数理解析研究所講究録
第 703 巻 1989 年 34-45

Regularisation in 3D BIE for Anisotropic Elastodynamic Crack Problems

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1. Introduction

Let Γ be a smooth piece of curved surface in R^3, having a smooth edge $\partial\Gamma$. The elastodynamic crack problem is formulated as follows: Find functions $u_i(\mathbf{x})$ and $\tau_{ij}(\mathbf{x})$ which satisfy the field equations

$$(1) \qquad \begin{cases} \tau_{ij,j} + \rho\omega^2 u_i = 0 \\ \frac{1}{2}(u_{i,j} + u_{j,i}) = D_{ijkl}\tau_{kl} \end{cases} \quad \text{in } R^3 \setminus \bar{\Gamma}$$

boundary condition

$$\tau_{ij}^{\pm} n_j = t_i \quad \text{on } \Gamma$$

regularity condition

$$[u_i] = 0 \quad \text{on } \partial\Gamma$$

and the radiation condition, where D_{ijkl} is a positive constant tensor which satisfies

$$D_{ijkl} = D_{jikl} = D_{klij}$$

ρ and ω are positive constants, and t_i is a function given on Γ. Also, n_i stands for the unit normal vector to Γ, superposed + and −, respectively, indicate the limit from the side of Γ into which $\mathbf{n}$ points and the limit from the other side, $_{,i} = \partial/\partial x_i$, and

$$[u_i] = u_i^+ - u_i^- .$$

In physical terms $u_i, \tau_{ij}, \rho, \omega$ and $\mathbf{D}$ represent the displacement, stress, density, frequency and elastic compliance, respectively.

The double layer potential approach for this problem uses an 'integral' equation

$$t_i(\mathbf{x}) = \text{p.f.} \int_\Gamma \Sigma_{ijkl}(\mathbf{x}-\mathbf{y}) n_j(\mathbf{x}) n_l(\mathbf{y}) f_k(\mathbf{y}) \mathrm{d}S_y, \quad \mathbf{x} \in \Gamma$$

where $f_i (= [u_i])$ is the unknown vector function on Γ, and Σ is a kernel function which satisfies

$$\text{(2)} \qquad \Sigma_{ikab,kj}(\mathbf{x}) + \Sigma_{jkab,ki}(\mathbf{x}) + 2\rho\omega^2 D_{ijkl}\Sigma_{klab}(\mathbf{x}) = -\rho\omega^2(\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja})\delta(\mathbf{x})$$

with Dirac's delta $\delta(\mathbf{x})$. With $\mathbf{f}$, one computes τ_{ij} by

$$\tau_{ij}(\mathbf{x}) = \int_\Gamma \Sigma_{ijkl}(\mathbf{x}-\mathbf{y}) n_k(\mathbf{x}) f_l(\mathbf{y}) \mathrm{d}S_y$$

and u_i by using (1).

A difficulty inherent to the numerical analysis based on this approach is the strong singularity of $\Sigma(\mathbf{x})$, which is of the order of $|\mathbf{x}|^{-3}$ as $|\mathbf{x}| \to 0$. This singularity is usually removed with the help of the "regularisation", or integration by parts in other words[1][2]. In [1] Nishimura & Kobayashi have shown that this regularisation is carried out in an automatic manner, once one finds a decomposition of the form

$$\text{(3)} \qquad \Sigma_{ijkl}(\mathbf{x}) = (\text{curl}\,)_i(\text{curl}\,)_j(\text{curl}\,)_k(\text{curl}\,)_l \Phi_{....}(\mathbf{x}) + \Psi_{ijkl}(\mathbf{x})$$

where Φ and Ψ are kernels which behave essentially as $O(|\mathbf{x}|)$ and $O(|\mathbf{x}|^{-1})$ as $|\mathbf{x}| \to 0$, respectively. Φ is called the stress function for Σ.

In this note we shall derive explicit formulae for Φ and Ψ in the general case of anisotropic elastodynamics. Also, we shall discuss the relation between Nédélec's regularisation technique and the present formulation.

