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Nonstandard representations of heat kernels on compact Riemannian manifolds

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In this note we show applications of nonstandard analysis to heat kernels on compact Riemannian manifolds. As our framework of nonstandard analysis, we adopt the nonstandard set theory UNST presented by T. Kawai [7]. In §1, we give some notational preliminaries together with the axioms of UNST. Section 2 concerns nonstandard construction of a Brownian motion on a compact Riemannian manifold. Its each sample path will be obtained as the standard part of the projection of the solution of a certain (internal) differential equation on the nonstandard extension of the bundle of orthonormal frames over the manifold. Then in §3, we get a nonstandard probabilistic representation of the heat kernel for a heat equation for functions on a compact Riemannian manifold. More generally, in §4, we obtain a nonstandard probabilistic representation of the heat kernel for a heat equation for differential forms with values in a Riemannian vector bundle. We mention here that sections 3 and 4 are motivated by S. Watanabe's work [13] giving probabilistic expressions of heat kernels by using the

Malliavin calculus (cf. [5], [6], [14], [1]).

Basically, the material (except the preliminary section 1) of this note is taken from [2; §2 and §4].

1. Preliminaries

We shall adopt T. Kawai's nonstandard set theory UNST ([7]), which is a conservative extension of ZFC. The language of UNST is given by adding three constant symbols $\mathfrak{U}$, $\mathcal{I}$ and $*$ to the language of ZFC. If $a \in \mathfrak{U}$ [resp. $a \in \mathcal{I}$], a is called *usual* [resp. *internal*]. For a formula φ in ZFC (that is, a formula in UNST without $\mathfrak{U}$, $\mathcal{I}$ and $*$), ${}^{\mathfrak{U}}\varphi$ [resp. ${}^{\mathcal{I}}\varphi$] denotes the relativisation of φ to $\mathfrak{U}$ [resp. $\mathcal{I}$]. We recall the axioms of UNST, the following (1)–(9):

(1) If φ is an axiom of ZFC, then ${}^{\mathfrak{U}}\varphi$ is an axiom of UNST.

(2) Each axiom of ZFC different from the axiom of regularity is an axiom of UNST.

(3) (Axiom of regularity in a restricted form)
$\forall A\ [A \neq 0 \wedge A \cap \mathcal{I} = 0 \longrightarrow \exists x \in A\ [x \cap A = 0]]$. ($0 = \phi$: the empty set.)

(4) $* : \mathfrak{U} \longrightarrow \mathcal{I}$ (map). When $a \in \mathfrak{U}$, we write ${}^{*}a$ for $*(a)$.

(5) (Transitivity of $\mathcal{I}$) $\forall A\ \forall B\ [A \in B \wedge B \in \mathcal{I} \longrightarrow A \in \mathcal{I}]$.

(6) (Transitivity of $\mathfrak{U}$) $\forall A\ \forall B\ [A \in B \wedge B \in \mathfrak{U} \longrightarrow A \in \mathfrak{U}]$.

(7) $\forall A\ \forall B\ [A \subset B \wedge B \in \mathfrak{U} \longrightarrow A \in \mathfrak{U}]$.

(8) (Transfer principle) Let $\varphi(x_1, \ldots, x_n)$ be an n-ary formula in ZFC (the free variables of φ are among

$x_1, \ldots, x_n$). Then

$$\forall x_1,\ldots,x_n \in \mathfrak{U}\ [{}^{\mathfrak{U}}\varphi(x_1,\ldots,x_n) \longleftrightarrow {}^{\mathcal{J}}\varphi({}^*x_1,\ldots,{}^*x_n)].$$

(9) (Saturation principle) Define $(D : \mathfrak{U}\text{-}size) \equiv \exists F\ (\text{map})\ [F : \mathfrak{U} \longrightarrow D\ (\text{onto})]$. Let $\varphi(a,b,x_1,\ldots,x_n)$ be an (n+2)-ary formula in ZFC (the free variables of φ are among $a, b, x_1, \ldots, x_n$). Then

$$\forall D : \mathfrak{U}\text{-size}\ \ \forall x_1,\ldots,x_n \in \mathcal{J}$$
$$\left[\begin{array}{l}\forall d \in \mathcal{J}\ \left[\begin{array}{l} d \text{ is finite} \wedge d \subset D \\ \longrightarrow \exists b \in \mathcal{J}\ \forall a \in d\ {}^{\mathcal{J}}\varphi(a,b,x_1,\ldots,x_n)\end{array}\right] \\ \longrightarrow \exists B \in \mathcal{J}\ \forall A \in D \cap \mathcal{J}\ {}^{\mathcal{J}}\varphi(A,B,x_1,\ldots,x_n)\end{array}\right].$$

