数理解析研究所講究録 138
第 708 巻 1989 年 138-153

Uniform distribution and means of distances on spheres

Gerold Wagner (Stuttgart)

1. Introduction. In the following we shall study a problem of irregularities of point distributions on spheres. Given an N point set $\omega_N = \{x_1, x_2, \dots, x_N\}$ on the surface $S = S^{d-1}$ of the unit sphere in d-dimensional ($d \geq 2$) Euclidean space E^d, we are interested in the relations which hold between the *distribution* of ω_N on S, and bounds for certain *distance functions* and *distance functionals* ("potentials" and "energies") generated by the set ω_N in a natural way.

We begin by introducing a basic concept. Denote by $\kappa = \kappa(x,\gamma) \subset S$ the spherical cap with "center" $x \in S$ and "angle" γ, i.e. $\kappa(x,\gamma) = \{y \in S : \langle x,y \rangle \geq \cos\gamma\}$, where $0 < \gamma < \pi$, and $\langle x,y \rangle$ denotes the scalar product in E^d. Let $A_\kappa(\omega_N)$ be the number of points x_j, $1 \leq j \leq N$, which are contained in κ, and let σ be the normalized surface measure on S, $\sigma(S) = 1$. Then we call

$$D(\omega_N) = \sup_{\kappa \subset S} |A_\kappa(\omega_N) - N\cdot\sigma(\kappa)|$$

the *discrepancy* of ω_N, and

$$\Delta(x,\gamma;\omega_N) = A_{\kappa(x,\gamma)}(\omega_N) - N\cdot\sigma(\kappa(x,\gamma)) \qquad (x \in S,\ 0<\gamma<\pi)$$

the *discrepancy function* of ω_N.

The following two important results should be mentioned.

Theorem A. ([S1],[B1]) For any point set $\omega_N \subset S^{d-1}$ the following lower estimate is true:

$$D(\omega_N) \geq c_d \cdot N^{\frac{1}{2}\frac{d-2}{d-1}}. \tag{1}$$

Here c_d is a positive constant depending on the dimension d.

We remark that W.M.Schmidt [S1] proved the slightly weaker result $D(\omega_N) \geqslant c_d(\varepsilon)\cdot N^{q-\varepsilon}$, $\varepsilon > 0$ arbitrary, $q = \frac{1}{2}\frac{d-2}{d-1}$, whereas J.Beck [B1] proved the relation

$$(2) \qquad \int_{\gamma=0}^{\pi} \int_S \Delta^2(x,\gamma;\omega_N)\ \sin^{d-2}\gamma\ d\sigma(x)\ d\gamma \ \geqslant\ c_d\cdot N^{\frac{1}{2}\frac{d-2}{d-1}},$$

from which (1) clearly follows. J.Beck [B2] also proved that the bound (1) is essentially best possible.

<u>Theorem B.</u> (see [B2]) For each $N \geqslant 1$ there exists an N point set $\omega_N^o \subset S^{d-1}$ such that

$$D(\omega_N^o) \ \leqslant\ c_d'\cdot N^{\frac{1}{2}\frac{d-2}{d-1}} \cdot \sqrt{\log N}$$

is true. Here c_d' is a positive constant depending on d only.

Note that Theorem A and B are trivial in the case d=2.

We may describe the irregularity of the distribution of ω_N in still another way! Given the point set $\omega_N = \{x_1, x_2, \ldots, x_N\} \subset S$, and a fixed parameter α, $1-d < \alpha < \infty$, we define on S a *distance function* $U_\alpha(x,\omega_N)$ as follows:

$$(3)\quad U_\alpha(x,\omega_N) = \begin{cases} \sum_{j=1}^{N} |x - x_j|^\alpha - N\cdot m(\alpha,d) & (x \in S^{d-1},\ \alpha \neq 0,2,4,\ldots) \\ \sum_{j=1}^{N} |x - x_j|^\alpha \log|x - x_j| - N\cdot m(\alpha,d) & (\alpha = 0,2,4,\ldots). \end{cases}$$

Here $|\ldots|$ denotes the Euclidean distance in E^d, and $m(\alpha,d)$ is the mean value of the kernel $|x-y|^\alpha$ on S^{d-1}, i.e.

$$m(\alpha,d) = \begin{cases} \int_S |x-y|^\alpha d\sigma(y) & (\alpha \neq 0,2,4,\ldots) \\ \int_S |x-y|^\alpha \log|x-y|\ d\sigma(y) & (\alpha = 0,2,4,\ldots). \end{cases}$$

The following observation is of basic importance. We may consider $U_\alpha(x,\omega_N)$ as the convolution of the kernel $|x-y|^\alpha$ with the discrete measure which assigns weight 1 to each of the points $x_j \in \omega_N$. if we replace the summation in (3) by an integration with respect to the uniform distribution measure $N\cdot\sigma$, the corresponding expression vanishes identically. Due to the fact that uniform distribution can be approximated by an N point distribution to a certain degree of accuracy only, we have certain "natural" lower bounds for, say, the integral norm $\|U_\alpha\|_1 = \int_S |U_\alpha(x,\omega_N)|\,d\sigma(x)$.

