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FAMILY OF OPERATORS DEFINED ON ANALYTIC FUNCTIONS

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1. Introduction.

Let $\mathcal{F}$ denote the class of analytic functions f which are holomorphic in the unit disk $E = \{|z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. On the other hand, let σ be a probability measure supported by the unit interval $I = [0, 1]$. Then a linear operator $\mathcal{L}$ is defined on $\mathcal{F}$ by means of

$$\mathcal{L} f(z) = \int_I \frac{f(zt)}{t} d\sigma(t) \qquad (f \in \mathcal{F}).$$

In a series of previous papers [5~19], we have dealt with various problems concerning $\mathcal{L}$. The present note is a survey of several results obtained on these problems in which they are rearranged in systematic and partly improved forms. The details of proofs are to be referred to respective original papers.

2. Additive family of operators.

Since $f \in \mathcal{F}$ implies $\mathcal{L}f \in \mathcal{F}$, the iteration $\{\mathcal{L}^n\}_{n=0}^{\infty}$ arises automatically within the class $\mathcal{F}$. However, this sequence can further be interpolated into a family $\{\mathcal{L}^{\lambda}\}$ depending on a continuous parameter $\lambda \geqq 0$ in such a manner that it is subject to the additivity $\mathcal{L}^{\lambda}\mathcal{L}^{\mu} = \mathcal{L}^{\lambda+\mu}$. In fact, by referring to [3] and [28], we have the following theorem.

THEOREM 2. 1. There exists always an additive family $\{\mathcal{L}^{\lambda}\}$ generated by σ. Further, if the sequence $\{\alpha_{\nu}^{\lambda}\}_{\nu}$ given by

$$\alpha_{\nu} = \int_I t^{\nu-1} d\sigma(t) \qquad (\nu = 1, 2, \ldots)$$

is fully monotone, there exists a probability measure σ_{λ} as an essentially unique solution of the moment problem of Hausdorff type

$$\int_I t^{\nu-1} d\sigma_{\lambda}(t) = \alpha_{\nu}^{\lambda} \qquad (\nu = 1, 2, \ldots)$$

such that $\mathcal{L}^\lambda$ admits the integral representation

$$\mathcal{L}^\lambda f(z) = \int_I \frac{f(zt)}{t} d\sigma_\lambda(t) \qquad (f \in \mathcal{F}).$$

THEOREM 2. 2. For any $f \in \mathcal{F}$, the limit relations

$$\lim_{\lambda \to +0} \mathcal{L}^\lambda f(z) = f(z) \quad \text{and} \quad \lim_{\lambda \to \infty} \mathcal{L}^\lambda f(z) = z$$

hold in E uniformly in the wider sense except the extreme cases where we have always $\mathcal{L} f(z) = z$ for the former relation and $\mathcal{L} f(z) = f(z)$ for the latter.

THEOREM 2. 3. The family generated by the measure $\sigma(t; a) = t^a$ with $a > 0$ is given by

$$\mathcal{L}(a)^\lambda f(z) = \frac{a^\lambda}{\Gamma(\lambda)} \int_I f(zt) t^{a-2} \left(\log \frac{1}{t}\right)^{\lambda-1} dt.$$

It is remarked that the operator $\mathcal{L}(a)$ with an integer a was dealt with by several authors; for instance, $\mathcal{L}(1)$ by Srivastava and Owa [29], $\mathcal{L}(2)$ by Libera [21] and Livingston [22], $\mathcal{L}(n)$ with n = 2, 3, ... by Bernardi [1], each in connection with some classes of functions univalent in E.

3. Relation to fractional calculus.

The operator $\mathcal{L}(a)$ is expressed in terms of ordinary integration operator.

THEOREM 3. 1. Any operator $\mathcal{L}$ under consideration is commutative with $\vartheta = d/d \log z$.