2. Notation and Preliminaries

(a) Fundamental Solution

We now introduce the following notation:

$$\begin{array}{lll} \tau_{11} \to T_1 & \tau_{22} \to T_2 & \tau_{33} \to T_3 \\ \tau_{23} \to T_4 & \tau_{31} \to T_5 & \tau_{12} \to T_6 \end{array}$$

$$\begin{array}{cccc} D_{1111} \to D_{11} & D_{1122} \to D_{12} & 2D_{2311} \to D_{41} & 2D_{3111} \to D_{51} \\ 4D_{2323} \to D_{44} & \cdots & 4D_{1212} \to D_{66} & \text{etc.} \end{array}$$

$$\begin{array}{cccc} \Sigma_{1111} \to \Sigma_{11} & \Sigma_{1122} \to \Sigma_{12} & \Sigma_{2311} \to \Sigma_{41} & \Sigma_{3111} \to \Sigma_{51} \\ \Sigma_{2323} \to \Sigma_{44} & \cdots & \Sigma_{1212} \to \Sigma_{66} & \text{etc.} \end{array}$$

With this convention (2) is easily seen to transform into

$$(4) \qquad \left\{ \begin{pmatrix} \partial_1^2 & & & & \partial_1\partial_3 & \partial_1\partial_2 \\ & \partial_2^2 & & \partial_2\partial_3 & & \partial_1\partial_2 \\ & & \partial_3^2 & \partial_2\partial_3 & \partial_1\partial_3 & \\ & \partial_2\partial_3 & \partial_2\partial_3 & \partial_2^2+\partial_3^2 & \partial_1\partial_2 & \partial_1\partial_3 \\ \partial_1\partial_3 & & \partial_1\partial_3 & \partial_1\partial_2 & \partial_1^2+\partial_3^2 & \partial_2\partial_3 \\ \partial_1\partial_2 & \partial_1\partial_2 & & \partial_1\partial_3 & \partial_2\partial_3 & \partial_1^2+\partial_2^2 \end{pmatrix} + \rho\omega^2\mathbf{D} \right\} \Sigma = -\rho\omega^2\mathbf{1}\delta.$$

The F.T. of (4) is written as

$$(\mathbf{K} - \rho\omega^2\mathbf{D})\hat{\Sigma} = \rho\omega^2\mathbf{1},$$

where ^ indicates the F.T. with respect to $\mathbf{x}$ $(\mathbf{x} \to \xi)$ and $\mathbf{K}$ is the matrix obtained by replacing ∂_i in the first matrix in (4) by the Fourier parameter ξ_i. Obviously one has

$$(5) \qquad \hat{\Sigma} = (\mathbf{K} - \rho\omega^2\mathbf{D})^{-1}\rho\omega^2 = \frac{\{\mathrm{cof}(\mathbf{K} - \rho\omega^2\mathbf{D})\}^T}{\det(\mathbf{K} - \rho\omega^2\mathbf{D})}\rho\omega^2.$$

(b) Some Matrices

In statics where $\omega = 0$, τ has a stress function representation given by

$$\tau_{ij} = e_{imk}e_{jnl}\xi_m\xi_n\phi_{kl},$$

where ϕ is the stress function. This relation is transformed into

$$\begin{pmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{pmatrix} = \begin{pmatrix} & \xi_3^2 & \xi_2^2 & -2\xi_2\xi_3 & & \\ \xi_3^2 & & \xi_1^2 & & -2\xi_3\xi_1 & \\ \xi_2^2 & \xi_1^2 & & & & -2\xi_1\xi_2 \\ -\xi_2\xi_3 & & & -\xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ & -\xi_3\xi_1 & & \xi_1\xi_2 & -\xi_2^2 & \xi_2\xi_3 \\ & & -\xi_1\xi_2 & \xi_1\xi_3 & \xi_2\xi_3 & -\xi_3^2 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \\ \phi_5 \\ \phi_6 \end{pmatrix},$$

namely,

$$\mathbf{T} = \mathbf{B}(\mathbf{1} \quad \mathbf{C})\phi$$

in the matrix form, where

$$\mathbf{B} = \begin{pmatrix} & \xi_3^2 & \xi_2^2 \\ \xi_3^2 & & \xi_1^2 \\ \xi_2^2 & \xi_1^2 & \\ -\xi_2\xi_3 & & \\ & -\xi_3\xi_1 & \\ & & -\xi_1\xi_2 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} \frac{\xi_1^2}{\xi_2\xi_3} & -\frac{\xi_1}{\xi_3} & -\frac{\xi_1}{\xi_2} \\ -\frac{\xi_2}{\xi_3} & \frac{\xi_2^2}{\xi_1\xi_3} & -\frac{\xi_2}{\xi_1} \\ -\frac{\xi_3}{\xi_2} & -\frac{\xi_3}{\xi_1} & \frac{\xi_3^2}{\xi_1\xi_2} \end{pmatrix}.$$