Similar to (3), we define a *distance functional* $E_\alpha(\omega_N)$ by setting

$$(4)\qquad E_\alpha(\omega_N) = \begin{cases} \sum\limits_{j=1}^{N}\sum\limits_{k=1}^{N}\left(|x_j - x_k|^\alpha - m(\alpha,d)\right) & (\alpha > 0) \\ \sum\limits_{j=1}^{N}\sum\limits_{\substack{k=1\\ j\neq k}}^{N}\left(|x_j - x_k|^\alpha - m(\alpha,d)\right) & (1-d < \alpha \leq 0)\ , \end{cases}$$

again with the agreement that a factor $\log|x_j - x_k|$ is added if $\alpha \in \{0,2,\dots\}$. Here we also note that if we replace the summation in (4) by a double integration with respect to the uniform distribution measure $N\cdot\sigma$, the integral vanishes. Approximating uniform distribution by a discrete distribution will lead, as before, to certain natural bounds for the "energy sums" $E_\alpha(\omega_N)$. We shall always refer to these natural bounds as to "lower bounds", although they may actually be upper bounds in certain cases.

Our aim is to prove analogues of Theorem A and B for the functions $U_\alpha(x,\omega_N)$ and the functionals $E_\alpha(\omega_N)$. Moreover, we will give some quantitative relations between the discrepancy $D(\omega_N)$ and the numbers $\int_S |U_\alpha(x,\omega_N)|\,d\sigma(x)$, $E_\alpha(\omega_N)$. Many of our results are trivial in the case d=2, i.e. for the unit circle. On the other hand, there are some more detailed questions concerning the logarithmic case $\alpha=0$ on the unit circle, and we shall devote the next section to give a brief account on this topic.

2. Products of distances on the unit circle.

2.1. Polynomials. With the point set $\omega_N = \{z_1, z_2, \dots, z_N\}$, $z_k = e^{i\varphi_k}$, we associate the polynomial $p(z,\omega_N) = \prod_{k=1}^{N}(z - z_k)$, $z = e^{i\varphi}$.

Clearly $\max_{z\in S^1} |p(z,\omega_N)| = \exp(\max_{x\in S^1} U_o(z,\omega_N)) \geqslant \exp(\frac{1}{2} \|u(\varphi,\omega_N)\|_1)$,

where $u(\varphi,\omega_N) = U_o(e^{i\varphi},\omega_N)$ and $\|u(\varphi,\omega_N)\|_1 = \frac{1}{2\pi} \int_0^{2\pi} |u(\varphi,\omega_N)| d\varphi$.

Hence lower bounds for $\|u(\varphi,\omega_N)\|_1$ yield also lower bounds for the maximum modulus of the polynomial $p(z,\omega_N)$ on the unit circle. Note the "natural" lower bound

(5) $$\|u(\varphi,\omega_N)\|_1 \geqslant c > 0 ,$$

which is essentially best possible by the choice $z_k = \exp(2\pi ik/N)$, $k=1,2,\ldots,N$.

The concept of a discrepancy function on S^1 will be slightly modified as follows: Put

$$\Delta^*(\varphi,\omega_N) = \sum_{k=1}^{N} f(\varphi-\varphi_k) ,$$

where $f(\varphi)$ is the saw-tooth function $f(\varphi) = \frac{1}{2\pi}(\pi - \varphi)$, $0 \leqslant \varphi < 2\pi$, 2π-periodically continued over the real axis.

We have

$$\frac{1}{2} D(\omega_N) \leqslant \sup_{0\leqslant \varphi < 2\pi} |\Delta^*(\varphi,\omega_N)| \leqslant D(\omega_N) ,$$

hence $\Delta^*(\varphi,\omega_N)$ is a perfect substitute for the discrepancy function $\Delta(\gamma,z;\omega_N)$ introduced in Section 1.

