

Ultra-hyperbolic approach to some multi-dimensional
inverse problems

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§1. Introduction. Our aim is to describe the basic idea of [16] to show the uniqueness of multi-dimensional inverse problems of some kind.

To fix the idea, let us recall the work [11] by A. Pierce in 1979, where uniqueness of some parabolic inverse problem was established via the theory of Gel'fand-Levitan [6]. Namely, for (p, R, H) and (g, j, J) in $C^1[0, 1] \times \mathbb{R} \times \mathbb{R}$, let $u = u(x, t)$ and $v = v(x, t)$ solve

$$(1.1) \quad \begin{cases} u_t - u_{xx} = p(x)u & (0 < x < 1, 0 < t < T), & u|_{t=0} = 0 & (0 < x < 1), \\ -u_x + Ru|_{x=0} = 0 & (0 < t < T), & u_x + Hu|_{x=1} = f(t) & (0 < t < T) \end{cases}$$

and

$$(1.2) \quad \begin{cases} v_t - v_{xx} = g(x)v & (0 < x < 1, 0 < t < T), & v|_{t=0} = 0 & (0 < x < 1), \\ -v_x + jv|_{x=0} = 0 & (0 < t < T), & v_x + Jv|_{x=1} = f(t) & (0 < t < T) \end{cases}$$

respectively. Then, the identity

$$(1.3) \quad v - u \Big|_{x=1} = 0 \quad (0 < t < T)$$

implies

$$(1.4) \quad (g, f, J) = (p, R, H),$$

provided that $f \neq 0$.

The proof is carried out in the following manner. First, in the case of $f \neq 0$ the function $g(t) \equiv u \Big|_{x=1}$ ($0 < t < T$) determines the spectral characteristics $\{\lambda_j, c_j\}_{j=0}^{\infty}$ of $A = A_{p, R, H}$, the differential operator $-\frac{d^2}{dx^2} + p(x)$ under the boundary condition $(-\frac{d}{dx} + R) \cdot \Big|_{x=0} = (\frac{d}{dx} + H) \cdot \Big|_{x=1} = 0$. Then, the conclusion (1.4) follows from the Gelfand-Levitan theory.

Here, $\{\lambda_j\}_{j=0}^{\infty}$ ($-\infty < \lambda_0 < \lambda_1 < \dots$) denotes the set of eigenvalues of $A = A_{p, R, H}$, while $c_j > 0$ is the norming constant; $c_j = \|g_j\|_{L^2(0,1)}^2$, where $g_j = g_j(x)$ is the eigenfunction of A corresponding to λ_j and normalized as $g_j \Big|_{x=0} = 1$. The Gelfand-Levitan theory implies that the spectral characteristics $\{\lambda_j, c_j\}$ determine the operator $A_{p, R, H}$.

Motivated by this, R. Murayama and the author have studied the equation

$$(1.5) \quad \begin{cases} u_t - u_{xx} = p(x)u & (0 < x < 1, 0 < t < T), & u|_{t=0} = a(x) & (0 < x < 1) \\ -u_x + Ru|_{x=0} = u_x + Hu|_{x=1} = 0 & (0 < t < T) \end{cases}$$

to determine $(p, R, H, a) \in C^2[0, 1] \times \mathbb{R} \times \mathbb{R} \times L^2(0, 1)$ through the boundary value $f_j(t) \equiv u|_{x=j}$ ($0 < t < T$; $j=0, 1$) of the solution ([17], [9], [13]).

For this problem the uniqueness holds in a generic situation.

There, we modified the idea of Gel'fand-Levitan to introduce the following deformation formula

$$(1.6) \quad g(x, \lambda; g, j) = g(x, \lambda; p, h) + \int_0^x K(x, y; g, j; p, h) g(y, \lambda; p, h) dy \quad (0 \leq x \leq 1)$$

for $\lambda \in \mathbb{R}$. Here, $g = g(x, \lambda; p, h)$ denotes the solution of

$$(1.7) \quad \left(-\frac{d^2}{dx^2} + p(x)\right)g = \lambda g \quad (0 \leq x \leq 1) \quad \text{with} \quad g|_{x=0} = 1 \quad \text{and} \quad \frac{d}{dx}g|_{x=0} = h.$$

The kernel $K = K(x, y) = K(x, y; g, j; p, h)$ is independent of λ and is characterized as the solution of the hyperbolic boundary value problem

$$(1.8) \quad \begin{cases} K_{xx} - K_{yy} + p(y)K = g(y)K & (\text{in } \Omega), & K_y(x, 0) = hK(x, 0) & (0 \leq x \leq 1), \\ K(x, x) = (y-R) + \frac{1}{2} \int_0^x (g(s) - p(s)) ds & (0 \leq x \leq 1), \end{cases}$$

where $\Omega = \{(x, y) \mid 0 < y < x < 1\}$.