A direct calculation shows that

(6) $$\mathbf{KB} = \mathbf{0}$$

holds. As a matter of fact, $\mathbf{K}$ is of rank 3, and the 3 columns of $\mathbf{B}$ span $ker\,\mathbf{K}$.

We now introduce

(7) $$\mathbf{F} = (\mathbf{B} \quad \mathbf{A})^T,$$

where $\mathbf{A}$ is an arbitrary (6×3) matrix s.t.

(8) $$\det \mathbf{F} = 1.$$

We then have the following results:

(9) • $\mathbf{K}_o := \mathbf{F}\mathbf{K}\mathbf{F}^T = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{K}}_o \end{pmatrix},$

where $\bar{\mathbf{K}}_o$ is a (3×3) matrix. $\bar{\mathbf{K}}_o$ satisfies

$$\det \bar{\mathbf{K}}_o = \frac{1}{\xi_1^2 \xi_2^2 \xi_3^2}. \tag{10}$$

Proof

We use (6) and (7) to have

$$\mathbf{FKF}^T = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}^T\mathbf{KA} \end{pmatrix},$$

which means $\bar{\mathbf{K}}_o = \mathbf{A}^T\mathbf{KA}$.

Let $\mathbf{b}_i (i = 1 \sim 3)$ be a set of orthonormal base vectors for $ker\, \mathbf{K}$. Also, let $\mathbf{a}_i\, (i = 1 \sim 3)$ be such that $(\mathbf{b}_i, \mathbf{a}_j)$ forms a system of orthonormal base vectors for R^6. Then $\mathbf{B}$ and $\mathbf{A}$ are written as

$$\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)\bar{\mathbf{B}}, \quad \mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)\bar{\mathbf{A}} + (\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)\bar{\mathbf{B}}',$$

where $\bar{\mathbf{B}}, \bar{\mathbf{B}}'$ and $\bar{\mathbf{A}}$ are (3×3) matrices. Also, we have from (6)~(8)

$$\mathbf{K} = \sum_i \kappa_i \mathbf{a}_i \otimes \mathbf{a}_i, \quad 1 = \det \mathbf{F} = \left| \begin{pmatrix} \bar{\mathbf{B}} & \bar{\mathbf{B}}' \\ \mathbf{0} & \bar{\mathbf{A}} \end{pmatrix} \right| = \det \bar{\mathbf{B}} \det \bar{\mathbf{A}},$$

$$\mathbf{A}^T\mathbf{KA} = \bar{\mathbf{A}}^T \begin{pmatrix} \kappa_1 & & \\ & \kappa_2 & \\ & & \kappa_3 \end{pmatrix} \bar{\mathbf{A}},$$

which imply

$$\det \mathbf{A}^T\mathbf{KA} = \frac{1}{(\det \bar{\mathbf{B}})^2} \kappa_1 \kappa_2 \kappa_3.$$

This result shows that the value of $\det \mathbf{A}^T\mathbf{KA}$ is independent of the choice of $\mathbf{A}$. Hence we may put

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} \\ \dfrac{\mathbf{1}}{(2\xi_1^2 \xi_2^2 \xi_3^2)^{1/3}} \end{pmatrix},$$

for example. This choice gives

$$\det \mathbf{A}^T\mathbf{K}\mathbf{A} = \frac{1}{\xi_1^2\xi_2^2\xi_3^2}. \qquad \square$$

3. Computation of $\hat{\Sigma}$

We shall compute Σ in several steps.

(a) Computation of $\det(\mathbf{K} - \rho\omega^2\mathbf{D})$

- $\det(\mathbf{K} - \rho\omega^2\mathbf{D}) = \sum_{i=1}^{4} d_i(\rho\omega^2)^{i+2}$, where d_i are polynomials of ξ.