Relations between $\max |p(z,\omega_N)|$ and the distribution of the root set ω_N are expressed by the following

Theorem 1. The following inequalities are true:

(a) $\|u(\varphi,\omega_N)\|_1 \gg D^2(\omega_N)/N$

(b) $\|u(\varphi,\omega_N)\|_1 \gg \frac{1}{\log N} \|\Delta^*(\varphi,\omega_N)\|_1$.

Remarks. 1.) Assertion (a) was first proved by P.Erdös and P.Turan in 1940 (see [E-T]) in the slightly weaker form

$$\max_{z\in S^1} |p(z,\omega_N)| \geqslant \exp(c\cdot D^2(\omega_N)/N) ,$$

using the theory of orthogonal polynomials. T.Ganelius [G] in 1959 and E.Hlawka [H] in 1968 gave simpler proofs. Assertion (b) is due to the author [W6] .

2.) In view of the natural bound (5) the assertion (a) is nontrivial only if $D(\omega_N) \geqslant K\sqrt{N}$ with K sufficiently large. In the following sense assertion (a) is best possible:

Choose $z_1 = z_2 = \dots = z_{[\sqrt{N}]} = \exp(\frac{\pi i}{N}[\sqrt{N}])$, $z_k = \exp(2\pi ik/N)$ for $k = [\sqrt{N}] + 1, \dots, N$. For such point sets ω_N we have

$$D(\omega_N) \geqslant c_1 \cdot \sqrt{N} ,$$

but still

$$\| u(\varphi, \omega_N) \|_1 \leqslant c_2 ,$$

where $c_1 > 0$, $c_2 > 0$ are absolute constants.

3.) Assertion (b) is probably true without the factor $(1/\log N)$, but this seems rather difficult to prove. Note the striking but useless relation

$$\| u(\varphi, \omega_N) \|_2 = \| \Delta^*(\varphi, \omega_N) \|_2 .$$

P.Erdös [E] asked the following question: let $\omega = (z_1, z_2, \dots)$ be an *infinite* sequence of points on the unit circle. For each section $\omega_n = (z_1, z_2, \dots, z_n)$ of ω consider the maximum $\mu_n(\omega) = \max_{z \in S^1} |p(z, \omega_n)|$. Can $(\mu_n(\omega))$ be bounded in n?

It is known that there are infinite sequences ω (f.e. the famous van der Corput-sequence, see [K-N]) satisfying the relations

$$D(\omega_n) \leqslant c \cdot \log n \quad \text{and} \quad \| \Delta^*(\varphi, \omega_n) \|_1 \leqslant c' \cdot \sqrt{\log n}$$

for all $n \geqslant 2$. Hence neither (a) nor (b) can be used to settle this question. Using a method introduced by W.M.Schmidt [S2] , the author [W1] was able to prove the following result:

Theorem 2. There are absolute constants $c > 0$ and $\delta > 0$ such that for any infinite sequence ω on S^1 the inequality

$$\mu_n(\omega) \geqslant c \cdot (\log n)^{\delta} \tag{6}$$

holds for an infinite number of indices n.

Using a reduction method to be found in [T-W] it is even possible to show that (6) holds for "almost all" n.

It is believed that (6) holds in the stronger form

$$\mu_n(\omega) \geqslant c\cdot n^{\delta}$$

for infinitely many n. The van der Corput sequence shows that $\mu_n(\omega) \leqslant n+1 \leqslant 2n$ for all n is possible. This is not the "best" sequence, however. C.N.Linden [L] gave an example of a sequence ω' for which

$$\mu_n(\omega') \leqslant c'\cdot n^{\delta'}$$

holds for each n and constants $c' > 0$ and $0 < \delta' < 1$.

2.2. Mutual distances. It is known that for each ω_N on S^1 the natural bound

$$E_o(\omega_N) = \sum_{j \neq k}\sum \log|z_j - z_k| \leqslant N\cdot\log N$$

is true, with equality holding if and only if the points $z_1, z_2, \ldots, z_N$ are the vertices of a regular N-gon. The following result is a counterpart to Theorem 1(a), see [W6].

Theorem 3. With certain numerical constants $c_1 > 0$ and $c_2 > 0$, the following inequality is true:

$$E_o(\omega_N) \leqslant N\cdot\log N + c_1\cdot N - \frac{c_2\cdot D^2(\omega_N)}{\log(2N/D(\omega_N))} . \tag{7}$$

Note that relation (7) is nontrivial as soon as $D(\omega_N) \geqslant K\cdot\sqrt{N\cdot\log N}$ holds with K sufficiently large.

