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An inequality for analytic functions

D.K.THOMAS (ウェールズ大学)

INTRODUCTION

Denote by A the class of functions which are analytic in the unit disc $D = \{z : |z| < 1\}$ and are normalised so that $f(0) = 0$ and $f'(0) = 1$. In [3], Obradovič showed that if $f \in A$ and satisfies $\operatorname{Re} f(z)/z > \alpha$ for $\alpha < 1$, then

$$\operatorname{Re} \frac{a+1}{z^{a+1}} \int_0^z t^{a-1} f(t)dt > \alpha + \frac{1-\alpha}{3+2a}$$

for $a > -1$ and $z \in D$. This result, (which is not sharp) was then used by Odradovič to establish certain non-sharp lower bound estimates for the real parts of some integral operators of functions in various classes of univalent functions.

In this note, we prove the sharp version of Obradivič's result and give a generalisation. The method is quite elementary. Other applications of the method have been given in [1] and [2].

RESULTS

Let $f \in A$ and $z \in D$. For $n = 1, 2...$, and $a > -1$, define

$$I_n(z) = \frac{a+1}{z^{a+1}} \int_0^z t^a I_{n-1}(t)dt,$$

where $I_0(z) = f(z)/z$.

We prove:

THEOREM. *Let $f \in A$ and $\alpha < 1$. Then for $n \geq 0$ and $z \in D$, the inequality* $\operatorname{Re} f(z)/z > \alpha$ *implies*

$$\operatorname{Re} I_n(z) \geq \gamma_n(r) > \gamma_n(1),$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

where

$$0 < \gamma_n(r) = 1 + 2(a+1)^n(1-\alpha)\sum_{j=1}^{\infty}\frac{(-r)^j}{(j+a+1)^n} < 1. \tag{1}$$

Equality is attained when $\dfrac{f(z)}{z} = \alpha + (1-\alpha)\dfrac{1-z}{1+z}$.

PROOF: The case $n = 0$ is trivial. Suppose that $n = 1$, then since $\operatorname{Re} f(z)/z > \alpha$, we have, with $z = re^{i\theta}$,

$$\frac{\partial}{\partial r}\int_0^z t^{a-1}f(t)dt = z^a\frac{f(z)}{z}e^{i\theta} = z^a e^{i\theta}[\alpha + (1-\alpha)h(z)],$$

where $\operatorname{Re} h(z) > 0$ for $z \in D$.

Thus integrating and noting that $\operatorname{Re} h(z) \geq \dfrac{1-\rho}{1+\rho}$ for $0 \leq \rho < 1$, it follows that if $a > -1$,

$$\begin{aligned}
\operatorname{Re}\frac{a+1}{z^{a+1}}\int_0^z t^{a-1}f(t)dt &\geq \frac{a+1}{r^{a+1}}\int_0^r \rho^a\left[\alpha + (1-\alpha)\left(\frac{1-\rho}{1+\rho}\right)\right]d\rho,\\
&= \frac{a+1}{r^{a+1}}\int_0^r \rho^a(1 + 2(1-\alpha)\sum_{j=1}^{\infty}(-\rho)^j)d\rho,\\
&= 1 + 2(a+1)(1-\alpha)\sum_{j=1}^{\infty}\frac{(-r)^j}{j+a+1},
\end{aligned}$$

which proves the theorem in the case $n = 1$.

Next note that writing $t = \rho e^{i\theta}$, we have

$$\begin{aligned}
\operatorname{Re} I_{n+1}(z) &= \operatorname{Re}\frac{a+1}{z^{a+1}}\int_0^z t^a I_n(t)dt,\\
&= \frac{a+1}{r^{a+1}}\int_0^r \rho^a \operatorname{Re} I_n(\rho e^{i\theta})d\rho,\\
&\geq \frac{a+1}{r^{a+1}}\int_0^r\left(\rho^a + 2(a+1)^n(1-\alpha)\sum_{j=1}^{\infty}\frac{(-1)^j\rho^{j+a}}{(j+a+1)^n}\right)d\rho,\\
&= \gamma_{n+1}(r),
\end{aligned}$$

where the inequality follows by induction.

Rearranging terms in the infinite series given in (1), shows that $0 < \gamma_n(r) < 1$ and the proof is complete.

References

1. S.A.Halim and D.K.Thomas, *A note on Bazilevič functions*, to appear.
2. S.A.Halim and D.K.Thomas, *Sharp estimates for some integral operators of convex functions of order alpha*, to appear.
3. M.Obradovič, *On certain inequalities for some regular functions in the unit disc*, Internat. J. Math. and Math. Sci. **8** (1985), 677–681.

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Department of Mathematics and Computer Science, University College, Swansea SA2 8PP, Wales, U.K.