<u>Proof</u>

It is clear from the definition that this determinant is a 6th order polynomial of $\rho\omega^2$ whose coefficients are polynomials of ξ. Hence it is sufficient to show that the coefficients of the 0th $\sim$ 2nd powers of $\rho\omega^2$ vanish. But one immediately shows this from the following calculation:

$$\begin{aligned}
\det(\mathbf{K} - \rho\omega^2\mathbf{D}) &= \det(\rho\omega^2\mathbf{F}\mathbf{D}\mathbf{F}^T - \mathbf{K}_o) \\
&= \det\left(\rho\omega^2\mathbf{F}\mathbf{D}\mathbf{F}^T - \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{K}}_o \end{pmatrix}\right) \\
&= (\rho\omega^2)^3 \det\left(\mathbf{F}\mathbf{D}\mathbf{F}^T_{\downarrow^1_3, 1\to 3}\right) \det \bar{\mathbf{K}}_o + O\left((\rho\omega^2)^4\right) \\
&= \frac{(\rho\omega^2)^3 \det(\mathbf{B}^T\mathbf{D}\mathbf{B})}{\xi_1^2\xi_2^2\xi_3^2} + \cdots,
\end{aligned}$$

where we have used (7)$\sim$(10). This calculation also shows

$$d_1 = \frac{\det(\mathbf{B}^T\mathbf{D}\mathbf{B})}{\xi_1^2\xi_2^2\xi_3^2}. \qquad \square \tag{11}$$

- $d_1 \neq 0$.

<u>Proof</u>

Suppose $\det(\mathbf{B}^T\mathbf{D}\mathbf{B}) = 0$ (see (11)). This means that there exists a nonzero vector $\mathbf{a}$ s.t.

$$\mathbf{a}^T\mathbf{B}^T\mathbf{D}\mathbf{B}\mathbf{a} = \mathbf{0}.$$

But this implies $\mathbf{Ba} = \mathbf{0}$ since $\mathbf{D}$ is positive. Hence the definition of $\mathbf{B}$ gives $\mathbf{a} = \mathbf{0}$, which is a contradiction. □

Finally we note that d_i is a polynomial (of ξ) of degree $8 - 2i$.

(b) Computation of $\operatorname{cof}(\mathbf{K} - \rho\omega^2\mathbf{D})$

(12) • $\operatorname{cof}(\mathbf{K} - \rho\omega^2\mathbf{D})^T = \mathbf{F}^T\operatorname{cof}(\mathbf{K}_o - \rho\omega^2\mathbf{F}\mathbf{D}\mathbf{F}^T)^T\mathbf{F}$.

<u>Proof</u>

Since

$$\begin{aligned}(\mathbf{K} - \rho\omega^2\mathbf{D})^{-1} &= \{\mathbf{F}^{-1}\mathbf{F}(\mathbf{K} - \rho\omega^2)\mathbf{F}^T\mathbf{F}^{-1}\}^{-1} \\ &= \mathbf{F}^T(\mathbf{K}_o - \rho\omega^2\mathbf{F}\mathbf{D}\mathbf{F}^T)^{-1}\mathbf{F},\end{aligned}$$

we divide the both sides of the above equation by

$$\det(\mathbf{K} - \rho\omega^2\mathbf{D}) = \det(\mathbf{K}_o - \rho\omega^2\mathbf{F}\mathbf{D}\mathbf{F}^T)$$

to obtain (12). □

• $\operatorname{cof}(\mathbf{K} - \rho\omega^2\mathbf{D}) = \sum_{i=1}^{4}(\rho\omega^2)^{i+1}\mathbf{S}_i$, where $\mathbf{S}_i$ is a matrix whose components are polynomials of ξ.