There is some contrast in the behaviour of $\|u(\varphi,\omega_N)\|_1$ and $E_o(\omega_N)$ with respect to irregularities of the distribution of the point set ω_N: If only two points z_j, z_k coincide, the sum $E_o(\omega_N)$ completely breaks down, whereas $\|u(\varphi,\omega_N)\|_1 \leqslant c_o$ may still hold. On the other hand, distributing $[\frac{N}{2} - \sqrt{N}]$ points equidistantly on the upper half circle, and doing so with $N - [\frac{N}{2} - \sqrt{N}]$ over the lower half circle, we still have

$$E_o(\omega_N) \geqslant N\cdot\log N - c_o\cdot N ,$$

whereas by Theorem 1(b) the values of the corresponding polynomial "explode": we have $\|u(\varphi,\omega_N)\|_1 \gg \sqrt{N}/\log N$.

As in the case of polynomials we may ask the following question: given an infinite sequence $\omega = (z_1, z_2, \dots)$ on S^1, what can be said about the behaviour of the sequence $(\varepsilon_n(\omega))$, where $\varepsilon_n(\omega) = \sum\sum_{\substack{1\leq j,k\leq n\\ j\neq k}} \log|z_j - z_k|$?

The following result is proved in [W6].

<u>Theorem 4</u>. Let $\omega = (z_1, z_2, \dots)$ be an infinite sequence on the unit circle. Then for infinitely many n the following inequality holds:

(8)
$$\varepsilon_n(\omega) \leq n\cdot\log n - c\cdot n \quad .$$

Here $c > 0$ is a numerical constant.

We remark (8) is best possible, apart from the value of the constant c. If ω denotes the van der Corput sequence, then the reverse inequality

$$\varepsilon_n(\omega) > n\cdot\log n - c'\cdot n$$

holds for all n and some numerical constant $c' > 0$.

3. <u>Lower bounds for distance functions on spheres</u>.

We recall the notation introduced in Section 1. Then the following counterparts to Theorem A and the Erdös-Turan inequality Theorem 1(a) are true.

<u>Theorem 5</u>. ([W3], [W2]) For any point set $\omega_N = \{x_1, \dots, x_N\} \subset S^{d-1}$ we have the natural bound

$$\|U_\alpha(x,\omega_N)\|_1 \geq c(d,\alpha)\cdot N^{-\alpha/(d-1)} \quad .$$

<u>Theorem 6</u>. ([W6]) For any point set $\omega_N \subset S^{d-1}$ with discrepancy $D(\omega_N)$ we have the inequality

$$\|U_\alpha(x,\omega_N)\|_1 \geq c'(d,\alpha)\cdot D(\omega_N)\cdot\left(\frac{D(\omega_N)}{N}\right)^{d+\alpha-1} \quad .$$

<u>Remarks</u>. 1.) In the special case $\alpha = 2-d$ (the classical "Newtonian case") Theorem 5 and 6 are implicitely contained in a more general result of P.Sjögren [Sj]. In fact, Sjögren proves such inequalities for a whole class of

sufficiently smooth closed surfaces.

2.) In threedimensional space Theorem 5 has the following physical interpretation: Suppose we distribute N electrons, each endowed with unit charge, at the places $x_1, x_2, \dots, x_N$ on a conducting sphere of radius 1. They generate the potential $\sum_{j=1}^{N} |x - x_j|^{-1}$ which has mean value N on S^2. By Theorem 5 there are points on S^2 at which the actual potential is by at least $c\cdot\sqrt{N}$ *below* the mean value. It can be proved (see Theorem 9 in Section 5) that this result is essentially best possible.

3.) Theorem 6 can be strengthened slightly for dimensions $d \geq 3$. Suppose that for some spherical cap $\kappa = \kappa(x,\gamma)$ we have

$$|A_\kappa(\omega_N) - N\cdot\sigma(\kappa)| \geq \frac{1}{2} D(\omega_N) .$$

We may assume $0 < \gamma \leq \frac{\pi}{2}$ (otherwise take the complementary cap). Put

$$\gamma_o = \max \left\{\gamma, \left\{D(\omega_N)/N\right\}^{1/(d-1)}\right\} .$$

Then the stronger inequality

$$\| U_\alpha(x,\omega_N)\|_1 \geq c(d,\alpha)\cdot D(\omega_N)\cdot \left\{\frac{D(\omega_N)}{N\cdot\gamma_o^{d-2}}\right\}^{d-1+\alpha}$$

holds true.