THEOREM 3. 2. For any $a > 0$ and $\lambda \geqq 1$, we have

$$\vartheta(z^{a-1}\mathcal{L}(a)^\lambda) = a z^{a-1}\mathcal{L}(a)^{\lambda-1} \quad \text{or} \quad \vartheta\mathcal{L}(a)^\lambda = a\mathcal{L}(a)^{\lambda-1} - (a-1)\mathcal{L}(a)^\lambda.$$

In particular, $\mathcal{L}(1)^\lambda$ coincides with the fractional integration of order λ with respect to log z.

THEOREM 3. 3. For any $a > 0$, we have

$$\mathcal{L}(a) = a\sum_{\kappa=1} \frac{(2-a)_{\kappa-1}}{(\kappa+1)!}\widetilde{\mathcal{J}}_\kappa; \quad \widetilde{\mathcal{J}}_\kappa = \frac{(\kappa+1)}{z^\kappa}\mathcal{J}^\kappa, \quad \mathcal{J}f(z) = \int_0^z f(\zeta)d\zeta,$$

where $(\cdot)_n$ denotes Pochhammer symbol. In particular, when $a = k > 1$ is an integer, the expression reduces to finite sum consisting of the beginning $k - 1$ terms.

4. Distortion of the real part.

Consider the functionals on $\mathcal{F}$ defined by

$$h(r) = \min_{|z|=r} \operatorname{Re} \frac{f(z)}{z} \quad \text{and} \quad H(r) = \max_{|z|=r} \operatorname{Re} \frac{f(z)}{z},$$

$r \in [0, 1)$ being a fixed parameter, and denote by $\hat{h}$ and $\hat{H}$ the corresponding quantities associated with $\hat{f} = \mathcal{L} f$. The monotoneity $h(r) \leqq \hat{h}(r) \leqq 1 \leqq \hat{H}(r) \leqq H(r)$ is sharpened as follows.

THEOREM 4. 1. $\hat{h}(r) \geqq h(r) + (1 - h(r))\Phi, \quad \hat{H}(r) \leqq H(r) - (H(r) - 1)\Phi;$

$$\Phi = \int_I \frac{1 - t}{1 + t} d\sigma(t).$$

The equality sign in either estimation does not appear for an $r \in (0, 1)$ unless $f(z) = z$, provided σ is not the measure concentrated at 1.

By means of a theorem due to Koebe [4], this theorem is slightly sharpened.

THEOREM 4. 2.

$$\frac{2(H(r) - h(r))}{\pi} \int_I \arctan \frac{T(r) - t}{1 - T(r)t} d\sigma(t) + \frac{H(r) + h(r)}{2} \leqq \hat{h}(r)$$

$$\leqq \hat{H}(r) \leqq \frac{2(H(r) - h(r))}{\pi} \int_I \arctan \frac{T(r) + t}{1 + T(r)t} d\sigma(t) + \frac{H(r) + h(r)}{2};$$

$$T(r) = -\tan\left(\frac{\pi}{4} \frac{H(r) + h(r) - 2}{H(r) - h(r)}\right).$$

5. Distortion on the value-range.

Let $\mathcal{F}(\alpha)$ with $\alpha < 1$ denote the subclass of $\mathcal{F}$ consisting of f such that $\operatorname{Re}(f(z)/z) > \alpha$ in E.

THEOREM 5. 1. For any $f \in \mathcal{F}(\alpha)$ we have

$$\left| \frac{\mathcal{L}f(z)}{z} - \frac{\mathcal{L}\varphi(r; \alpha)}{r} \right| \leqq \frac{\mathcal{L}\psi(r; \alpha)}{r} - 1 \qquad (|z| \leqq r < 1),$$

where φ and ψ are elementary functions in $\mathcal{F}(\alpha)$ defined by

$$\frac{\chi(z;\alpha)}{z} = (1-\alpha)\frac{1+z}{1-z} + \alpha,$$

$$\frac{\varphi(z;\alpha)}{z} = \frac{\chi(z^2;\alpha)}{z^2} \quad \text{and} \quad \frac{\psi(z;\alpha)}{z} = 1 + (1-2\alpha)z + \frac{\chi(z^2;\alpha)}{z}.$$

The extremal functions are of the form $f(z) = \bar{\varepsilon}\chi(\varepsilon z;\alpha)$ with $|\varepsilon| = 1$, unless σ coincides with the point measure concentrated at 0.