This method of integral transformation, sometimes is referred to as the transmutation theory (Carroll [5], e.g.), has been useful in the study of one-space dimensional inverse problems ([14]). However, it seemed to be quite difficult to extend the idea to multi-dimensional cases. In 1986, the author showed a uniqueness result for those cases with analytic coefficients, utilizing Holmgren's theorem ([15]). Now we have established it within the C^∞ -category, noting some key identity. This is the object of the present article.

We refer to some works by the Novosibirsk school for other approaches to multi-dimensional inverse problems, especially for hyperbolic equations (Bukgeim-Jahno [2], Romanov [12], Bukgeim-Klibanov [1]).

Henceforth, $\Omega \subset \mathbb{R}^n$ denotes a bounded domain whose boundary $\partial\Omega$ is smooth. The differential operator $Pu \equiv \nabla \cdot (a \nabla u) + cu$ is symmetric, uniformly elliptic, and second order with smooth real coefficients $a = (a_{ij}(x))$ and $c = c(x)$ on $\bar{\Omega}$, and $d = d(x)$ is a given smooth function on $\partial\Omega$. To fix the idea, consider the parabolic problem

$$(1.9) \quad \begin{cases} \frac{\partial u}{\partial t} - Pu = 0 & (x \in \Omega, 0 < t < T), \quad u|_{t=0} = 0 \quad (x \in \Omega), \\ \left(\frac{\partial}{\partial \nu} + d \right) u|_{\partial\Omega} = f & (0 < t < T) \end{cases}$$

where $\frac{\partial}{\partial \nu_P} = \sum \nu_i a_{ij} \frac{\partial}{\partial x_j}$ denotes the differentiation along the co-normal vector, $\nu = (\nu_i)$ being the outer unit normal vector on $\partial\Omega$.

Regarding the function $F = F(\mathcal{J}, t)$ as an input, we wish to determine the coefficients (a, c, d) through the boundary output $g = g(\mathcal{J}, t) = u(\mathcal{J}, t)$ ($\mathcal{J} \in P, 0 < t < T$), where $P \subset \partial\Omega$ with $|P| > 0$. Our conclusion assures a generic uniqueness result provided that the input $F = f(\mathcal{J})R(t)$ is given in the same area as the output g , that is, $R \not\equiv 0$ and $\text{supp } f \subset P$. Thus, we can extend the result by A. Pièrce to the multi-space dimensional case.

A similar phenomenon can be seen for the interior input-output problem. Namely, in the parabolic problem

$$(1.10) \quad \left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \Delta u = f(x)R(t) \quad (x \in \Omega, 0 < t < T), \quad u|_{t=0} = 0 \quad (x \in \Omega), \\ \left(\frac{\partial}{\partial \nu_P} + d\right) u|_{\partial\Omega} = 0 \quad (0 < t < T), \end{array} \right.$$

the output $g = u|_w$ ($0 < t < T$) determines generically (a, c, d) , provided that the input $F \equiv f(x)R(t)$ is taken as $R \not\equiv 0$ and $\text{supp } f \subset w$, where $w \subset \Omega$ is a non-empty open set. However, the cases where inputs and outputs are taken in different areas remain as problems in future.

