

Strongly compatible total orders on free monoids

T. Saito, T. Iwamoto, Y. Kobayashi, K. Kajitori

斎藤 立彦 (水産大・教養) 岩本 徳郎 (宇部工専)
小林 菱治 (徳島大・総合科学) 楫取 和明 (水産大・教養)

In this paper, we deal with a combinatorial problem which the first author raised in relation to term-rewriting systems. To 'complete' a grammar (as a term-rewriting system), one uses a compatible total order on the set X^* of all words (an order \leq on X^* is said compatible if $x < y$ implies $uxv < uyv$ for all u, v) which does not have an infinite descending sequence. An order \leq on X^* is said length-sensitive if $|x| < |y|$ implies $x < y$, where $|x|$ for a word x denotes the length of x . The length-sensitivity assures the non-existence of an infinite descending sequence. Usually, a total order that is used for the above purpose is the length-sensitive lexicographic order (which is clearly compatible). The problem raised by the first author was to describe the all length-sensitive compatible total orders on X^* . He soon noticed that if we replace the compatibility by a somewhat stronger property which we shall call the strong compatibility, then the description of the orders becomes strikingly simple especially when $\text{Card}(X) = 2$ ($\text{Card}(X)$ denotes the cardinality of the set X). In this paper, we take the strong compatibility as a basic property on orders, and present some results on such orders.

Let $X = \{a_1, a_2, \dots, a_n\}$ be a set. We call X the alphabet and its elements letters. The free monoid X^* on X is the set of all words over X (that is, finite sequences of letters including the empty sequence, denoted 1, with concatenation as the operation). We treat X itself as a subset of X^* by identifying each letter a with the word consisting of 'a' alone. For any word x in X^* , the length of a word x is the number of letters occurring in x , and denoted by $|x|$. For each natural number n , X^n denotes the set $\{w \in X^* : |w| = n\}$, while $X^{[n]}$ denotes the set $\{w \in X^* : |w| \leq n\}$. We basically follow the notations in [B-P].

We consider an order relation \leq on X^* with the following conditions:

(P1) for letters, $a_1 < a_2 < \dots < a_n$ ($x < y$ means that $x \leq y$ and $x \neq y$), and for any words $x, y \in X^*$,

(P2) (The length sensitivity) if $|x| < |y|$, then $x < y$.

(P3) (The compatibility) if $x < y$, then $uxv < uyv$ for all $u, v \in X^*$.

(P4) (The strong compatibility) if $x = x_1x_2$, $y = y_1y_2$ and $|x_1| = |y_1|$, $|x_2| = |y_2|$, then $x < y$ implies $x_1ux_2 < y_1uy_2$ for all $u \in X^*$, where $x_1, x_2, y_1, y_2 \in X^*$.

(P5) \leq is a total order on X^* .

By (P1), we mean that we may (and do) fix 'the' ordering on the letters of X without loss of generality. The length-sensitivity (P2) is our another presumption on orders on X^* and will be implicit on any orders which will appear in the paper. It is easy to check that (P4) implies (P3).

By the lexicographic order, we mean the order \leq on X^* determined by the rule: if $x = x_1x_2 \dots x_m$, $y = y_1y_2 \dots y_m$ and $x_1 \dots x_{i-1} = y_1 \dots y_{i-1}$ and $x_i < y_i$, then $x < y$, where $x_i, y_i \in X$ ($i = 1, \dots, m$).

We denote this order by \leq_{lex} .

If $x = x_1x_2 \dots x_m \in X^*$, where $x_i \in X$ ($i = 1, \dots, m$), then we denote $x_m \dots x_2x_1$ by \tilde{x} .

By the anti-lexicographic order, we mean the order \leq on X^* such that for any $x, y \in X^*$, $x <_{\text{lex}} y$ implies $\tilde{x} < \tilde{y}$. We denote this order by \leq_{al} .

It is easy to see that both $<_{\text{lex}}$ and \leq_{al} are strongly compatible.