4.) The proof of Theorem 5 as given in [W3] uses the method of "test functions". A remarkable variant of this method was introduced by T.Ganelius [G] (see also [Sj]) into potential theory, allowing a unified approach to both Theorem 5 and 6 (see [W6]).

4. Lower bounds for distance functionals.

In the case of distance functionals $E_\alpha(\omega_N)$ our results are less complete. We restrict ourselves to values of α satisfying $1-d < \alpha < 2$, and we obtain essentially best possible natural bounds for $0 < \alpha < 2$ in all dimensions, and for $-2 < \alpha \leq 0$ in dimension d=3 (see also the remarks following Theorem 9 in Section 5). We have

Theorem 7. (see [W3]) The following inequalities are true:

(a) $-E_\alpha(\omega_N) \geqslant c(\alpha,d)\cdot N^{1-(\alpha/(d-1))}$ for $0<\alpha<2$,

(b) $E_\alpha(\omega_N) \geqslant -c(\alpha,d)\cdot N^{1-(\alpha/(d-1))}$ for $1-d<\alpha<\min(0,3-d)$,

(c) $E_\alpha(\omega_N) \geqslant -c(\alpha,d)\cdot N^{2(1-\alpha)/(2-\alpha)}$ for $3-d<\alpha<0$,

(d) $E_0(\omega_N) \leqslant \frac{N}{2}\log N + O(N)$ for $d\geqslant 3$.

In all cases the constants $c(\alpha,d)$ are positive, and independent of ω_N.

Remarks.

1.) The case $\alpha=1$ deserves special mentioning. K.B.Stolarsky [St] proved the following remarkable identity for dimensions $d\geqslant 3$:

(9) $$-E_1(\omega_N) = c_d\cdot \int_{\gamma=0}^{\pi} \int_{S^{d-1}} \Delta^2(x,\gamma)\, \sin^{d-2}\gamma\, d\gamma\, d\sigma(x) \ ,$$

where $c_d>0$, and $\Delta(x,\gamma)$ denotes the discrepancy function introduced in Section 1. J.Beck [B1] used the identity (9) and his version of Theorem A to derive Theorem 7(a) in the special case $\alpha = 1$. Before K.B.Stolarsky [St], using W.M.Schmidt's version of Theorem A, gave nontrivial bounds for $E_1(\omega_N)$ for dimensions $d\geqslant 5$. On the other hand, there is a direct potential theoreti proof of Theorem 7(a) (see the remark below), from which the Beck-Schmidt result Theorem A follows via Stolarsky's identity (9).

2.) The proof of Theorem 7(a) uses a device which is well-known in potential theory: In order to prove that certain energy integrals are positive definite, we express them as a quadratic norm of a potential, generated by the same distribution, but with respect to a different kernel.
In our case we have the identity

$$-E_\alpha(\omega_N) = c(d,\alpha)\cdot \int_S \varphi_\alpha^2(x,\omega_N)\, d\sigma(x) \qquad (0<\alpha<2) \ ,$$

where φ_α is a potential, generated by a kernel k_α which is essentially given by

$$k_\alpha(|x-y|) \sim |x-y|^{(1+\alpha-d)/2} \ .$$

It can be shown that Theorem 5 remains true for this modified kernel k_α, hence

$$-E_\alpha(\omega_N) \;=\; c(d,\alpha)\cdot\|\varphi_\alpha(x,\omega_N)\|_2^2 \;\geqslant\; c(d,\alpha)\cdot\|\varphi_\alpha(x,\omega_N)\|_1^2 \;\geqslant$$
$$\geqslant\; c'(d,\alpha)\cdot\left(N^{-\frac{1+\alpha-d}{2(d-1)}}\right)^2 \;=\; c'(d,\alpha)\cdot N^{1-(\alpha/(d-1))}\;.$$

3.) The proofs of the assertions (b) - (d) in Theorem 7 are of elementary nature. In particular, there is a short and elegant proof of (d) for the sphere S^2 in E^3, and this result is essentially best possible, see [W2]. For dimensions $d\geqslant 4$, one expects the sharper inequality

$$E_o(\omega_N) \;\leqslant\; \frac{N}{d-1}\log N \;+\; O(N)\;,$$

but we have not been able to prove it.

