数理解析研究所講究録
第 698 巻 1989 年 70-73

Local L^∞ Estimate and Uniqueness of Solutions for the Generalized Korteweg-de Vries Equation

広島大学 総合科学部 堤 誉志雄 (Yoshio TSUTSUMI)
Faculty of Integrated Arts and Sciences, Hiroshima University
Higashisenda-machi, Naka-ku, Hiroshima 730, Japan

In this note we consider the regularity and uniqueness of solutions for the following generalized Korteweg-de Vries equation:

(1) $u_t + D^3u + a(u)Du = 0, \quad t > 0, \quad x \in \mathbb{R},$

(2) $u(0,x) = u_0(x), \quad x \in \mathbb{R},$

where $D = \partial/\partial x$ and $a(\lambda) \in C(\mathbb{R})$, and the author presents some recent results on that problem obtained in collaboration with J. Ginibre and G. Velo. The result in this note is part of [2] and [4].

In [5] and [6] it is shown that for any $u_0 \in L^2(\mathbb{R})$ there exists a global weak solution $u(t)$ of (1)-(2) with $a(\lambda) = \lambda$. Recently, that existence result has been extended to the general case that $|a(\lambda)| \le C\ |\lambda|^p$ for $1 \le p < 4$ in [1] and [4]. But the uniqueness problem for the weak solution is still largely open, while it is well known that the solution of (1)-(2) in $L^\infty(0,T;H^s(\mathbb{R}))$ $(s > 3/2)$ is unique, that is, the classical solution of (1)-(2) is unique. In [5] and [6] the new type of uniqueness results for (1)-(2) with $a(\lambda) = \lambda$ are also presented. In this note we state the regularity and uniqueness result of solutions for (1)-(2), which is an extension of the results due to Kato [5] and Kruzhkov and Faminskii [6].

We introduce the following Banach space:

$$\ell^m(L^q(0,T;L^r)) = \{ f \in L^q(0,T;L^r_{loc}(\mathbb{R})) ;$$

$$[\sum_{j=-\infty}^{\infty} [\int_0^T (\int_j^{j+1} |f(t,x)|^r \, dx)^{\frac{q}{r}} dt]^{\frac{m}{q}}]^{\frac{1}{m}} < \infty \},$$

$$1 \le m, q, r, \le \infty.$$

We put $\beta(p) = \max \{ \frac{1}{2p} - \frac{1}{8} , \frac{1}{8} \}$ and $x_+ = \max \{0 , x\}$.

We have the following theorem.

<u>Theorem.</u> Assume that for some p with $1 \le p < 7/2$

(3) $|a(\lambda)| \le C |\lambda|^p$.

If $(1 + x_+)^{\beta(p)} u_0 \in L^2(\mathbb{R})$, then there exists a unique solution u(t) of (1)-(2) such that for any R, T > 0

(4) $(1 + x_+)^{\beta(p)} u(t) \in L^\infty(0,T;L^2(\mathbb{R}))$,

(5) $u(t) \in L^2(0,T;H^1(-R,R))$,

(6) $u(t) \in \ell^\infty(L^q(0,T;L^\infty))$, $1 \le q < 6$,

(7) $u(t) \in L^q(0,T;L^\infty(0,\infty))$, $1 \le q < 6$.

<u>Remark.</u> (i) The above theorem is an extension of the results in [5] and [6], because it covers a(u) satisfying (3) and the decay condition at $x = +\infty$ of the initial data is relaxed.

(ii) The local L^∞ estimates (6) and (7) seem to be new and interesting. Note that (6) and (7) do not follow directly from the Sobolev imbedding theorem and (5).

(iii) It is conjectured that the upper bound p in (3) can be brought up from 7/2 to 4.

The proof of the above theorem is based on three types of estimats associated with the linear part of the equation (1), that is, (I) the space time integrability estimates similar to those valid for the Schrödinger equation and the Klein-Gordon equation, (II) the pointwise estimates of the propagator, which is the classical Airy function, and (III) the modified L^2 inequalities expressing the smoothing property of the equation.

REFERENCES

[1] J. L. Bona and J. C. Saut, (a) Singularités dispersives de solutions d'équations de type Korteweg-de Vries, C. R. Acad. Sci. Paris, 303 (1986), 101-103. (b) Dispersive blow up solutions of non inear dispersive wave equations, in preparation.

[2] J. Ginibre and Y. Tsutsumi, Uniqueness of solutions for the generalized Korteweg-de Vries equation, preprint, Orsay (1988).

[3] J. Ginibre, Y. Tsutsumi and G. Velo, Existence et unicité des solutions de l'équation de Korteweg-de Vries généralisée, to appear in C. R. Acad. Sci. Paris.

[4] J. Ginibre, Y. Tsutsumi and G. Velo, Existence and uniqueness of solutions for the generalized Korteweg-de de Vries equation, preprint, Orsay (1988).

[5] T. Kato, On the Cauchy problem for the (generalized) Korteweg-de Vries equation, Studies in Applied Mathematics, Adv. Math. Supplementary Studies, 18 (1983), 93-128.

[6] S. N. Kruzhkov and A. V. Faminskii, Generalized solutions of the Cauchy problem for the Korteweg-de Vries equation, Math. USSR Sbornik, 48 (1984), 391-421.