Proposition 1. Suppose that \leq is a strongly compatible total order on X^* . If \leq coincides with \leq_{lex} on $X^{[3]}$, then \leq in fact equals \leq_{lex} on the entire X^* . In particular, if $\text{Card}(X) \geq 3$, then $X^{[3]}$ above can be replaced by $X^{[2]}$.

Proof. We show the first assertion by induction on the length of words. Suppose that on $X^{[m-1]}$, \leq coincides with \leq_{lex} ($m \geq 4$). Let $x, y \in X^m$ be $x = x_1x_2 \dots x_m$ and $y = y_1y_2 \dots y_m$ and let i be the least such that $x_i \neq y_i$. WLOG, we may assume that $x_i < y_i$. Then, since $x <_{\text{lex}} y$, we have to show that $x < y$.

Case 1 : $i \geq 2$. Then, $x_i \dots x_m < y_i \dots y_m$ by the induction hypothesis.

So, using the strong compatibility, we have that $x = x_1 \cdots x_{i-1} x_i \cdots x_m < y_1 \cdots y_{i-1} y_i \cdots y_m = y$.

Case 2 : $i = 1$ and $x_j \leq y_j$ for some $j \neq i$. Then, $x_1 \cdots x_{j-1} x_{j+1} \cdots x_m < y_1 \cdots y_{j-1} y_{j+1} \cdots y_m$ by the induction hypothesis. So, using the strong compatibility, we have that $x = x_1 \cdots x_{j-1} x_j x_{j+1} \cdots x_m < y_1 \cdots y_{j-1} y_j y_{j+1} \cdots y_m = y$.

Case 3 : $i = 1$ and $x_j > y_j$ for all $j \neq i$. Since by the case 1, we have that $x \leq x_1 a_n^{m-1}$ and $y_1 a_1^{m-1} \leq y$, it suffices to show that $x_1 a_n^{m-1} < y_1 a_1^{m-1}$. Suppose on the contrary, that $y_1 a_1^{m-1} < x_1 a_n^{m-1}$. Then, by the strong compatibility, we have that $y_1 x_1 a_1^{m-1} < x_1^2 a_n^{m-1}$. But, on the other hand, we have:

$$\begin{aligned} x_1^2 a_n^{m-1} &= x_1(x_1 a_n) a_n^{m-2} \\ &< x_1(a_n x_1) a_n^{m-2} = x_1 a_n(x_1 a_n^{m-2}) \\ &< x_1 a_n(a_n^{m-3} a_1^2) = (x_1 a_n^{m-2}) a_1^2 \\ &< (y_1 x_1 a_1^{m-3}) a_1^2 = y_1 x_1 a_1^{m-1}, \text{ a contradiction.} \end{aligned}$$

We have proved the first assertion.

To show the second assertion, suppose that \leq coincides with \leq_{lex} on $X^{[2]}$. It suffices to show that \leq coincides with \leq_{lex} on $X^{[2]}$. Since in the above argument, m could be 3 until we showed the claim that $x_1 a_n^{m-1} < y_1 a_1^{m-1}$, we need only show the claim for $m = 3$, i.e., that $x_1 a_n^2 < y_1 a_1^2$. Suppose that $y_1 a_1^2 < x_1 a_n^2$. Then, we have that $y_1 a_n a_1^2 < x_1 a_n^3$. But, we have, if $x_1 > a_1$,

$$\begin{aligned} x_1 a_n^3 &= (x_1 a_n) a_n^2 \\ &< (y_1 a_1) a_n^2 = y_1 (a_1 a_n) a_n \\ &< y_1 (x_1 a_1) a_n = y_1 x_1 (a_1 a_n) \\ &< y_1 x_1 (a_n a_1) = y_1 (x_1 a_n) a_1 \\ &< y_1 (a_n x_1) a_1 = y_1 a_n a_1^2, \quad \text{a contradiction,} \end{aligned}$$

and if $x_1 = a_1$,

$$\begin{aligned} x_1 a_n^3 &= (x_1 a_n) a_n^2 \\ &< (a_2 a_1) a_n^2 = a_2 (a_1 a_n) a_n \\ &< a_2 (a_2 a_1) a_n = a_2^2 (a_1 a_n) \\ &< a_2^2 (a_n a_1) = a_2 (a_2 a_n) a_1 \\ &< a_2 (a_n a_1) a_1 \leq y_1 a_n a_1^2, \quad \text{again a contradiction.} \end{aligned}$$

We have proved the proposition.