It is known that the following hold in UNST:

(Extension principle)

$$\forall A \in \mathcal{J}\ \forall B \in \mathcal{J}\ \forall a\ \forall f$$
$$\left[\begin{array}{l} a \subset A \wedge a : \mathfrak{U}\text{-size} \wedge f : a \longrightarrow B\ (\text{map}) \\ \longrightarrow \exists F \in \mathcal{J}\ [F : A \longrightarrow B\ (\text{map}) \wedge \forall x \in a\ [F(x) = f(x)]]\end{array}\right].$$

By the transfer principle, the map $* : \mathfrak{U} \longrightarrow \mathcal{J}$ is injective. For $a \in \mathfrak{U}$, we often identify a with ${}^*a$ (and use a instead of ${}^*a$) when we do not need to take account of the set-structure of a. (For example, each $r \in R$ is identified with ${}^*r$. Then $R \subsetneqq {}^*R$.) If $f : A \longrightarrow B$ is a map with $A, B, f \in \mathfrak{U}$, we have ${}^*f : {}^*A \longrightarrow {}^*B$ and ${}^*(f(x)) = {}^*f({}^*x)$ for all $x \in A$; we often write $f : {}^*A \longrightarrow {}^*B$ rather than ${}^*f : {}^*A \longrightarrow {}^*B$.

When a is a point of a topological space X and $x \in$

Mon(a) (= the monad of a), we write $x \approx a$. (If $x, y \in {}^*R^n$ with $n \in N-\{0\}$ and $|x-y|$ is infinitesimal where $|\cdot|$ denotes Euclidean norm in ${}^*R^n$, we write $x \approx y$.) If X is a Hausdorff space and $x \in {}^*X$ is near-standard, the standard part of x is denoted by ${}^\circ x$ (cf. [7], [10], [11]).

Now we prepare a hyperfinite random wark in ${}^*R^n$, where $0 < n \in N$. Let $K_0 \in {}^*N-N$ and put $K = K_0!$ (factorial in ${}^*N$). Set $\Omega = \{-1, 1\}^{Kn}$. Then Ω is a hyperfinite set and each $\omega \in \Omega$ is expressed as

$$\omega = (\omega_i^\alpha),\ \omega_i^\alpha = \pm 1\ (i = 1, 2, \cdots, K;\ \alpha = 1, 2, \cdots, n).$$

Let $\mathcal{A}$ be the internal algebra of all internal subsets of Ω. Define an internal probability measure $\nu : \mathcal{A} \longrightarrow {}^*[0,1]$ (= $\{x \in {}^*R\ ;\ 0 \leq x \leq 1\}$) by $\nu(A) = |A|/|\Omega|$ for all $A \in \mathcal{A}$, where $|\cdot|$ denotes internal cardinality; that is, ν is the normalized counting measure on $(\Omega, \mathcal{A})$. Let $\{e_1, \ldots, e_n\}$ be the canonical basis of R^n (and thus of ${}^*R^n$). Define a hyperfinite random walk

$$w : {}^*[0,1] \times \Omega \ni (t,\omega) \longmapsto w(t,\omega) = w_t(\omega) \in {}^*R^n,$$

$$w(t,\omega) = \sum_{\alpha=1}^{n} w_t^\alpha(\omega) e_\alpha ,$$

by

$$w_t^\alpha(\omega) = \frac{1}{\sqrt{K}}\left(\sum_{i=1}^{[Kt]} \omega_i^\alpha(s) + (Kt-[Kt])\omega_{[Kt]+1}^\alpha\right),$$

$$\alpha = 1, 2, \cdots, n,$$

where $[Kt]$ stands for the greatest hyperinteger not exceeding Kt. The internal process w_t may be called *Anderson's n-dimensional process*.

