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Boundary Value Theory for Pseudo-Differential Operators

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In this note we introduce some fundamental properties of boundary values of holomorphic pseudo-differential operators. This work was motivated by the recent works of Bronštein concerning his new method of constructing fundamental solutions to weakly hyperbolic operators. Further, this is also deeply connected with the recent works of Uchikoshi.

Let X be a complex manifold ($=\mathbf{C}^n \ni z$), and T*X be its cotangent bundle ($=\mathbf{C}^n \times \mathbf{C}^n \ni (z;\zeta)$). For a point $p_0=(z_0;\zeta_0 dz) \in T^*X$, we call a domain of the following type "a conic neighborhood of p_0" :

$$V_r = \{(z;\zeta dz) \in \mathbf{C}^n \times \mathbf{C}^n;\ |z-z_0|<r,\ |\zeta/|\zeta|-\zeta_0/|\zeta_0||<r,\ |\zeta|>r^{-1}\}$$

if $\zeta_0 \neq 0$, where r is a small positive number. Then the stalk of our pseudo-differential operators at p_0 is defined as follows :

$$\mathcal{E}^{\mathbf{R}} = \varinjlim_{r\to+0} [\{P(z,\zeta) \in \mathcal{O}(V_r);\ \forall \varepsilon>0,\ \exists C_\varepsilon>0,\ |P(z,\zeta)| \leq C_\varepsilon e^{\varepsilon|\zeta|} \text{ on } V_r\}$$
$$/ \{P(z,\zeta) \in \mathcal{O}(V_r);\ \exists \delta>0,\ \exists C>0,\ |P(z,\zeta)| \leq Ce^{-\delta|\zeta|} \text{ on } V_r\}]$$

It is clear that $\{\mathcal{E}^{\mathbf{R}}|_*\}$ constitute a sheaf $\mathcal{E}^{\mathbf{R}}$ on $S^*X=(T^*X\setminus X)/\mathbf{R}_+$. Further it is well-known that this is a sheaf of rings. In fact, if we write the equivalence class of $P(z,\zeta)$ as

$$P(z,D_z) \quad (D_z=(\partial_{z_1},\ldots,\partial_{z_n})) \text{ or } :P(z,\zeta):,$$

then the defining function of the composition $P(z,D_z)Q(z,D_z)$ has an asymptotic expansion

$$\sum_{J\geq 0,\, J\in\mathbf{Z}^n} \frac{1}{J!}\cdot\frac{\partial^{|J|}P}{\partial\zeta^J}\cdot\frac{\partial^{|J|}Q}{\partial z^J} \qquad \text{(Leibniz rule)} .$$

<u>FACT</u> Sections of $\mathcal{E}_X^{\mathbf{R}}$ have a unique-continuation property on S*X.

Hence, on the basis of this fact, we can construct a "hyperfunction theory" for $\mathcal{E}_X^{\mathbf{R}}$; that is, a formal boundary value theory of sections of $\mathcal{E}_X^{\mathbf{R}}$ defined in a wedge-like domain in S*X with edge S_M^*X, where M is a real analytic manifold, X is its complexification. However, there is a great difficulty because a section $P(z,D_z)$ on an open set $W\subset S^*X$ does not necessarily have any global defining function (even if $W=\mathbf{R}_+W\subset T^*\mathbf{C}^n$ is a convex set). Though it has a defining function on each relatively compact convex subdomain of W. So we must restrict ourselves to the case of "boundary values" of sections of $\mathcal{E}_X^{\mathbf{R}}$ having global defining functions.

Consider the following wedge-like conic domain :

$$\Omega(r,R) = \{(z;\zeta dz)\in T^*\mathbf{C}^n;\ |z-\mathring{x}|<r,\ |\zeta/|\mathrm{Im}\zeta|-i\mathring{\eta}|<r,$$
$$|\mathrm{Im}\zeta|>r^{-1},\ -R\cdot\mathrm{Re}\zeta_1>|\mathrm{Re}\zeta'|+|\mathrm{Im}\zeta||\mathrm{Im}z|\ \}$$

for $r,R>0$ and a point $(\mathring{x};i\mathring{\eta}dx)\in iS^*M$ with $|\mathring{\eta}|=1$. Then,

"$P(z,\zeta)$ is a (simple) symbol" $\Leftrightarrow$ $P(z,\zeta)\in\mathcal{O}(\Omega(r,R))$ for some $r,R>0$

s.t. $\forall K\subset\subset\Omega(r,R)$ (in conic sense), $\forall\varepsilon>0$,

$$\sup_K |P(z,\zeta)|e^{-\varepsilon|\zeta|}<+\infty.$$

Formal symbols are also defined similarly.