An entirely similar proof shows the following:

Proposition 2. Suppose that \leq is a strongly compatible total order

on X^* . If \leq coincides with \leq_{al} on $X^{[3]}$, then \leq in fact equals \leq_{al} on the entire X^* .

In particular, if $\text{Card}(X) \geq 3$, then $X^{[3]}$ above can be replaced by $X^{[2]}$.

From now on until the end of the proof of Theorem 7, we restrict our attention to orders on X^* with $\text{Card}(X) = 2$. We define four orders other than $<_{lex}$ or $<_{al}$ and will show that these four orders together with $<_{lex}$ and $<_{al}$ exhaust all the strongly compatible total orders on X^* when $\text{Card}(X) = 2$.

Let $c \in X$. For $x \in X^*$, the number of c occurring in x is denoted by $|x|_c$. We consider the following conditions on an order \leq on $X^* = \{a, b\}^*$ ($a < b$): for $x, y \in X^*$ with $|x| = |y|$,

- (a) if $|x|_b < |y|_b$, then $x < y$,
- (b) if $|x|_b = |y|_b$ and $x <_{lex} y$, then $x < y$,
- (c) if $|x|_b = |y|_b$ and $y <_{al} x$, then $x < y$,
- (b') if $|x|_b = |y|_b$ and $x <_{al} y$, then $x < y$,
- (c') if $|x|_b = |y|_b$ and $y <_{lex} x$, then $x < y$.

It is clear that an order on X^* satisfying (a) and one of (b), (c), (b'), (c') for all $x, y \in X^*$ is a strongly compatible total order on X^* . The next four lemmas show that similar assertions to Proposition 1,2 are true for these four orders.

Lemma 3. Let $X = \{a, b\}$ with $a < b$. Suppose that \leq is a strongly compatible total order on X^* . Then, if \leq coincides with the ordering defined by the conditions (a) and (b) on $X^{[4]}$, then they coincide on the entire X^* .

Lemma 4 Let X, \leq be as in the lemma 3. Then, if \leq coincides with the ordering defined by the conditions (a) and (c) on $X^{[4]}$, then they coincide on the entire X^* .

Lemma 5. Let X, \leq be as in the lemma 3. Then, if \leq coincides with the ordering defined by the conditions (a) and (b') on $X^{[4]}$, then they coincide on the entire X^* .

Lemma 6. Let X, \leq be as in the lemma 3. Then, if \leq coincides with the ordering defined by the conditions (a) and (c') on $X^{[4]}$, then they coincide on the entire X^* .

Because the proofs of these lemmas are similar, we exhibit only

the proof of Lemma 3.

Proof of Lemma 3.

We show by induction on the length of a word. Suppose that on $X^{[m-1]}$, \leq coincides with the order defined by the conditions (a) and (b) ($m \geq 5$). Let $x, y \in X^m$ be $x = x_1x_2 \cdots x_m$ and $y = y_1y_2 \cdots y_m$. We have to show that if either $|x|_b = |y|_b$ and $x <_{\text{lex}} y$, or $|x|_b < |y|_b$, then $x < y$.

Case 1 : $|x|_b = |y|_b = k$ and $x <_{\text{lex}} y$. If $x_1 \cdots x_{i-1} = y_1 \cdots y_{i-1}$ and $x_i < y_i$ ($2 \leq i \leq m$), then $|x_i \cdots x_m|_b = |y_i \cdots y_m|_b$ and $x_i \cdots x_m <_{\text{lex}} y_i \cdots y_m$, so that by the induction hypothesis, $x_i \cdots x_m < y_i \cdots y_m$. By the compatibility of \leq , $x = x_1 \cdots x_{i-1} x_i \cdots x_m < y_1 \cdots y_{i-1} y_i \cdots y_m = y$. From this fact, we can assume that x, y do not have a common initial segment, and obtain that $x < ab^ka^{m-k-1}$ and $ba^{m-k}b^{k-1} < y$. Therefore, it suffices to show that $ab^ka^{m-k-1} < ba^{m-k}b^{k-1}$. If $2k > m$, then we have :