As is well known, the saturation principle implies that the finitely additive measure $°\nu : \mathscr{A} \ni A \mapsto °(\nu(A)) \in [0,1]$ $(= \{x \in \mathbf{R};\ 0 \le x \le 1\})$ is countably additive on $\mathscr{A}$ and is extended uniquely to a probability measure $\tilde{\nu}$ on the σ-algebra $\sigma(\mathscr{A})$ generated by $\mathscr{A}$. The completion $(\Omega, L(\mathscr{A}), \nu_L)$ of the probability space $(\Omega, \sigma(\mathscr{A}), \tilde{\nu})$ is called the (uniform) *Loeb probability space* of $(\Omega, \mathscr{A}, \nu)$ (see [9], [10], [11]).

<u>**Proposition 1.1**</u> ([3]). Set

$$b(t,\omega) = \sum_{\alpha=1}^{n} b^\alpha(t,\omega)e_\alpha = °(w(t,\omega)), \quad (t,\omega) \in [0,1] \times \Omega.$$

$$(b^\alpha(t,\omega) = °(w^\alpha(t,\omega)), \quad (t,\omega) \in [0,1] \times \Omega.)$$

Then $b(t,\omega)$ is continuous in t and finite for almost all ω, and (a continuous and finite version of) $(b(t,\omega))_{t\in[0,1]}$ is a Brownian motion, called *Anderson's Brownian motion*.

We refer to [3], [10] for applications of Anderson's Brownian motion to stochastic integrals.

2. Brownian motion on a compact Riemannian manifold — nonstandard approach —

Let (M, g) be a compact, connected, C^∞ Riemannian manifold of dimension n $(0 < n \in \mathbf{N})$, and let $\pi : O(M) \longrightarrow M$ be the bundle of orthonormal frames for TM over M. Then $O(M)$ is compact. Moreover, it is endowed with the connection induced naturally from the Riemannian metric g. For $\xi \in \mathbf{R}^n$, the basic vector field on $O(M)$ corresponding to ξ is denoted by $B(\xi)$; for $r \in O(M)$, $B(\xi)_r \in T_rO(M)$ is the horizontal lift of $r\xi \in T_{\pi(r)}M$. Here $r\xi$ is the tangent vector at $\pi(r)$ whose components with respect to r are just the components of ξ. It is a fundamental fact that if C is the integral curve of $B(\xi)$ through $r \in O(M)$ then $\pi(C)$ is the geodesic to which $r\xi$ is tangent ([8]).

Let $\mathfrak{X}(O(M))$ be the space of C^∞ vector fields on $O(M)$. Then we have the map $B : \mathbf{R}^n \ni \xi \longmapsto B(\xi) \in \mathfrak{X}(O(M))$, so that ${}^*B : {}^*\mathbf{R}^n \ni \xi \longmapsto {}^*B(\xi) \in {}^*(\mathfrak{X}(O(M)))$. As stated in §1, for $\xi \in {}^*\mathbf{R}^n$, we write $B(\xi)$ instead of ${}^*B(\xi)$.

For each $\omega \in \Omega$, consider the following (internal) ordinary differential equation on ${}^*(O(M))$:

$$\frac{dr_t}{dt} = B\left(\frac{dw(t,\omega)}{dt}\right)r_t , \tag{2.1}$$

where $dw(t,\omega)/dt = \sum_{\alpha=1}^{n} (dw_t^{\alpha}(\omega)/dt)e_{\alpha}$. Since $w(\cdot,\omega)$ is "hyper"-piecewise smooth, the equation (2.1) has a correct meaning. The solution of (2.1) with the condition $r_0 = r \in {}^*(O(M))$ is denoted by $r_t(r,\omega)$ or $r_t(r)$ (with ω missing). More precisely, r_t is the continuous curve (in ${}^*(O(M))$) starting from r at time $t = 0$ and satisfying (2.1) for each time-interval $(k/K, (k+1)/K) \subset {}^*[0,1]$, $k = 0, 1, \cdots, K-1$. Since M and $O(M)$ are both compact, $r_t(r,\omega)$ and $\pi(r_t(r,\omega))$ are both near-standard, and it holds that

$$^{\circ}\pi(r_t(r,\omega)) = \pi(^{\circ}r_t(r,\omega)).$$

Note that $\pi(r_t(r,\omega))$ is a (*-)broken geodesic in ${}^*M$. We put

$$X_t(r,\omega) = {}^{\circ}\pi(r_t(r,\omega)) = \pi(^{\circ}r_t(r,\omega)).$$

Let $\mathbb{E}[\cdot]$ denote expectation and Δ_g be the Laplace-Beltrami operator. Then we have the following:

<u>Lemma 2.1</u>. Let $h : M \longrightarrow \mathbb{R}$ be a C^{∞} function. Then

$$u(t,x) = \mathbb{E}[h(X_t(r,\omega))], \qquad t \in (0,1),\ x \in M,\ r \in O(M),\ \pi(r) = x, \tag{2.2}$$

is well-defined and is the solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta_g u, \qquad \lim_{t \downarrow 0} u(t, \cdot) = h. \tag{2.3}$$

Proof. [2].

<u>**Corollary 2.2**</u>. The stochastic process $(X_t(r))_{t \in [0,1]}$, $r \in O(M)$, is a Brownian motion on (M, g) such that $X_0(r) = \pi(r)$; that is, $(X_t(r))$ is a diffusion process on M with generator $(1/2)\Delta_g$.

Thus we have the following diagram:

$$r_t(r,\omega) \xrightarrow{\ \circ\ } {}^{\circ}r_t(r,\omega)$$

solution of (2.2) — stochastic moving frame

$\pi(={}^*\pi)$ (cf. §1) $\downarrow$ — $\pi \downarrow$

$$\pi(r_t(r,\omega)) \xrightarrow{\ \circ\ } X_t(r,\omega)$$

(*-)broken geodesic in ${}^*M$ — Brownian motion on M

$t \in [0,1]$, $r \in O(M)$, $\omega \in \Omega$.

3. Nonstandard representation of the heat kernel for (2.3)

Let (M, g) be as in §2, and $|\Lambda|(M)$ the C^∞ real line bundle of densities (of order one) over M (cf. [4]). We denote both $\int_M$ and ${}^*\int_{{}^*M}$ simply by $\int$. Let $dv_g \in C^\infty(|\Lambda|(M))$ (= {C^∞ sections of $|\Lambda|(M)$} = {C^∞ densities (of order one) on M}) be the Riemannian volume density (the volume density of the Riemannian metric g, cf. [12, Chapter 3, §3]); it is a *positive* density. Then there exists a non-negative $\hat{\delta}_{dv_g} \in {}^*(C^\infty(M\times M))$ such that

$$h(x) \approx \int \hat{\delta}_{dv_g}(x,\cdot)h\ dv_g = \int \hat{\delta}_{dv_g}(\cdot,x)h\ dv_g$$

for every C^∞ function $h : M \longrightarrow \mathbf{R}$ and $x \in M$, where $C^\infty(M\times M)$ is the space of $\mathbf{R}$-valued C^∞ functions on $M\times M$. Let $X_t(r)$ be as in §2.

We note that the map $C^\infty(M) \ni h \longmapsto \mathbb{E}[h(X_t(r))] \in \mathbf{R}$ is in $\mathfrak{U}$ (see §1). Thus we have the map

$${}^*(C^\infty(M)) \ni \hat{\delta}_{dv_g}(\cdot,z) \longmapsto \mathbb{E}[\hat{\delta}_{dv_g}(X_t(r),z)] \in {}^*\mathbf{R}, \quad z \in {}^*M.$$

Now we give a nonstandard representation of the heat kernel of the heat equation (2.3) with respect to dv_g.

<u>**Theorem 3.1**</u>. For $t \in (0,1) \subset \mathbf{R}$, $x, y \in M$, and $r \in O(M)$ with $\pi(r) = x$, the quantity $\mathbb{E}[\hat{\delta}_{dv_g}(X_t(r),y)]$ is near-standard and the heat kernel $e_0(t,x,y)$ with respect to dv_g for the equation (2.3) is given by