There are also inequalities for $E_\alpha(\omega_N)$ of the Erdös-Turan type. For values of α satisfying $0<\alpha<2$ or $\alpha = 2-d$ they are as follows:

Theorem 8. (a) For $0<\alpha<2$ and $d\geqslant 2$ we have the inequality

$$-E_\alpha(\omega_N) \;\geqslant\; c(d,\alpha)\cdot D^2(\omega_N)\cdot\left(\frac{D(\omega_N)}{N}\right)^{d+\alpha-2}\;.$$

(b) For $d\geqslant 3$ we have the inequality

$$E_{2-d}(\omega_N) \;\geqslant\; -c_1(d)\cdot N^{1+\frac{d-2}{d-1}} \;+\; c_2(d)\cdot\frac{D^2(\omega_N)}{\log(2N/D(\omega_N))}\;.$$

The constants $c(d,\alpha)$, $c_1(d)$ and $c_2(d)$ are positive.

Note that in the case d=2 the inequality which corresponds to Theorem 8(b) is given by Theorem 3.

5. Constructing good point sets.

The natural lower bounds in Theorem 5 and Theorem 7(a) are best possible, apart from the values of the constants. We first state the result (see [W4]).

Theorem 9. (a) For each $\alpha > 1-d$ and each $N\geqslant 1$ there exists a point set ω_N^o (depending on α) such that

$$\|U_\alpha(x,\omega_N^o)\|_{(\infty)} \;\leqslant\; c'(d,\alpha)\cdot N^{-\alpha/(d-1)}$$

holds with some positive constant $c' = c'(d,\alpha) > 0$.

Here $\| \quad \|_{(\infty)}$ denotes the "one-sided" maximum norm, defined by $\|U_\alpha\|_{(\infty)} = \max_S |U_\alpha|$ if $\alpha > 0$, $\|U_0\|_{(\infty)} = \max_S U_0$, and $\|U_\alpha\|_{(\infty)} = -\min_S U_\alpha$ if $1-d < \alpha < 0$.

(b) For each α, $0 < \alpha < 2$, and each $N \geqslant 2$ there exists a set ω_N^0 (depending on α) such that

$$-E_\alpha(\omega_N^0) \leqslant c'(d,\alpha) \cdot N^{1-(\alpha/(d-1)}$$

holds with some positive constant $c'(d,\alpha) > 0$.

Remarks.

1.) Note first that assertion (b) is an immediate consequence of assertion (a). For $\alpha = 1$, (b) was already proved by K.B.Stolarsky [St] .

2.) Theorem 9 corresponds to Theorem B in Section 1. J.Beck's"construction"of good point sets (see [B2]) is based on an ingenious variant of the probabilistic method. In contrast, our method is constructive, making use of the existence of so-called "quadrature formulas with equal weights". It should be noted, however, that this latter method does not provide an alternative proof of Theorem B.

3.) The problem concerning energy sums in the case $1-d < \alpha \leqslant 0$ is not settled by Theorem 9. Here the construction of good point sets requires a completely different approach. For dimension d=3 a highly computational proof ([W] , unpublished) shows that Theorem 7(b) (for the logarithmic case (d) see [W2]) is in fact also best possible, but for $d \geqslant 4$ the situation is not clear.

Let us briefly describe our method of how to construct the "good" point sets. First, we divide the sphere S^{d-1} into mutually disjoint domains $B_1, B_2, \ldots, B_m$, subject to the following conditions:

(1) Each B_μ is approximately a "square", i.e. for certain constants $c_1(d) > 0$, $c_2(d) > 0$ we have

$$c_1(d)\cdot\sigma(B_\mu) \leqslant (\operatorname{diam} B_\mu)^{d-1} \leqslant c_2(d)\cdot\sigma(B_\mu), \quad \mu = 1,2, \ldots ,m.$$

(2) For certain positive constants $q_1 = q_1(d,\alpha)$, $q_2 = q_2(d,\alpha)$ we have the inequalities

$$q_1 \leqslant N\cdot\sigma(B_\mu) \leqslant q_2, \quad \mu = 1,2, \ldots ,m ,$$

hence the subdomains B_μ are neither too small nor too big in size.

(3) The numbers $s_\mu := N\cdot\sigma(B_\mu)$ $(\mu = 1,2, \ldots ,m)$ are positive integers.

Now we distribute s_μ points $x_1^{(\mu)}, \ldots ,x_{s_\mu}^{(\mu)}$ on each B_μ in such a way that the relation

$$(10) \qquad \frac{s_\mu}{\sigma(B_\mu)} \int_{B_\mu} p(x)\, d\sigma(x) = \sum_{j=1}^{s_\mu} p(x_j^{(\mu)})$$

holds for each polynomial $p = p(x_1,x_2, \ldots ,x_d)$ of degree $\leqslant r$, $r = r(d,\alpha)$.