$$|ab^{m-k}a^{m-k-1}| = |ba^{m-k}b^{m-k-1}| = 2m - 2k < m.$$

Since $|ab^{m-k}a^{m-k-1}|_b = |ba^{m-k}b^{m-k-1}|_b = m-k$ and $ab^{m-k}a^{m-k-1} <_{\text{lex}} ba^{m-k}b^{m-k-1}$, we have that $ab^{m-k}a^{m-k-1} < ba^{m-k}b^{m-k-1}$. Then, from the strong compatibility of \leq , we have :

$$ab^ka^{m-k-1} = (ab^{m-k})b^{2k-m}a^{m-k-1} < (ba^{m-k})b^{2k-m}b^{m-k-1} = ba^{m-k}b^{k-1}.$$

If $2k < m$, then we similarly obtain that $ab^ka^{k-1} < ba^kb^{k-1}$, so that

$$ab^ka^{m-k-1} = (ab^k)a^{m-2k}a^{k-1} < (ba^k)a^{m-2k}b^{k-1} = ba^{m-k}b^{k-1}.$$

In the case of $2k = m$, suppose that $ba^{m-k}b^{k-1} < ab^ka^{m-k-1}$. Then, $ba^kb^{k-1} < ab^ka^{k-1}$, so that $ba^{k+1}b^{k-1} < a^2b^ka^{k-1}$. On the other hand, we have :

$$\begin{aligned} a^2b^ka^{k-1} &= a(ab^{k-2}b^2a)a^{k-2} \\ &< a(bb^{k-2}a^2b)a^{k-2} && \text{(since } abba < baab) \\ &= (ab^{k-1})(a^2b)a^{k-2} \\ &< (ba^{k-1})(a^2b)b^{k-2} && \text{(since } ab^{k-1}a^{k-2} < ba^{k-1}b^{k-2}) \\ &= ba^{k+1}b^{k-1}, \text{ a contradiction.} \end{aligned}$$

Case 2 : $|x|_b < |y|_b$. Put $|x|_b = s$ and $|y|_b = t$. with $s < t$. If $s+t < m$, then $x_i = y_i = a$ for some i ($1 \leq i \leq m$), and if $s+t > m$, then $x_i = y_i = b$ for some i ($1 \leq i \leq m$). In the both cases, we have :

$$\begin{aligned} |x_1 \cdots x_{i-1} x_{i+1} \cdots x_m|_b &< |y_1 \cdots y_{i-1} y_{i+1} \cdots y_m|_b, \text{ so that} \\ x_1 \cdots x_{i-1} x_{i+1} \cdots x_m &< y_1 \cdots y_{i-1} y_{i+1} \cdots y_m. \end{aligned}$$

Thus, $x = x_1 \cdots x_{i-1} x_i x_{i+1} \cdots x_m < y_1 \cdots y_{i-1} y_i y_{i+1} \cdots y_m = y$.

So, suppose that $s+t = m$. Then, from the case 1, we obtain that $x \leq b^s a^t$ and $a^s b^t \leq y$. Thus, it suffices to show that $b^s a^t <$

$a^s b^t$. If $t-s \geq 2$, then since $b^s a^{t-1} < a^s b^{t-1}$ by induction, we have $b^s a^t < a^s b^t$. Hence, let $t = s+1$ (so $m = 2s+1$), and assume on the contrary to our purpose that $a^s b^{s+1} < b^s a^{s+1}$. Then, $a^s a^s b b^{s+1} < a^s b^s b a^{s+1}$. But, on the other hand, we have that

$$\begin{aligned} a^s b^s b a^{s+1} &= (a^s b^{s+1}) a^{s+1} \\ &< (b^s a^{s+1}) a^{s+1} \\ &= (b^{s-1}) b a^{s-1} (a^{s+1}) a^2 \\ &< (a^{s-1}) b a^{s-1} (a b^s) a^2 \quad (\text{since } b^{s-1} a^{s+1} < a^{s-1} a b^s) \\ &< (a^{s-1}) a a^{s-1} (a b^s) b^2 \quad (\text{since } b a^{s-1} a^2 < a a^{s-1} b^2) \text{ ---} (*) \\ &= a^s a^s b b^{s+1}, \quad \text{a contradiction.} \end{aligned}$$