$$e_0(t,x,y) = {}^\circ\mathbb{E}[\hat{\delta}_{dv_g}(X_t(r),y)],$$

$$t \in (0,1),\ x, y \in M,\ r \in O(M),\ \pi(r) = x.$$

(Therefore the solution of the heat equation (2.3) is given by

$$u(t,x) = \int_M {}^\circ\mathbb{E}[\hat{\delta}_{dv_g}(X_t(r),y)]h(y)\, dv_g(y),$$

$$t \in (0,1),\ x, y \in M,\ r \in O(M),\ \pi(r) = x.)$$

Proof. Let $t \in (0,1)$, $x, y \in M$, $r \in O(M)$, $\pi(r) = x$. By Lemma 2.1 and Corollary 2.2, the solution of (2.3) is

$$u(t,x) = \mathbb{E}[h(X_t(r))] = {}^\circ\mathbb{E}[\int \hat{\delta}_{dv_g}(X_t(r),\cdot)h\, dv_g]$$

$$= {}^\circ\int \mathbb{E}[\hat{\delta}_{dv_g}(X_t(r),\cdot)]h\, dv_g\ .$$

This shows that if $\mathbb{E}[\hat{\delta}_{dv_g}(X_t(r),y)]$ is near-standard then its standard part is the heat kernel with respect to dv_g for the equation (2.3). But in differential geometry the heat kernel $e_0(t,x,y)$ is known to exist, and so we have

$$\mathbb{E}[\hat{\delta}_{dv_g}(X_t(r),y)] = \int \hat{\delta}_{dv_g}(\cdot,y)e_0(t,x,\cdot)dv_g \approx e_0(t,x,y).$$

This means that $\mathbb{E}[\hat{\delta}_{dv_g}(X_t(r),y)]$ is near-standard; moreover, it holds that $^{\circ}\mathbb{E}[\hat{\delta}_{dv_g}(X_t(r),y)] = e_0(t,x,y)$.

4. Nonstandard representation of a heat kernel for differential forms with values in a Riemannian vector bundle

In this section we will give nonstandard representation of the heat kernel of a heat equation for differential forms with values in a Riemannian vector bundle. First we need some preliminaries.

For a C^∞ real vector bundle $\pi_E : E \longrightarrow M$ over a C^∞ compact manifold M, we denote by $\pi_{E^*} : E^* \longrightarrow M$ the dual vector bundle of E. Let η be an E^*-valued random variable defined on a complete probability space, and put $F = \pi_{E^*}\circ\eta$, which is an M-valued random variable. We write the evaluation between E^* [resp. ${}^*E^*$] and E [resp. ${}^*E$] as the multiplication sign " $\square$ "; for example, if $\sigma \in C^\infty(E)$, then $\eta\square(\sigma\circ F)$ (= ${}_{E^*}(\eta,\ \sigma\circ F)_E$) is a real-valued random variable. We assume $\mathbb{E}[|\eta\square(\sigma\circ F)|] < \infty$ for all $\sigma \in C^\infty(E)$. As before, we denote both $\int_M$ and ${}^*\int_{{}^*M}$ simply by $\int$.

If we take up a *positive* density $\rho \in C^\infty(|\Lambda|(M))$, then there exists a non-negative $\hat{\delta}_\rho \in {}^*(C^\infty(M\times M))$ such that

$$h(x) \approx \int \hat{\delta}_\rho(x,\cdot)h\rho = \int \hat{\delta}_\rho(\cdot,x)h\rho$$

for every C^∞ function $h : M \longrightarrow \mathbb{R}$ and $x \in M$, where $C^\infty(M\times M)$ is the space of $\mathbb{R}$-valued C^∞ functions on $M \times M$.

<u>**Lemma 4.1**</u>. Let η, F, ρ, and $\hat{\delta}_\rho$ be as above. Let $\mathbb{E}[\ |F=\cdot]$ denote conditional expectation under $F = \cdot$. Then for $\sigma \in C^\infty(E)$,

$$\mathbb{E}[\eta\square(\sigma\circ F)] \approx \int \mathbb{E}[\hat{\delta}_\rho(F,\cdot)]\mathbb{E}[\eta\square(\sigma\circ F)|F=\cdot]\rho.$$