§2. Reduction to the spectral problem. To state the result, let P and Q be second order symmetric uniformly elliptic differential operators and α and β be smooth functions on $\partial\Omega$. We suppose that these are temporally homogeneous and depend only on space variables. Given a family of inputs $F_\ell = \int_S f_\ell(t) \ell \in S$, we consider the parabolic equations

$$(2.1) \quad \begin{cases} \frac{\partial u_\ell}{\partial t} - P u_\ell = 0 & (x \in \Omega, 0 < t < T), & u_\ell|_{t=0} = 0 & (x \in \Omega), \\ \left(\frac{\partial}{\partial \nu_P} + \alpha \right) u_\ell \Big|_{\partial\Omega} = F_\ell & (0 < t < T), \end{cases}$$

and

$$(2.2) \quad \begin{cases} \frac{\partial v_\ell}{\partial t} - Q v_\ell = 0 & (x \in \Omega, 0 < t < T), & v_\ell|_{t=0} = 0 & (x \in \Omega), \\ \left(\frac{\partial}{\partial \nu_Q} + \beta \right) v_\ell \Big|_{\partial\Omega} = F_\ell & (0 < t < T). \end{cases}$$

For a given area $P \subset \partial\Omega$ with $|P| > 0$, we suppose that

$$(2.3) \quad v_\ell - u_\ell \Big|_P = 0 \quad (0 < t < T, \ell \in S).$$

We wish to establish some criterion for (2.3) to imply the uniqueness

$$(2.4) \quad (Q, \beta) = (P, \alpha).$$

For the moment, we drop the suffix $l \in S'$. Henceforth $\{\lambda_j\}$, $\{\varphi_j\}$, $\{\mu_j\}$, $\{\psi_j\}$ denote the eigenvalues and eigenfunctions of $-P_a$, the differential operator $-P$ with $(\frac{\partial}{\partial \nu_P} + d) \cdot |_{\partial \Omega} = 0$ and $-Q_p$, the differential operator $-Q$ with $(\frac{\partial}{\partial \nu_Q} + \beta) \cdot |_{\partial \Omega} = 0$, respectively. Here, $-\infty < \lambda_1 < \lambda_2 \leq \dots \rightarrow +\infty$, $-\infty < \mu_1 < \mu_2 \leq \dots \rightarrow +\infty$, and $\|\varphi_j\|_{L^2(\Omega)} = \|\psi_j\|_{L^2(\Omega)} = 1$. Then,

$$G(x, y; t) = \sum_j e^{-\lambda_j t} \varphi_j(x) \varphi_j(y)$$

is nothing but the Green function of $-\frac{\partial}{\partial t} + P_a$, and the solution $u = u(x, t)$ of (2.1) is given as

$$\begin{aligned} u(x, t) &= \int_0^t d\tau \int_{\partial \Omega} G(x, \xi; t - \tau) F(\xi, \tau) \\ &= \int_0^t r(x, t - \tau) h(\tau) d\tau, \end{aligned}$$

where

$$(2.5) \quad r(x, t) = \sum_j e^{-\lambda_j t} \varphi_j(x) \int_{\partial \Omega} \varphi_j(\xi) f(\xi) d\omega_\xi$$

because of $F(\xi, t) = f(\xi) h(t)$. Similarly, we have

$$v(x, t) = \int_0^t s(x, t - \tau) h(\tau) d\tau$$

with

$$(2.6) \quad s(\alpha, t) = \sum_j e^{-\lambda_j t} \gamma_j(\alpha) \int_{\Omega} \gamma_j(\zeta) f(\zeta) d\sigma_\zeta,$$

so that (2.3) reads as

$$(2.7) \quad \int_0^t s(\alpha, t-\tau) h(\tau) d\tau = \int_0^t r(\alpha, t-\tau) h(\tau) d\tau \quad (\alpha \in \mathcal{P}, 0 < t < T).$$

Assuming

$$(A.1) \quad \operatorname{Re} \neq 0,$$

we arrive at

$$s(\alpha, t) = r(\alpha, t) \quad (\alpha \in \mathcal{P}, 0 < t < T),$$

which extends to the full-range $t \in (0, \infty)$ by analytic continuation:

$$(2.8) \quad s(\alpha, t) = r(\alpha, t) \quad (\alpha \in \mathcal{P}, 0 < t < \infty).$$

Here, we compare the behavior as $t \rightarrow +\infty$ of both sides of (2.8).

To this end we introduce the sets $J_\lambda = \{j \mid \lambda_j = \lambda\}$ and $L_\lambda = \{j \mid \mu_j = \lambda\}$ for real numbers $\lambda \in \mathbb{R}$ and impose that

(A.2) for each $\lambda \in \mathbb{R}$ with $J_\lambda \neq \emptyset$ and $L_\lambda \neq \emptyset$, the matrices $(A_{je})_{j \in J_\lambda, e \in S'}$ and $(B_{je})_{j \in L_\lambda, e \in S'}$ are of full-rank, respectively, where $A_{je} \equiv \int_{\Omega} g_j(\zeta) f_e(\zeta) d\zeta_j$ and $B_{je} \equiv \int_{\Omega} \tau_j(\zeta) f_e(\zeta) d\zeta_j$.