At (*), we can use the induction hypothesis since $|b a^{s-1} a^2| = s+2 < 2s+1 (= m)$ by our hypothesis that $m \geq 5$. We have proved the lemma.

Now, our main theorem:

Theorem 7. Let $X = \{a, b\}$ with $a < b$. Suppose that \leq is a strongly compatible total order on X^* . Then, \leq must be one of the following six orders on X^* :

- (1) the lexicographic order,
- (2) the anti-lexicographic order,
- (3)~(6) one which satisfies the condition (a) and one of the conditions (b),(c),(b'),(c').

Proof. For the sake of Proposition 1-2 and Lemma 3-6, it suffices to check that if \leq is a strongly compatible total order on $\{a, b\}^*$, then it coincides with $<_{lex}$ or $<_{al}$ on $X^{[3]}$, or with one of the other four orders on $X^{[4]}$. So, let \leq be a strongly compatible total order on $\{a, b\}^*$. Then, since $a < b$, by the compatibility of \leq , there are only two possibilities on X^2 :

- I. $aa < ab < ba < bb$, and
- II. $aa < ba < ab < bb$.

Suppose that the possibility I is the case. Then, the strong compatibility of \leq imposes on the ordering on X^3 that $aaa < aab < aba < (abb \text{ or } baa) < bab < bba < bbb$. The strong compatibility alone can not determine if $abb < baa$ or $baa < abb$. Therefore, we get two possibilities:

- I-(i). $aaa < aab < aba < abb < baa < bab < bba < bbb$, or
- I-(ii). $aaa < aab < aba < baa < abb < bab < bba < bbb$.

If I-(i) is the case, then \leq coincides with $<_{lex}$ on $X^{[3]}$, and we are done. So, suppose that I-(ii) is the case. Then, the strong compatibility of \leq imposes on the ordering on X^4 that

$$aaaa < aaab < aaba < abaa < baaa < aabb < abab <$$

< (abba or baab) <

<baba < bbaa < abbb < babb < bbab < bbba < bbbb.

Here, again the strong compatibility fails to determine if abba < baab or baab < abba. If abba < baab, then \leq coincides on $X^{[4]}$ with the order defined by the conditions (a) and (b), and if baab < abba, then \leq coincides on $X^{[4]}$ with the order defined by the conditions (a) and (c). Hence, we have shown the case I above. The case II can be treated similarly.

When $\text{Card}(X) \geq 3$, a description of the strongly compatible total orders becomes much more complicated even when $\text{Card}(X) = 3$. We are going to show that there are infinitely many strongly compatible total orders on X^* when $\text{Card}(X) \geq 3$.

Let $X = \{a_1, a_2, \dots, a_n\}$ ($n \geq 3$). Let $f : X^* \rightarrow \mathbf{R}$ be a mapping from X^* into the set of real numbers satisfying :

(1) for $a_i \in X$, $f(a_1) \leq f(a_2) \leq \dots \leq f(a_n)$,

(2) for $x = x_1 x_2 \dots x_m \in X^*$, where $x_i \in X$ ($i = 1, \dots, m$), $f(x) =$

$$\sum_{i=1}^m f(x_i).$$

Then, let \leq_f be the order on X^* defined by:

(i) if $|x| < |y|$, then $x \leq_f y$,

(ii) if $|x| = |y|$ and $f(x) < f(y)$, then $x \leq_f y$,

(iii) if $|x| = |y|$, $f(x) = f(y)$ and $x <_{\text{lex}} y$ then $x \leq_f y$.