<u>Proof</u>

Since

$$\begin{aligned}\operatorname{cof}(\mathbf{K}_o - \rho\omega^2\mathbf{F}\mathbf{D}\mathbf{F}^T) &= \operatorname{cof}\left[\begin{pmatrix}0 & 0\\ 0 & \bar{\mathbf{K}}_o\end{pmatrix} - \rho\omega^2\mathbf{F}\mathbf{D}\mathbf{F}^T\right] \\ &= (\rho\omega^2)^2\begin{pmatrix}\operatorname{cof}(\mathbf{B}^T\mathbf{D}\mathbf{B})\det\bar{\mathbf{K}}_o & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix} + O((\rho\omega^2)^3),\end{aligned}$$

we use (12) to obtain the required result. □

The explicit expression for $\mathbf{S}_1$ is obtained without difficulty. Indeed, we have

$$\mathbf{S}_1 = \mathbf{F}^T \begin{pmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{F} = (\mathbf{B} \quad \mathbf{A}) \begin{pmatrix} \mathbf{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{B}^T \\ \mathbf{A}^T \end{pmatrix} = \mathbf{B}\mathbf{S}\mathbf{B}^T, \tag{13}$$

where

$$\mathbf{S} = \frac{\operatorname{cof}(\mathbf{B}^T\mathbf{D}\mathbf{B})}{\xi_1^2\xi_2^2\xi_3^2}. \tag{14}$$

Since $\mathbf{S}$ is written explicitly as

$$S_{ij} = \frac{1}{2\xi_1^2\xi_2^2\xi_3^2} e_{ipq}e_{jrs}B_{Ap}D_{AB}B_{Br}B_{Cq}D_{CD}B_{Ds},$$

we use (13) and (14) to have

$$(\mathbf{S}_1)_{IJ} = \frac{1}{2}\frac{e_{ipq}B_{Ii}B_{Ap}B_{Cq}}{\xi_1\xi_2\xi_3}D_{AB}D_{CD}\frac{e_{jrs}B_{Jj}B_{Br}B_{Ds}}{\xi_1\xi_2\xi_3}. \tag{15}$$

Finally we note that $\mathbf{S}_i$ is a polynomial (of ξ) of degree $8 - 2i$.

(c) Stress Function

From (15) and the "quotient law" one expects that

$$\frac{B_{Ii}B_{Jj}B_{Kk}e_{ijk}}{\xi_1\xi_2\xi_3}$$

is a tensor of the 6th order. Indeed, an "experiment" shows that the $(ij) \to I, (st) \to J, (mn) \to K$ component of the above expression is given as follows:

$$\frac{1}{4}e_{ipk}e_{jql}\xi_p\xi_q[(\delta_{ks}\delta_{lm} + \delta_{km}\delta_{ls})e_{tun} + (\delta_{kt}\delta_{lm} + \delta_{km}\delta_{lt})e_{sun}$$
$$+ (\delta_{ks}\delta_{ln} + \delta_{kn}\delta_{ls})e_{tum} + (\delta_{kt}\delta_{ln} + \delta_{kn}\delta_{lt})e_{sum}]\xi_u$$

Therefore the general expression for the F.T. of the stress function (see (3)) is

$$
(16)\quad
\begin{aligned}
\hat{\Phi}_{klij} &= \frac{(\delta_{ks}\delta_{lm}+\delta_{km}\delta_{ls})(\delta_{ia}\delta_{jc}+\delta_{ic}\delta_{ja})e_{tun}e_{bvd}\xi_u\xi_v D_{stab}D_{mncd}}{2[\det(\mathbf{K}-\rho\omega^2\mathbf{D})/(\rho\omega^2)^3]} \\
&= \frac{(D_{ktib}D_{lnjd}+D_{ltib}D_{knjd}+D_{ktjb}D_{lnid}+D_{ltjb}D_{knid})e_{tun}e_{bvd}\xi_u\xi_v}{2[\det(\mathbf{K}-\rho\omega^2\mathbf{D})/(\rho\omega^2)^3]}.
\end{aligned}
$$

Example : Isotropy. In this case the compliance tensor $\mathbf{D}$ is given in terms of the Lamé's constants (λ, μ) as

$$D_{ijkl}=\frac{1}{4\mu}\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}-\frac{2\lambda}{3\lambda+2\mu}\delta_{ij}\delta_{kl}\right).$$