THEOREM 5. 2. If $\mathcal{L}\varphi(r;\alpha)$ is bounded as $r \to 1-0$, then $\mathcal{L}\psi(r;\alpha)$ is also bounded and both possess their respective finite limits. The value-range of $\mathcal{L}f(z)/z$ in E is then given by

$$\left|\frac{\mathcal{L}f(z)}{z} - \mathcal{L}\varphi(1-0;\alpha)\right| \leqq \mathcal{L}\psi(1-0;\alpha) - 1.$$

If $\mathcal{L}\varphi(r;\alpha)$ is unbounded as $r \to 1-0$, then $\mathcal{L}\psi(r;\alpha)$ is also unbounded and the value-range is given by

$$\mathrm{Re}\,\frac{\mathcal{L}f(z)}{z} > B(\alpha) \equiv \lim_{r\to 1-0}(\mathcal{L}\varphi(r;\alpha) - \mathcal{L}\psi(r;\alpha)) + 1.$$

6. Length and area distortions.

For the mapping $w = f(z)/z$ with $f \in \mathcal{F}(\alpha)$, let $L(r;f)$ denote the length of the image curve of $\{|z| = r < 1\}$ and $A(r;f)$ the area of the image domain of $\{|z| < r < 1\}$ according to multiplicity. In connection with a theorem of Rogosinski [27], following estimations are derived.

THEOREM 6. 1. $L(r;\mathcal{L}f) \leqq \int_I L(rt;f)d\sigma(t)$, $L(\tau;f) \leqq (1-\alpha)\dfrac{4\pi\tau}{1-\tau^2}$;

$$A(r;\mathcal{L}f) \leqq \int_I A(rt;f)d\sigma(t), \quad A(\tau;f) \leqq \frac{1}{4\pi}L(\tau;f)^2 \leqq (1-\alpha)^2\frac{4\pi\tau^2}{(1-\tau^2)^2}.$$

7. Some classes of univalent functions.

Let $*$ denote the Hadamard product. The particular function $\chi(z) = z/(1-z)$ plays the role of identity with respect to $*$ within the class of holomorphic functions vanishing at 0. Since within $\mathcal{F}$ the operators $\mathcal{L}$ and $*$ are commutative, we get

$$\mathcal{L}f = \mathcal{L}(f * \chi) = f * \mathcal{L}\chi.$$

Consequently, $\mathcal{L}$ is represented also by $*\mathcal{L}\chi$ or $\mathcal{L}\chi *$.

Let $\mathcal{S}$, $\mathcal{S}t$ and $\mathcal{K}$ denote the familiar classes of univalent functions. Among several results we state here a typical one.

THEOREM 7. 1. In case of $\mathcal{L}$ generated by t, if $f \in \mathcal{S}$ then $\mathcal{L}^{\lambda}f \in \mathcal{S}t$ at least for $\lambda \geq \lambda_0$ where λ_0 is a certain number less than 4.

In connection with this theorem, it seems plausible that $f \in \mathcal{S}$ implies $\mathcal{L}^{\lambda}f \in \mathcal{S}$ for $\lambda \geq 1$. But it has been pointed out by Owa [26], by referring to a result of Krzyz-Lewandowski [20], that this is not the case. However, another conjecture that $f \in \mathcal{K}$ implies $\mathcal{L}^{\lambda}f \in \mathcal{K}$ for $\lambda \geq 1$ has been affirmed. Related problems have been variously observed by Owa [24], [25], [26].