Proof. This is shown as follows:

$$\mathbb{E}[\eta\square(\sigma\circ F)] \approx \mathbb{E}[\int \hat{\delta}_\rho(F,\cdot)\mathbb{E}[\eta\square(\sigma\circ F)|F=\cdot]\rho]$$

$$= \int \mathbb{E}[\hat{\delta}_\rho(F,\cdot)]\mathbb{E}[\eta\square(\sigma\circ F)|F=\cdot]\rho\ .$$

Now, let (M, g) be as in §2, and let Q be a C^∞ Riemannian vector bundle of rank $\ell \in \mathbb{N}-\{0\}$ over M, so that Q is endowed with a Riemannian fiber metric a and a metric linear connection $\nabla^Q : C^\infty(Q) \times \mathfrak{X}(M) \ni (\sigma, X) \longmapsto \nabla^Q_X \sigma \in C^\infty(Q)$, where $\mathfrak{X}(M)$ denotes the space of C^∞ vector fields on M. Recall that ∇^Q satisfies the following conditions:

(1) $\nabla^Q_X \sigma$ is $C^\infty(M)$-linear in X and $\mathbb{R}$-linear in σ,

(2) $\nabla^Q_X(f\sigma) = (Xf)\sigma + f\nabla^Q_X\sigma, \qquad f \in C^\infty(M),$

(3) $X(a(\sigma_1,\sigma_2)) = a(\nabla^Q_X\sigma_1,\sigma_2) + a(\sigma_1,\nabla^Q_X\sigma_2), \quad \sigma_1, \sigma_2 \in C^\infty(Q)$.

Consider the vector bundle $\pi_E : E = \bigoplus_{m=0}^{n} \{(\Lambda^m T^*M)\otimes Q\}$ (Whitney sum) $\longrightarrow M$, where $T^*M = (TM)^*$ (= the cotangent bundle over M). Then $C^\infty(E) := \{C^\infty$ sections of E$\} = \{$Q-valued C^∞ differential forms on M$\}$. Let ∇^E and $\nabla^{T^*M\otimes E}$ denote linear connections in E and $T^*M\otimes E$, respectively, induced naturally from the Levi-Civita connection and the given connection ∇^Q in Q, and let "Trace" denote the trace operator with respect to the fiber metric in $T^*M\otimes T^*M\otimes E$ induced naturally from g and the given fiber metric a in Q.

We consider the following heat equation for Q-valued differential forms:

$$\frac{\partial\sigma}{\partial t} = \frac{1}{2}\Delta_Q\sigma \left(:= \frac{1}{2}\,\mathrm{Trace}\,(\nabla^{T^*M\otimes E}\nabla^E\sigma)\right), \quad t \in (0,1),$$

$$\lim_{t\downarrow 0} \sigma(t,\cdot) = \sigma_0 \in C^\infty(E). \tag{4.1}$$

As in §3, we denote by $dv_g \in C^\infty(|\Lambda|(M))$ the Riemannian volume density; it is a positive density. Let $\hat{\delta}_{dv_g}$ be as in §3. Let $\pi_{O(M)} : O(M) \longrightarrow M$ [resp. $\pi_{O(Q)} : O(Q) \longrightarrow M$] be the bundle of orthonormal frames for TM [resp. Q] over M. Denote by P the fiber product of $\pi_{O(M)}$ and $\pi_{O(Q)}$; that is, $P = \{p = (r_1, r_2) \in O(M) \times O(Q) \; ; \; \pi_{O(M)}(r_1) = \pi_{O(Q)}(r_2)\}$. Let π_1 and π_2 denote the projections from P onto O(M) and

$O(Q)$, respectively. Then P is a principal fiber bundle over M with structure group $O(n)\times O(\ell)$ and projection $\tilde{\pi} = \pi_{O(M)}\circ\pi_1$, and E is associated with P. Since M is compact, so is P. As usual, we regard each $p \in P$ as a diffeomorphism of the standard fiber of P onto E_x or onto E_x^*, where $x = \tilde{\pi}(p)$.