This gives

$$
\begin{aligned}
(4\mu)^2 D_{stab}D_{mncd}e_{tun}e_{bvd}\xi_u\xi_v &= e_{tun}\xi_u e_{bvd}\xi_v \\
\times &\left(\delta_{sa}\delta_{tb}+\delta_{sb}\delta_{ta}-\frac{2\lambda}{3\lambda+2\mu}\delta_{st}\delta_{ab}\right)\left(\delta_{mc}\delta_{nd}+\delta_{md}\delta_{nc}-\frac{2\lambda}{3\lambda+2\mu}\delta_{mn}\delta_{cd}\right) \\
= &\Big[2\delta_{sa}\delta_{mc}\delta_{nv}+\delta_{sa}e_{buc}e_{bvm}-\frac{2\lambda}{3\lambda+2\mu}\delta_{sa}e_{bum}e_{bvc} \\
&+e_{aud}e_{svd}\delta_{mc}+e_{auc}e_{svm}-\frac{2\lambda}{3\lambda+2\mu}e_{aum}e_{svc} \\
&-\frac{2\lambda}{3\lambda+2\mu}e_{sud}e_{avd}\delta_{mc}-\frac{2\lambda}{3\lambda+2\mu}e_{suc}e_{avm}+\left(\frac{2\lambda}{3\lambda+2\mu}\right)^2 e_{sum}e_{avc}\Big]\xi_u\xi_v \\
\sim &\,|\xi|^2\Big[2\delta_{sa}\delta_{mc}+\delta_{sa}\delta_{cm}-\frac{2\lambda}{3\lambda+2\mu}\delta_{sa}\delta_{cm}+\delta_{as}\delta_{mc} \\
&+(\delta_{as}\delta_{cm}-\delta_{am}\delta_{cs})-\frac{2\lambda}{3\lambda+2\mu}(\delta_{as}\delta_{mc}-\delta_{ac}\delta_{ms})-\frac{2\lambda}{3\lambda+2\mu}\delta_{sa}\delta_{mc} \\
&-\frac{2\lambda}{3\lambda+2\mu}(\delta_{sa}\delta_{mc}-\delta_{sm}\delta_{ca})+\left(\frac{2\lambda}{3\lambda+2\mu}\right)^2(\delta_{sa}\delta_{mc}-\delta_{sc}\delta_{ma})\Big],
\end{aligned}
$$

where $\sim$ indicates an equality modulo terms proportional to either ξ_s or ξ_a or ξ_m or ξ_c. The symmetrisation (δ terms) in (16) transforms the δ terms in the above formula into

$$\delta_{sa}\delta_{mc}\to 2(\delta_{ki}\delta_{lj}+\delta_{kj}\delta_{il})$$

$$\delta_{sc}\delta_{ma}\to 2(\delta_{ki}\delta_{lj}+\delta_{kj}\delta_{il})$$

$$\delta_{sm}\delta_{ca}\to 4\delta_{kl}\delta_{ij}.$$

Hence the stress function for this case is

$$\hat{\Phi}_{klij} = \frac{|\xi|^2}{4\mu^2(3\lambda+2\mu)} \frac{(\lambda+2\mu)(\delta_{ki}\delta_{lj}+\delta_{kj}\delta_{li})+2\lambda\delta_{kl}\delta_{ij}}{\det(\mathbf{K}-\rho\omega^2\mathbf{D})/(\rho\omega^2)^3}.$$

4. Remarks

1 It is not difficult to evaluate d_1 in terms of tensor components. Indeed,

$$\begin{aligned}\frac{\det \mathbf{B}^T\mathbf{D}\mathbf{B}}{\xi_1^2\xi_2^2\xi_3^2} &= \frac{1}{6}\frac{e_{ikm}B_{Ii}B_{Kk}B_{Mm}}{\xi_1\xi_2\xi_3}D_{IJ}D_{KL}D_{MN}\frac{e_{jln}B_{Jj}B_{Ll}B_{Nn}}{\xi_1\xi_2\xi_3}\\ &= \frac{2}{3}e_{ipc}e_{aqm}e_{kre}e_{jsd}e_{btn}e_{luf}\xi_p\xi_q\xi_r\xi_s\xi_t\xi_u D_{iajb}D_{kcld}D_{menf}.\end{aligned}$$

2 It is noted that the present formulation transforms the "cofactor" in (5) only. In addition the stress function is given in a form of

$$\frac{\text{polynomials in } \xi}{\det(\mathbf{K}-\rho\omega^2\mathbf{D})/(\rho\omega^2)^3}.$$

Hence this process does not introduce anything artificial to the final results in that the functions Φ and Ψ maintain the correct causality in the time domain.