8. Distortion properties on some linear combinations.

Any $f \in \mathcal{F}$ yields together with its derivatives some related functions within $\mathcal{F}$. In fact, any function F of the form

$$F(z) = \sum_{\kappa=0}^{K} A_{\kappa}\vartheta^{\kappa}f(z) \qquad (f \in \mathcal{F}), \qquad \vartheta = \frac{d}{d \log z},$$

belongs to $\mathcal{F}$, provided complex coefficients A's satisfy $\sum_{\kappa=0}^{K} A_{\kappa} = 1$. Every distortion property on $\mathcal{L}$ valid in $\mathcal{F}$ can be, of course, effectively applied to such F.

For instance, in case of K = 2, generic expression of F is given by

$$F(z) = A_0 f(z) + (1 - A_0 - A_2)\vartheta f(z) + A_2\vartheta^2 f(z)$$

with arbitrary constants A_0 and A_2. Hence we have

$$\mathcal{L}F(z) = A_0\mathcal{L}f(z) + (1 - A_0 - A_2)\vartheta\mathcal{L}f(z) + A_2\vartheta^2\mathcal{L}f(z).$$

In the particular case generated by t, the last expression is simplified in view of $\vartheta\mathcal{L} = \mathrm{id}$. In putting $A_0 = a_0 + ib_0$, $A_2 = a_2 + ib_2$ with real a_0, b_0, a_2 and b_2, we have

$$\mathrm{Re}\,\frac{F(z)}{z} = a_0\,\mathrm{Re}\,\frac{f(z)}{z} + (1 - a_0)\mathrm{Re}\,f'(z) + a_2\,\mathrm{Re}\,zf''(z)$$

$$- b_0\,\mathrm{Im}\,\frac{f(z)}{z} + b_0\,\mathrm{Im}\,f'(z) - b_2\,\mathrm{Im}\,zf''(z),$$

$$\mathrm{Re}\frac{\mathcal{L}F(z)}{z} = a_0\,\mathrm{Re}\frac{1}{z}\int_0^z \frac{f(\zeta)}{\zeta}d\zeta + (1 - a_0 - a_2)\mathrm{Re}\frac{f(z)}{z} + a_2\,\mathrm{Re}\,f'(z)$$

$$- b_0\,\mathrm{Im}\frac{1}{z}\int_0^z \frac{f(\zeta)}{\zeta}d\zeta + (b_0 + b_2)\mathrm{Im}\frac{f(z)}{z} - b_2\,\mathrm{Im}\,f'(z),$$

the value of Φ in Theorem 4. 1 being equal to 2 log 2 - 1.

9. Product of operators.

In dealing with the product of operators of the type $\mathcal{L}(a)$ with several different a's, we restrict ourselves to positive integral powers, and put for the sake of brevity

$$\mathcal{K}(a)^h = \frac{1}{a^h}\mathcal{L}(a)^h \qquad (h = 1, 2, \ldots).$$

It can be shown that for any polynomial P of n variables, the operator of the form $\Omega = P(\mathcal{K}(a_1), \ldots, \mathcal{K}(a_n))$ is expressible as a linear form of $\mathcal{K}(a_\nu)$ $(\nu = 1, \ldots, n)$ together with their derivatives of order less than respective degree in P. Further, a concrete way of describing such a linear form is given as well.

10. Miscellaneous supplements.

By considering the operator $\mathcal{L}[\rho]$ generated by a measure σ of the form

$$\sigma(t) = \int_0^\infty t^a \rho(a)da,$$

Theorem 3. 3 is generalized as follows.

THEOREM 10. 1. By putting $\varphi(t) = \rho(t)/t$, we have

$$\mathcal{L}[\rho] = \sum_{\kappa=1}^{\infty} \frac{(-1)^{\kappa-1}}{(\kappa + 1)!}\varphi^{(\kappa-1)}(1)\mathcal{J}_\kappa .$$

Next, we return to the family $\{\mathcal{L}(a)^\lambda\}$, in which the parameters have initially been subject to $\lambda \geqq 0$ and $a > 0$. The problem is to deal with analytic prolongation with these parameters.

THEOREM 10. 2. The operator $\mathcal{L}(a)^\lambda$ is analytically prolongable with respect to λ and a within single-valuedness into the whole complex pair cut along the negative real axis on the a-plane.