Then, in particular for each $\lambda \in \mathbb{R}$ with $J_\lambda \neq \emptyset$, there exist some $j \in J_\lambda$ and $l \in S'$ such that $A_{jl} \neq 0$ and so is true for L_λ and B_{jl} . On the other hand, by virtue of Calderón's uniqueness theorem ([47]), we have the following

Proposition. Each of $\{g_j\}_{j \in J_\lambda}$ and $\{\tau_j\}_{j \in L_\lambda}$ forms a system of linearly independent functions on P .

In fact, g_j 's ($j \in J_\lambda$) satisfy the same equation $(P + \lambda)g = 0$ in Ω . Therefore, supposing $g \equiv \sum_{j \in J_\lambda} a_j g_j = 0$ on P for real constants a_j ($j \in J_\lambda$) we have $g|_P = \frac{\partial}{\partial \bar{z}} g|_P = 0$ and hence $g = 0$ near P by Calderón's theorem. Now the unique continuation property assures us of $g \equiv 0$ in Ω and hence $a_j = 0$ ($j \in J_\lambda$).

Therefore, for each λ with $J_\lambda \neq \emptyset$ there exists some l such that

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$\sum_{j \in J_\lambda} g_j(\alpha) A_{je} \neq 0$ on Γ and so is true for L_λ and $\sum_{j \in L_\lambda} \gamma_j(\alpha) B_{je}$.

In particular, (2.8) implies that

$$(2.9) \quad \{ \lambda_j \mid j \in J_\lambda \} = \{ \mu_j \mid j \in L_\lambda \}$$

and also

$$(2.10) \quad \sum_{j \in J_\lambda} g_j(\alpha) A_{je} = \sum_{j \in L_\lambda} \gamma_j(\alpha) B_{je} \quad (\alpha \in \Gamma, \lambda \in S).$$

Again, the assumption (A2) improve (2.9) as

$$(2.11) \quad J_\lambda = L_\lambda \quad (\lambda \in \mathbb{R})$$

via Proposition and (2.10). Furthermore, (2.10) reduces to the relation

$$(2.12) \quad g_j(\alpha) = \sum_{k \in L_\lambda} \delta_{jk} \gamma_k(\alpha) \quad (\alpha \in \Gamma, j \in J_\lambda)$$

for some real numbers $\{ \delta_{jk} \}$.

Henceforth, we suppose the following important assumption:

$$(A.2) \quad \text{supp. } f_\ell \subset P \quad (\ell \in S').$$

Then, we have

$$A_{j\ell} = \int_P g_j(x) f_\ell(x) dx$$

and hence can substitute (2.12) into (2.10). Noting Proposition, we get

$$\sum_{j,m \in J_\lambda} \delta_{jk} \delta_{jm} B_{me} = B_{ke} \quad (k \in L_\lambda, e \in S')$$

or

$$T(\delta_{jB}) (\delta_{jB}) = (\delta_{jB})$$

by (A.2). Hence $\{\tilde{\psi}_j\}_{j \in L_\lambda}$ forms an orthonormal system in $L^2(\Omega)$ where $\tilde{\psi}_j = \sum_{B \in L_\lambda} \delta_{jB} \psi_B$. Without loss of generality we may suppose that

$$(2.13) \quad \mu_j = \lambda_j \quad \text{and} \quad \psi_j(x) = g_j(x) \quad (x \in P) \quad \text{for all } j.$$

§3. Iso-spectral deformation. Now, we shall show that the condition (2.13) implies the uniqueness

$$(3.1) \quad (Q, \beta) = (P, \alpha),$$

which corresponds exactly to Gelfand-Levitan's uniqueness result in one-space dimension. The author has shown the fact in a rather special case in [15] of analytic coefficients operators. An extension to C^∞ coefficients operators has been performed by A. Nachman, J. Sylvester and G. Uhlmann in [10] motivated by Borg-Levinson's work ([3], [8]), for $-P = -\Delta + p(x)$, $-Q = -\Delta + q(x)$, $\beta = \alpha = 0$ and $\Gamma = \partial\Omega$ through a function theoretic method. Here, we take a different approach of deformation formula described in §1 for one-space dimensional case.