Then, it is clear that \leq_f is a strongly compatible total order on X^* for every f satisfying (1) and (2) above.

Proposition 8. Let $X = \{a_1, a_2, \dots, a_n\}$ ($n \geq 3$). Then, the cardinality of the set of all strongly compatible total orders on X^* is that of the set of real numbers.

Proof. Let a, b, c be three distinct letters in X with $a < b < c$. For each real number $s \in [0, 1]$, let f_s be a map : $X^* \rightarrow \mathbf{R}$ satisfying (1),(2) above and such that $f_s(a) = 0$, $f_s(b) = s$, and $f_s(c) = 1$. Denote \leq_{f_s} by \leq_s . We claim that if $0 < s < t < 1$, then \leq_s and \leq_t are not the same orders, which will show the theorem. To show the claim, let p, q be natural numbers such that $s < \frac{q}{p} < t$ (hence, $ps < q$

$< pt$). Consider the words $x = b^p$ and $y = c^q a^{p-q}$. Since $f_s(x) = ps < q = f_s(y)$, $x <_s y$. On the other hand, since $f_t(x) = pt > q = f_t(y)$, $y <_t x$. Hence, \leq_s and \leq_t are not the same.

Finally, we answer the following question :

Q. Does there exist a compatible total order which is not strongly compatible?

The answer is 'positive'. To show an example, we modify the above \leq_f so that an evaluation of a word also counts a weight on the 'location' of each letter in the word.

Let the alphabet $X = \{ a_1, a_2, \dots, a_n \}$. As before, f puts weight (a real number) on each letter:

$$f(a_1) < f(a_2) < \dots < f(a_n).$$

Let s be a positive real. Then, f_s evaluates a word $x = x_1 x_2 \dots x_m$ (where $x_i \in X$) by :

$$f_s(x) = \sum_{i=1}^m f(x_i) s^{i-1}.$$

Then, let \leq_f^s be the order on X^* defined by:

- (i) if $|x| < |y|$, then $x <_f^s y$,
- (ii) if $|x| = |y|$ and $f_s(x) < f_s(y)$, then $x <_f^s y$.

It is clear that if s is a transcendental real, then \leq_f^s is a total order. To see that \leq_f^s is compatible, it suffices to observe that $f_s(uv) = f_s(u) + f_s(v)s^{|u|}$ for $u, v \in X^*$. But, for some s , \leq_f^s is not strongly compatible. For example, let s be a transcendental number in the interval $(1, \frac{1+\sqrt{5}}{2})$, and let $f(a) = 0$, $f(b) = 1$ for $a, b \in X$.

Then, it is easy to see that $s^2 < s+1$ and $aab <_f^s bba$. If \leq_f^s were strongly compatible, then $aaa^{k-2}b <_f^s bba^{k-2}a$ for every k . But, for a sufficiently large k , $f_s(aaa^{k-2}b) = s^k > s+1 = f_s(bba^{k-2}a)$ and $bba^{k-2}a <_f^s aaa^{k-2}b$. Hence, this \leq_f^s is not strongly compatible.

Appendix.

In this appendix, we present some complimentary results on \leq_f^s - orders in the case that the alphabet $X = \{a, b\}$.

First of all, we get rid of an unnecessary complication on f .

Suppose that we are given an order \leq_f^s on $\{a,b\}^*$ for some f and (a transcendental) s . Then, for $x = x_1x_2 \cdots x_m$, (where $x_i \in \{a,b\}$), we have:

$$f_s(x) = \sum_{i=1}^m f(x_i) s^{i-1} = (f(b)-f(a)) \cdot \left[\sum_{i=1}^m \delta_i(x) s^{i-1} \right] + \sum_{i=1}^m f(a) s^{i-1},$$

where $\delta_i(x) = 1$ if $x_i = b$, else 0. From this, we conclude that an order \leq_f^s on $\{a,b\}^*$ with $f(a) < f(b)$ is equivalent to the order \leq_g^s with $g(a) = 0$ and $g(b) = 1$. Hence, in the followings, we consider only orders \leq_f^s with $f(a) = 0$ and $f(b) = 1$, and omit the subscript f as \leq^s .

