xi_i^2). \tag{4.5}$$

For each $I \subset \{1,\dots,\ell+1\}$, we define $m(I) = \min\{i\ ;\ i \in I\}$. Let $\mathcal{F} = \{I_1,\dots,I_k\} \in \mathcal{E}_1$ and let $\xi \in \mathcal{F}$. We define $\tilde{\xi} \in M(\mathcal{F})$ by

$$\tilde{\xi}_j = \begin{cases} \xi_{m(I_i)} & \text{if } j \in I_i \\ 0 & \text{if } j \in \mathcal{F}^c. \end{cases} \tag{4.6}$$

It is easy to see that $\tilde{\xi}$ is in the W-orbit of ξ. We also define

$$\widetilde{\frac{\partial}{\partial \xi_{I_i}}} = \frac{1}{|I_i|} \sum_{j \in I_i} \left(\xi_j/\xi_{m(I_i)}\right) \frac{\partial}{\partial \xi_j}.$$

Note that $\xi_j/\xi_{m(I_i)} = \pm 1$ and $\xi_j^2 = \xi_{m(I_i)}^2 = \tilde{\xi}_{I_i}^2$ for each $j \in I_i$. It is easy to see that $\widetilde{\frac{\partial}{\partial \xi_{I_i}}} \in T_\xi M(\mathcal{F})$ and $d\Phi_\xi(\widetilde{\frac{\partial}{\partial \xi_{I_i}}}) = 2\tilde{\xi}_{I_i} X'(\tilde{\xi}_{I_i}^2)$. Now Proposition (2.11) and Lemma (2.15) can be translated into the following form.

PROPOSITION (4.7). *Let $\mathcal{F} = \{I_1,\dots,I_k\} \in \mathcal{E}_1$. Then*
(i) *The dimension of $T_\xi M(\mathcal{F})$ is k and it is generated by $\left\{\widetilde{\frac{\partial}{\partial \xi_{I_i}}}\ ;\ i = 1,\dots,k\right\}$.*
(ii) *The restriction $\phi : M(\mathcal{F}) \to V(|\mathcal{F}|)$ is a finite covering.*
(iii) *$V(|\mathcal{F}|)$ is non-singular and $T_{\phi(\xi)}V(|\mathcal{F}|)$ is generated by $\{X'(\tilde{\xi}_{I_i}^2)\ ;\ i = 1,\dots,k\}$.*

LEMMA (4.8). *Let $\mathcal{F}$ be as in Proposition (4.7). Then*

$$X_s'(\tilde{\xi}_{I_1}^2,\dots,\tilde{\xi}_{I_t}^2) \in T_{\phi(\xi)}V(|\mathcal{F}|) \quad \text{for } s = 1,\dots,k.$$

Let $\mathcal{F} \succeq \mathcal{G}$ and let $\mathcal{G} = \{J_1,\dots,J_m\}$. We can write $\mathcal{F} = \{J_{i,j}\ ;\ i = 0,\dots,m,\ j = 1,\dots,\nu_i\}$ so that $J_{i,j} \subset J_i$ where $J_0 = \mathcal{G}^c$ by definition. Let $\bar{p}(u)$, $\bar{q}(u)$, q, $p(u)$, $q(u)$, η, T and γ be as §3. We consider the equality $\bar{p}(u) - \bar{q}(u) = \sum_{i=0}^m \sum_{j=1}^{\nu_i} |J_{i,j}|(X(\widetilde{p(u)}^2_{J_{i,j}}) - X(\widetilde{q(u)}^2_{J_i}))$. Then using Lemma (4.8), we do the same argument as for the $A_{\ell+1}$-discriminant to obtain

THEOREM (4.9). *$\overline{S}$ is a regular stratification with the unique tangential limits property. Namely (i) T is generated by $\{\ X^{(j)}(\tilde{\eta}_{J_i}^2)\ ;\ i = 0,\dots,m,\ j = 1,\dots,\nu_i\}$. (ii) (Regularity) $\gamma \in T$.*

For the stratification of discriminant variety of D_ℓ, see [8].

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