The formula (1.6) reads as

$$(3.2) \quad \gamma(x, \lambda) = \varphi(x, \lambda) + \int_0^1 H(x-y) K(x, y) \varphi(y, \lambda) dy,$$

where $H = H(x)$ is the Heaviside function. Then, the commutator $[\frac{d^2}{dx^2}, H]$ will produce the δ -function and the deformation (1.5) is achieved. In spite that there is no reasonable extension of the Heaviside function in multi-dimensional spaces, the above relation means

$$(3.3) \quad \psi_j = (1 + K) g_j$$

for the iso-spectral case $\mu_j = \lambda_j$ ($j=1, 2, \dots$). The operator K is formally an integral operator with the kernel

$$(3.4) \quad K(x, y) = \sum_j \{ \psi_j(x) - g_j(x) \} g_j(y).$$

In view of (3.2), $K = K(x, y)$ has the Heaviside-function like discontinuity on the diagonal $D \equiv \{ (x, x) \mid x \in \Omega \}$ when the space-dimension is one.

For the moment, we shall develop a formal theory which will be justified later. Hence we note the following key "identity"

$$(3.5') \quad K(x, y) \equiv \sum_j \{ \psi_j(x) - g_j(x) \} g_j(y) = \sum_j \psi_j(x) \{ g_j(y) - \psi_j(y) \}.$$

In fact we have

$$\sum_j g_j(x) g_j(y) = \sum_j \psi_j(x) \psi_j(y) = \delta(x - y).$$

The "function" $K^* = K^*(x, y) = \sum_j \psi_j(x) g_j(y)$ satisfies the ultra-hyperbolic equation

$$\square K^* = 0,$$

where

$$(3.6) \quad \square = -Q_x + P_y.$$

In fact, we have $\lambda_j = \mu_j$ ($j=1, 2, \dots$). Hence

$$(3.7) \quad \square K = 0 \quad (x \neq y)$$

by $K(x, y) = K^*(x, y) - f(x-y)$. On the other hand we have

$$"K|_{P \times \Omega} = K|_{\Omega \times P} = 0"$$

by $g_j(x) = \gamma_j(x)$ ($x \in P, j=1, 2, \dots$).

However, noting (3.7) we have

$$(3.8') \quad Q_x^m K|_{P \times \Omega} = P_y^m K|_{P \times \Omega} = 0,$$

where $m=0, 1, 2, \dots$. In other words,

$$" \sum_j \lambda_j^m \gamma_j(x) \{g_j(y) - \gamma_j(y)\} = 0 \quad (m=0, 1, 2, \dots; x \in P, y \in \Omega). "$$

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From the "Weierstrass approximation theorem" we get

$$\sum_{j \in \mathbb{Z}_x} \gamma_j(x) \{ \beta_j(y) - \gamma_j(y) \} = 0 \quad (x \in P, y \in \Omega)$$

for every $\lambda \in \mathbb{R}$. Again by Proposition in the previous section, we have $\beta_j \equiv \gamma_j$ ($j=1, 2, \dots$). Now, both of eigenvalues and eigenfunctions coincide, we arrive at $P_\alpha = Q_\beta$, which means that $Q \equiv P$ and $\beta \equiv \alpha$.

The above formal argument can be justified in the following way. First, for any given integer $k \geq 0$, we chose large numbers λ and s so that

$$\zeta_s(x, y; \lambda) \equiv \sum_j \{ \gamma_j(x) - \beta_j(x) \} \beta_j(y) (\lambda_j + \lambda)^{-s} \in C^k(\overline{\Omega} \times \overline{\Omega}).$$

Then,

$$\mathcal{L}(x, y) \equiv (-P_y + \lambda)^s \zeta_s(x, y; \lambda) \in C^k(\overline{\Omega}_x \rightarrow \mathcal{D}'_y(\Omega))$$

is independent of λ and s , and $\mathcal{L} = \mathcal{L}(x, y) \in C^\infty(\overline{\Omega}_x \rightarrow \mathcal{D}'_y(\Omega))$