Proposition A. For $s \geq 2$ (resp. $0 < s \leq \frac{1}{2}$), \leq^s coincides with \leq_{al} (resp. \leq_{lex}).

Proof. We only prove the case for $s \geq 2$. It suffices to show that for every $k \geq 1$, $1 + s + s^2 + \cdots + s^{k-1} < s^k$ because L.H.S. = $f_s(b^ka)$ and R.H.S. = $f_s(a^kb)$. But, since $s \geq 2$, $\frac{s^k-1}{s-1} \leq s^{k-1}$, i.e., $\frac{s^k-1}{s-1} < s^k$, i.e., $1 + s + s^2 + \cdots + s^{k-1} < s^k$.

Proposition B. For $s \in (\frac{1}{2}, 1) \cup (1, 2)$, \leq^s is not strongly compatible.

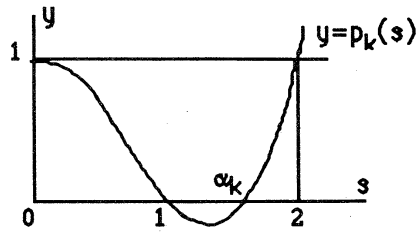
Proof. We prove only the case for $s \in (1, 2)$. We first show that $\leq^s \neq \leq_{al}$, for which it suffices to prove that there is a $k \geq 1$ such that $s^k < 1 + s + s^2 + \cdots + s^{k-1}$ since then $a^kb <^s b^ka$. But, if $s \in (1, 2)$, then for a sufficiently large k , it holds that $1 < (2-s)s^k$, i.e., $1 - s^k < (1-s)s^k$, i.e., $\frac{s^k-1}{s-1} > s^k$, i.e., $1 + s + s^2 + \cdots + s^{k-1} > s^k$. On the other hand, there is an m such that $1 + s + s^2 + \cdots + s^{k-1} < s^{k+m}$. For such an m , we have that $a^ka^mb^s > b^ka^ma$. Hence, \leq^s is not strongly compatible.

Put for each $k \geq 2$, $p_k(s) = s^{k+1} - 2s^k + 1$. Then, as in the above proof, for each $k \geq 2$, and for each $s \in (1, 2)$,

$$p_k(s) > 0 \quad \text{iff} \quad a^kb <^s b^ka$$

(<) (>)

For each $k \geq 2$, since $p_k(s) = s^k\{(k+1)s - 2k\}$, it is easy to see that there is a unique root α_k of $p_k(s) = 0$ in the interval $(1, 2)$.



Then, $\{\alpha_k\}_{k \geq 2}$ is a strictly increasing sequence (converging to 2). For, if $2 \leq k < m$, then $p_m(\alpha_k) = \alpha_k^{m+1} - 2\alpha_k^m + 1 = \alpha_k^{m-k}(\alpha_k^{k+1} - 2\alpha_k^k + 1) + 1 - \alpha_k^{m-k} = 1 - \alpha_k^{m-k} < 0$, which means that $\alpha_k < \alpha_m$. The convergence is proved by a similar argument to the proof of Proposition B.

Proposition C. There are at least countably many compatible total orders on $\{a,b\}^*$.

Proof. Let $2 \leq k < m$ and $s \in (\alpha_k, \alpha_{k+1})$, $t \in (\alpha_m, \alpha_{m+1})$. Then, it suffices to show that $<^s$ and $<^t$ are distinct orders. Since $s < \alpha_{k+1} \leq \alpha_m$, $p_m(s) < 0$, i.e., $b^m a <^s a^m b$. But, since $\alpha_m < t$, $p_m(t) > 0$, i.e., $a^m b <^t b^m a$. Hence, $<^t \neq <^s$.

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T. Saito, K. Kajitori
Shimonoseki University of Fisheries
Yoshimi, Shimonoseki, JAPAN

K. Iwamoto
Ube Technical College
Tokiwadai, Ube, JAPAN

Y. Kobayashi
Tokushima University
Tokushima, JAPAN