3 In general the regularisation process goes as follows: i) Write

$$\hat{\Sigma}_{ijkl} = \frac{e_{ipa}e_{jqb}e_{krc}e_{lsd}\varphi_{abcd}\xi_p\xi_q\xi_r\xi_s + \rho\omega^2\psi_{ijkl}}{\det(\mathbf{K}-\rho\omega^2\mathbf{D})/(\rho\omega^2)^3} \tag{17}$$

where φ is the 'stress function' part of the cofactor. Notice that φ and ψ are polynomials in ξ. ii) Compute the Fourier inversions given by

$$\Phi := F^{-1}\left(\frac{\varphi}{\det(\mathbf{K}-\rho\omega^2\mathbf{D})/(\rho\omega^2)^3}\right), \quad \Psi := F^{-1}\left(\frac{\psi}{\det(\mathbf{K}-\rho\omega^2\mathbf{D})/(\rho\omega^2)^3}\right),$$

and use the regularisation techniques proposed elsewhere[1].

4 Nédélec's technique is interpreted as follows: One uses an identity

$$|\xi|^2\delta_{ij} = -e_{ipq}e_{qrj}\xi_p\xi_q + \xi_i\xi_j$$

to have

(18)

$$\hat{\Sigma}_{ijkl} = \frac{1}{|\xi|^4}e_{iPQ}\xi_P e_{lAB}\xi_A \underline{\underline{e_{QRS}\xi_R e_{BCD}\xi_C\hat{\Sigma}_{SjkD}}}$$
$$-\frac{1}{|\xi|^4}\left(\underline{e_{iPQ}\xi_P e_{QRS}\xi_R\xi_l\hat{\Sigma}_{ShjD}\xi_D + e_{lAB}\xi_A e_{BCD}\xi_C\xi_i\hat{\Sigma}_{SjkD}\xi_S - \xi_i\xi_l\hat{\Sigma}_{SjkD}\xi_S\xi_D}\right)$$

Since $\hat{\Sigma}_{ijkl}\xi_l \sim O(1/|\xi|)$ as one shows from (17), the expression in the $(\cdots)$ in (18) gives an integrable kernel. In order to show that the $1/|\xi|^4$ does not destroy the correct causality in the time domain, however, one would have to show that the underlined parts in (18)$\times$ det$(\mathbf{K}-\rho\omega^2\mathbf{D})$ could be factored out by $|\xi|^4$. Unfortunately, this is not always the case. To see this we use (17) ane (18) to have

$$\hat{\Sigma}_{ijkl} = \frac{1}{|\xi|^4}\frac{e_{iPQ}\xi_P e_{lAB}\xi_A\{|\xi|^4 e_{jqb}e_{krc}\xi_q\xi_r\varphi_{QbcA} + \rho\omega^2 e_{QRS}\xi_R e_{BCD}\xi_C\psi_{SjkD}\}}{\det(\mathbf{K}-\rho\omega^2\mathbf{D})/(\rho\omega^2)^3}$$
$$-\frac{\rho\omega^2}{|\xi|^4}\frac{e_{iPQ}\xi_P e_{QRS}\xi_R\xi_l\psi_{SjkD}\xi_D + e_{lAB}\xi_A e_{BCD}\xi_C\xi_i\psi_{SjkD}\xi_S - \xi_i\xi_l\psi_{SjkD}\xi_S\xi_D}{\det(\mathbf{K}-\rho\omega^2\mathbf{D})/(\rho\omega^2)^3}$$

This shows that it is impossible to eliminate the $1/|\xi|^4$ factor except in the static case. A possible remedy for this artificialty is to use Nédélec's technique to the Φ term only. This method will give exactly the same result as does the technique mentioned in 3 †. When one is interested only in a time harmonic analysis for a particular ω, however, the artificialty of the original Nédélec formulation may not cause numerical problems. In addition, the original Nédélec formulation works in statics regardless of the material symmetry.

† Notice, however, that the present proof that "φ is a polynomial" is necessary to claim that the modified Nédélec formulation is free of artificialty.

References

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