We consider the sum $\sum_{j=1}^{s_\mu} |x - x_j^{(\mu)}|^\alpha$ as an approximation of the integral $\frac{s_\mu}{\sigma(B_\mu)} \int_{B_\mu} |x - y|^\alpha\, d\sigma(y)$. In order to estimate the error, we expand the integrand (for fixed x) $|x - y|^\alpha$ into a Taylor polynomial with remainder term about some point $z \in B_\mu$. Such an expansion is possible for all but a bounded number of B_μ's in the neighbourhood of the point x.

Using the averaging property (10) of the sets $\{x_1^{(\mu)}, \ldots ,x_s^{(\mu)}\}$, we find that the total (onesided for $\alpha \leqslant 0$) error

$$N \int_S |x - y|^\alpha d\sigma(y) - \sum_{\mu=1}^{m} \sum_{j=1}^{s_\mu} |x - x_j^{(\mu)}|^\alpha$$

is of the same order as the maximal "local" (onesided) error

$$\frac{s_\nu}{\sigma(B_\nu)} \int_{B_\nu} |x - y|^\alpha d\sigma(y) - \sum_{j=1}^{s_\nu} |x - x_j^{(\nu)}|^\alpha ,$$

where B_ν is the subdomain containing the point x. This maximal local error in turn is obtained by simply inserting the diameter $\operatorname{diam} B_\nu \sim N^{-1/(d-1)}$ into the kernel $|x - y|^\alpha$, yielding Theorem 9(a).

The main problem consists in proving the existence of points $x_1^{(\mu)}, \dots, x_{s_\mu}^{(\mu)}$ with the property (10). By choosing the subdomains B_μ as coordinate boxes with respect to the spherical coordinates $\theta_1, \theta_2, \dots, \theta_{d-2}, \varphi$, and by separating coordinates, this crucial part may be derived from the following effective result, concerning the existence of quadrature formulas with equal weights.

Theorem 10. ([W5]). Let $w(x) \geq 0$ be an integrable weight function on the interval $[-1,1]$, satisfying the conditions $\int_{-1}^{1} w(x)\,dx = 1$ and

$L_2(1-|x|)^\beta \leq w(x) \leq L_1 \quad (\beta > 0,\ 0 < L_2 \leq 1 \leq L_1)$.

Let $\Phi = \{f_1, f_2, \dots, f_s\} \subset C^3[-1,1]$ be a set of functions such that the derivatives $f_1', f_2', \dots, f_s'$ form an orthonormal system with respect to $w(x)$.

Set $K := \max_{\mu,\nu} \max_{[-1,1]} \left(|f_\mu'|, |f_\mu''|, |(f_\mu' \cdot f_\nu')'|, |(f_\mu' \cdot f_\nu')''| \right)$.

Then for each $n > n_o := (24 \cdot s \cdot K^2 \cdot L_1/L_2)^{\beta+2}$ there exist points $-1 < \xi_1 < \dots < \xi_n < 1$ such that

$$\frac{1}{n} \sum_{j=1}^{n} f_\mu(\xi_j) = \int_{-1}^{1} f_\mu(x)\, w(x)\, dx$$

holds true for all $f_\mu \in \Phi$ simultaneously.

Remarks.

1.) The mere existence of the bound n_o was proved (under much more general assumptions) by P.D.Seymour and T.Zaslavsky [S-Z] in 1984. Their result, however, cannot be used for our purpose, as it does not include any effective bounds for n_o. What we need for the proof of Theorem 9 is a bound n_o which is independent of certain parameters implicit in the size and location of the subdomains B_μ.

2.) The proof of Theorem 10 is constructive, and is based on a variant of Newton's iteration method. It is suitable for numerical calculations.

Theorem 10 has other interesting applications, we mention two of them.

3.) P.Tschebyschew (Collected Works 1873) considered the following problem: For each $n \geqslant 1$, find points $-1 \leqslant \xi_1 \leqslant . . . \leqslant \xi_n \leqslant 1$ such that

$$\frac{2}{n} \sum_{j=1}^{n} \xi_j^{\nu} = \int_{-1}^{1} x^{\nu}\, dx \tag{11}$$

is true for $\nu = 1,2, \dots ,n$. As there are n variables $\xi_1, \xi_2, \dots ,\xi_n$ for the n conditions (11), this problem is likely to have a solution. P.Tschebyschew and others gave examples for $n = 1,\dots,7$ and 9. Surprisingly, S.N.Bernstein [Bs] showed in 1937 that (11) has no solution for $n > 9$. More precisely, he proved that in order to have a relation of the form

$$\frac{2}{n} \sum_{j=1}^{n} \xi_j^{\nu} = \int_{-1}^{1} x^{\nu}\, dx \qquad (\nu = 1,2, \dots ,s) \,, \tag{12}$$

we need at least $n > \text{const}\cdot s^2$ interpolation points ξ_j. Whether $c\cdot s^2$ such points are sufficient is not known yet. By using a more direct variant of Theorem 10 it can be shown, however, that (12) is solvable for any $n > n_o = \text{const}\cdot s^7$.