Let γ_1 [resp. γ_2] be the connection form on $O(M)$ [resp. $O(Q)$]. Define a connection form γ and an R^n-valued 1-form θ on P, respectively, by

$$\gamma = \pi_1^*\gamma_1 \oplus \pi_2^*\gamma_2 : TP \longrightarrow \mathfrak{o}(n)\oplus\mathfrak{o}(\ell) \text{ (Lie algebra)},$$

$$\theta(Y) = p_1^{-1}(\tilde{\pi}_*Y) \in R^n, \qquad Y \in T_pP, \quad p \in P, \quad p_1 = \pi_1(p),$$

where $\pi_i^*\gamma_i = \gamma_i\circ(\pi_i)_*$, $i=1,2$. Then for each $\xi \in R^n$ there exists uniquely a C^∞ vector field $\tilde{B}(\xi)$ on P such that $\gamma(\tilde{B}(\xi)) = 0$, $\theta(\tilde{B}(\xi)) = \xi$. Thus $\tilde{B}(\xi)_p \in T_pP$, $p \in P$, is the holizontal lift of $p_1\xi \in T_{\tilde{\pi}(p)}M$ with $p_1 = \pi_{O(M)}(p)$. For $\xi \in {}^*R^n$, we write $\tilde{B}(\xi)$ instead of ${}^*\tilde{B}(\xi)$.

Consider, for each $\omega \in \Omega$ (see §1), the following ordinary differential equation on ${}^*P$:

$$\frac{dp_t}{dt} = \tilde{B}\left(\frac{dw(t,\omega)}{dt}\right)_{p_t}. \tag{4.2}$$

For $t \in {}^*[0,1]$, let $p_t(p, \omega)$ be the solution of (4.2) starting from $p \in {}^*P$ at time $t = 0$. Set, for $p \in P$ and

$t \in [0,1]$,

$$X_t(p, \omega) := {}^\circ(\tilde{\pi}(p_t(p, \omega))) = \tilde{\pi}({}^\circ p_t(p, \omega)).$$

(Since both of P and M are compact, $p_t(t, \omega)$ and $\tilde{\pi}(p_t(p, \omega))$ are both near-standard.) Henceforth we omit ω. For $t \in (0,1)$, define $\tau_t = {}^\circ p_t(p) \circ p^{-1} : E_x = \pi_E^{-1}(x) \longrightarrow E_{X_t(p)}$ and $\tau_t' = {}^\circ p_t \circ p^{-1} : E_x^* \longrightarrow E_{X_t(p)}^*$, where $x = \tilde{\pi}(p)$, $p \in P$.

We see that the stochastic process $(X_t(p))_{t\in[0,1]}$ is a Brownian motion (starting from $\tilde{\pi}(p)$ at time $t=0$) on M, and τ_t [resp. τ_t'] defines the stochastic parallel displacement of fibers of E [resp. E^*] along the Brownian curve $(X_.(p))$. Then we obtain the following lemma (cf. [2]).

<u>**Lemma 4.2**</u>. The solution of (4.1) is given by

$$\sigma(t,x) = \mathbb{E}[\tau_t^{-1}\sigma_0(X_t(p))], \qquad p\in\tilde{\pi}^{-1}(x),\ x\in M,\ t\in(0,1). \quad (4.3)$$

(Here $\mathbb{E}[\tau_t^{-1}\sigma_0(X_t(p))]$ does not depend on the choice of $p \in \tilde{\pi}^{-1}(x)$ and thus (4.3) is well-defined.)

We note that the map $C^\infty(E) \ni \sigma_0 \longmapsto \mathbb{E}[\tau_t^{-1}\sigma_0(X_t(p))] \in E_x$ is in $\mathfrak{A}$ (see §1). Using $X_t(p)$, we rewrite Theorem 3.1 as follows:

<u>**Theorem 4.3**</u> . The heat kernel $e_0(t,x,y)$ with respect to

dv_g for the equation (2.3) is given by

$$e_0(t,x,y) = {}^{\circ}\mathbb{E}[\hat{\delta}_{dv_g}(X_t(p),y)],$$

$$t \in (0,1),\ x, y \in M,\ p \in P,\ \tilde{\pi}(p) = x.$$

We will now apply the previous results to the problem of getting a nonstandard representation of the heat kernel with respect to dv_g for the equation (4.1).