THEOREM 10. 3. The operator $\mathcal{L}(a)$ is inverse to $\Theta(a) = a^{-1}(d/d \log z + a - 1)$.

THEOREM 10. 4. The operator $\mathcal{L}(a)^{\lambda}$ with $\mathrm{Re}\,\lambda \leqq 0$ is expressible in terms of $\mathcal{L}(a)^{\mu}$ with $\mathrm{Re}\,\mu > 0$ in the form $\mathcal{L}(a)^{\lambda} = \Theta(a)^{m}\mathcal{L}(a)^{\lambda+m}$, where m is any positive integer satisfying $m > [-\mathrm{Re}\,\lambda]$.

Finally, we note that it is reasonable to define $\Theta(a)^{\lambda}$ with complex λ by means of $\Theta(a)^{\lambda} = \mathcal{L}(a)^{-\lambda}$. On the other hand, the operator $z^{-1}\Theta(a)^{\lambda}$ with $\lambda > 0$ and $a > 0$ may be regarded as a particular Gel'fond-Leont'ev derivative introduced in [2].

REFERENCES

[1] S.D. BERNARDI, Trans. Amer. Math. Soc. 135 (1969), 429-446. [2] A.D. GEL'FOND-A.F. LEONT'EV, Math. Sbornik 29 (71) (1951), 477-500. [3] F. HAUSDORFF, Math. Z. 9 (1921), 74-109; 280-299. [4] P. KOEBE, Math. Z. 6 (1920), 52-84. [5] Y. KOMATU, Bull. Fac. Sci.-Engrg., Chuo Univ. 22 (1979), 1-22. [6] Y. KOMATU, Lecture Notes in Math. No. 768 (1980), 292-300. [7] Y. KOMATU, Bull. Fac. Sci.-Engrg., Chuo Univ. 24 (1981), 1-6. [8] Y. KOMATU, Bull. Fac. Sci.-Engrg., Chuo Univ. 24 (1981), 7-12. [9] Y. KOMATU, Bull. Fac. Sci.-Engrg., Chuo Univ. 25 (1982), 45-49. [10] Y. KOMATU, Ann. Polon. Math. 46 (1985), 141-145. [11] Y. KOMATU, J. Math. Kyoto Univ. 25 (1985), 627-633. [12] Y. KOMATU, Compl. Var. 7 (1986), 97-106. [13] Y. KOMATU, Compl. Anal. Appl. '85 Sofia (1986), 327-334. [14] Y. KOMATU, Kodai Math. J. 10 (1987), 20-38. [15] Y. KOMATU, Mat. Vesnik 39 (1987), 399-404. [16] Y. KOMATU, Saitama Math. J. 6 (1988), 1-7. [17] Y. KOMATU, Compl. Anal. Appl. '87 Sofia (to appear). [18] Y. KOMATU, Comm. in Conf. Compl. Anal. '88 Halle. [19] Y. KOMATU, Proc. Colloq. Compl. Anal. '89 Bucharest (to appear). [20] J. KRZYŻ-Z. LEWANDOWSKI, Bull. Acad. Polon. Sci. 11 (1963), 447-448. [21] R.J. LIBERA, Proc. Amer. Math. Soc. 16 (1965), 755-758. [22] A.E. LIVINGSTON, Proc. Amer. Math. Soc. 17 (1966), 352-357. [23] C. NEUMANN, Vorlesungen über Riemanns Theorie der Abelschen Integrale, Leipzig (1884). [24] S. OWA, Math. Japon. 29 (1984), 859-873. [25] S. OWA, J. Korean Math. Soc. 21 (1984), 177-181. [26] S. OWA, Diss., Nihon Univ. (1986). [27] W. ROGOSINSKI, Math. Z. 35 (1932), 93-121. [28] J.A. SHOHAT-J.D. TAMARKIN, The problem of moments, New York (1943). [29] H.M. SRIVASTAVA-S. OWA, J. Math. Appl. 118 (1986), 80-87.

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