4.) One main purpose of Seymour and Zaslavsky's paper [S-Z] was to prove the existence of so-called spherical designs. By a d-dimensional *spherical design of size n and strength t* we mean a set of n points $\xi_1, \xi_2, \dots ,\xi_n$ on the sphere $S^d \subset E^{d+1}$ with the property that

$$\frac{1}{n} \sum_{j=1}^{n} p(\xi_j) = \int_{S^d} p(x)\, d\sigma(x)$$

holds for each polynomial $p(x_1, x_2, \dots ,x_{d+1})$ of degree not exceeding t. Applying Theorem 10 to sets of ultraspherical polynomials yields the existence of such designs for all sizes $n > n_o = c_d\cdot t^{12d^4}$, see [W5] .

References.

[B1] J.Beck — On the sum of distances between N points on a sphere. Mathematika 31 (1984), 33 - 41.

[B2] J.Beck — Some upper bounds in the theory of irregularities of distribution. Acta A ith., 43 (1984), 115 - 130.

[Bs] S.N.Bernstein — On the quadrature formulas of Cotes and Tchebyshev. Doklady Akad. Nauk SSSR, (1937).

[E] P.Erdös — Problems and results in diophantine approximation. Rêpartition Modulo 1, Springer Lecture Notes 475, 89 - 99, Springer-Verlag Berlin 1975.

[E-T] P.Erdös and P.Turan — On the uniformly dense distribution of certain sequences of points. Annals of Math. 41 (1940), 162 - 173.

[G] T.Ganelius — Some applications of a lemma on Fourier series. Publ. de l'Inst. Math. de l'Acad. Serbe des Sciences 11 (1957), 9 - 18.

[H] E.Hlawka — Interpolation analytischer Funktionen auf dem Einheitskreis. Abhandlungen aus Zahlentheorie und Analysis, VEB Akademie der Wissenschaften Berlin 1968.

[K-N] L.Kuipers and H.Niederreiter — Uniform distribution of sequences. Wiley and Sons, New York 1974.

[L] C.N.Linden — The modulus of polynomials with zeros on the unit circle. Bull. London Math. Soc. 9 (1977), no.1, 65 - 69.

[S1] W.M.Schmidt — Irregularities of distribution IV. Invent. Math. 7 (1969), 55 - 82.

[S2] W.M.Schmidt — Irregularities of distribution VII. Acta Arith. 21 (1972), 45 - 50.

[Sj] P.Sjögren — Estimates of mass distributions from their potentials and energies. Arkiv för Math. 10 (1972), 59 - 77.

[St] K.B.Stolarsky — Sums of distances between points on a sphere II. Proc. AMS 41 (1973), 575 - 582.

[S-Z] P.D.Seymour / T.Zaslavsky — Averaging Sets. Advances of Math. 52 (1984), 213 - 240.

[T-W] R.Tijdeman / G.Wagner — A sequence has almost nowhere small discrepancy. Mh. Math. 90 (1980), 315 - 329.

[W1] G.Wagner — On a problem of Erdös in diophantine approximation. Bull. London Math. Soc. 12 (1980), 81 - 88.

[W2] G.Wagner — On the product of distances to a point set on a sphere. to appear in: J. Australian Math. Soc.

[W3] G.Wagner — On means of distances on the surface of a sphere. to appear in: Pacific J. of Math.

[W4] G.Wagner — Upper bounds for distance functions on spheres. (to appear)

[W5] G.Wagner — Quadrature formulas with equal weights. (to appear)

[W6] G.Wagner — Erdös-Turan inequalities for distance functions on spheres. to appear in: Michigan Math. Journal

Gerold Wagner
Mathematisches Institut A
Pfaffenwaldring 57
7 Stuttgart 80
West Germany

Faculty of Science and Technology
Keio University
14-1 Hiyoshi 3 Chome
Kohoku-ku, Yokohama 223
Japan