<u>**Theorem 4.4**</u>. The heat kernel with respect to dv_g for the heat equation (4.1) is given by

$$e(t,x,y) = {}^{\circ}\mathbb{E}[\hat{\delta}_{dv_g}(X_t(p),y)]\cdot\mathbb{E}[\tau_t^{-1}|X_t(p)=y]$$

$$(= e_0(t,x,y)\mathbb{E}[\tau_t^{-1}|X_t(p)=y]),$$

$$t \in (0,1),\ x, y \in M,\ p \in P,\quad \tilde{\pi}(p) = x,$$

where the map $\mathbb{E}[\tau_t^{-1}|X_t(p)=y] \in \mathrm{Hom}(E_y,E_x)$ is defined in such a way that

$$\mathbb{E}[\tau_t^{-1}|X_t(p)=y]\sigma_0(y) = \mathbb{E}[\tau_t^{-1}\sigma_0(y)|X_t(p)=y]$$

$$= \mathbb{E}[\tau_t^{-1}\sigma_0(X_t(p))|X_t(p)=y] \in E_x\ ,\quad \sigma_0 \in C^\infty(E),$$

and thus $e(t,x,y) \in \mathrm{Hom}(E_y,E_x)$ for each $t \in (0,1)$.
(Note that

$$°\mathbb{E}[\hat{\delta}(X_t(p),y)] = e_0(t,x,y),$$

$$\mathbb{E}[\tau_t^{-1}|X_t(p)=y] \in \mathrm{Hom}(E_y,E_x),\ \sigma_0(y) \in E_y ,$$

so

$$°\mathbb{E}[\hat{\delta}(X_t(p),y)]\mathbb{E}[\tau_t^{-1}|X_t(p)=y] \in \mathrm{Hom}(E_y,E_x),$$

$$°\mathbb{E}[\hat{\delta}(X_t(p),y)]\mathbb{E}[\tau_t^{-1}|X_t(p)=y]\sigma_0(y) \in E_x ,$$

$$\int_M °\mathbb{E}[\hat{\delta}(X_t(p),y)]\mathbb{E}[\tau_t^{-1}|X_t(p)=y]\sigma_0(y)\ dv_g(y) \in E_x.)$$

Proof. Using the fiber-wise norms defined by the induced metrics in E and E^*, we have

$$\mathbb{E}[|\tau_t'\xi\square\sigma_0(X_t(p))|] \le \mathbb{E}[|\tau_t'\xi|_{E^*_{X_t(p)}}\cdot|\sigma_0(X_t(p))|_{E_{X_t(p)}}]$$

$$\le \max_{y\in M}|\sigma_0(y)|_{E_y}\cdot\mathbb{E}[|\tau_t'\xi|_{E^*_{X_t(p)}}] = \max_{y\in M}|\sigma_0(y)|_{E_y}\cdot|\xi|_{E^*_x} < \infty$$

for $\xi \in E^*_x$, $p \in \tilde{\pi}^{-1}(x)$, $x \in M$, $t \in (0,1)$, $\sigma_0 \in C^\infty(E)$. Fixing t, x and p, we regard $X_t(p)$, $\tau_t'\xi$ and dv_g, respectively, as F, η and ρ. Apply Lemma 4.1. Then we have

$$\xi\square\sigma(t,x) = \mathbb{E}[(\tau_t'\xi)\square\sigma_0(X_t(p))] \qquad \text{[by (4.3)]}$$

$$\approx \int \mathbb{E}[\hat{\delta}_{dv_g}(X_t(p),\cdot)]\mathbb{E}[(\tau_t^{-}\xi)\square\sigma_0(X_t(p))|X_t(p)=\cdot]\ dv_g$$

$$= \int \mathbb{E}[\hat{\delta}_{dv_g}(X_t(p),\cdot)](\xi\square\mathbb{E}[\tau_t^{-1}\sigma_0(X_t(p)|X_t(p)=\cdot])\ dv_g$$

$$= \xi\square\left(\int \mathbb{E}[\hat{\delta}_{dv_g}(X_t(p),\cdot)]\mathbb{E}[\tau_t^{-1}|X_t(p)=\cdot]\sigma_0\ dv_g\right).$$

Thus we obtain

$$\sigma(t,x) = \int_M {}^{\circ}\mathbb{E}[\hat{\delta}_{dv_g}(X_t(p),y)]\mathbb{E}[\tau_t^{-1}|X_t(p)=y]\sigma_0(y)\ dv_g(y).$$

This proves the theorem.

For further applications of nonstandard analysis, see [2].

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