$\mathcal{L} \in \mathcal{D}'(\Omega \times \Omega)$ can be defined. Similarly, from the function

$$M_S(\alpha, \gamma; \lambda) \equiv \sum_j \gamma_j(\alpha) \{ \beta_j(\gamma) - \gamma_j(\gamma) \} (\mu_j + \lambda)^{-S} \in C^{\infty}(\bar{\Omega} \times \bar{\Omega})$$

we can define $M = M(\alpha, \gamma) \in C^{\infty}(\bar{\Omega}_\gamma \rightarrow \mathcal{D}'_x(\Omega)) \subset \mathcal{D}'(\Omega \times \Omega)$ through

$$M(\alpha, \gamma) \equiv (-\Delta_x + \lambda)^S M_S(\alpha, \gamma; \lambda).$$

Now the key identity (3.5) is justified as

$$(3.5) \quad L(\alpha, \gamma) = M(\alpha, \gamma) \quad (:= K(\alpha, \gamma)) \quad \text{as } \mathcal{D}'(\Omega \times \Omega).$$

Namely, the distribution $K = K(\alpha, \gamma) \in \mathcal{D}'(\Omega \times \Omega)$ has an "uncertain" character $K(\alpha, \gamma) = L(\alpha, \gamma) \in C^{\infty}(\bar{\Omega}_x \rightarrow \mathcal{D}'_y(\Omega))$ and $K(\alpha, \gamma) = M(\alpha, \gamma) \in C^{\infty}(\bar{\Omega}_\gamma \rightarrow \mathcal{D}'_x(\Omega))$. By the hypothesis of iso-spectral: $\lambda_j = \mu_j$ ($j=1, 2, \dots$), ultra-hyperbolic relation

$$(3.7) \quad \square K = 0$$

holds in $\bar{\Omega} \times \Omega \setminus D$ and $\Omega \times \bar{\Omega} \setminus D$ as $K = L$ and $K = M$, respectively, where $D = \{(\alpha, x) \mid x \in \Omega\}$. Hence from $L_S|_{\Omega \times \Omega} = 0$ we obtain

$$(3.8) \quad Q_x^m K|_{\overline{\Omega} \times \Omega} = 0 \quad (m=0, 1, 2, \dots)$$

as elements in $C^\infty(\overline{\Omega} \times \Omega \rightarrow \mathcal{D}'_y(\Omega))$.

Here, we introduce the function

$$(3.9) \quad F_t(x, y) = \sum_j e^{-\lambda_j t} \gamma_j(x) \{g_j(y) - \gamma_j(y)\} \in C^\infty(\overline{\Omega} \times \overline{\Omega})$$

for $t > 0$. The Fey identity (3.5) deduces

$$(3.10) \quad F_t(x, y) = \sum_{m=0}^{\infty} \frac{t^m}{m!} Q_x^m K(x, y) \quad \text{in } \mathcal{D}'(\Omega \times \Omega).$$

In fact, we have in $\mathcal{D}'(\Omega \times \Omega)$ that

$$\begin{aligned} F_t(x, y) &= \sum_j \sum_m \frac{t^m}{m!} (-\lambda_j)^m \gamma_j(x) \{g_j(y) - \gamma_j(y)\} \\ &= \sum_m \sum_j \frac{t^m}{m!} (-\lambda_j)^m \gamma_j(x) \{g_j(y) - \gamma_j(y)\}, \end{aligned}$$

while

$$Q_x^m K(x, y) = \sum_j (-\lambda_j)^m \gamma_j(x) \{g_j(y) - \gamma_j(y)\}$$

in $\mathcal{D}'(\Omega \times \Omega)$ by (3.5). However, the right-hand side of (3.10) converges in $C^\infty(\overline{\Omega} \times \Omega \rightarrow \mathcal{D}'_y(\Omega))$ and hence (3.10) holds as

a relation there. Hence

$$(3.11) \quad F_t(x, y) = 0 \quad (0 < t < \infty, x \in P, y \in \Omega)$$

by (3.8).

Comparing the behavior as $t \rightarrow \infty$ and utilizing Proposition in the preceding section, we arrive at

$$(3.12) \quad y_j = g_j \quad (j=1, 2, \dots).$$

Then, $(Q, \beta) = (R, \alpha)$ follows.

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Note... An important remark has been given by H. Isozaki for multi-dimensional iso-spectral problems ([7]).