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# ON UNIVALENT FUNCTIONS IN MULTIPLY CONNECTED DOMAINS*

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The present article gives an account of some results on univalent functions in multiply connected domains obtained by author. The contents are

1. Two very simple proofs of Villat's formula
2. Schwarz's formula, Poisson's formula and Poisson-Jensen formula in multiply connected domains
3. Differentiability with respect to the parameter of analytic function family containing one parametric variable
4. Variation theorem and parametric representation theorem
5. Extremal problem of differentiable functionals

## 1. TWO VERY SIMPLE PROOFS OF VILLAT'S FORMULA

By Schwarz's formula of analytic functions in disks we obtain

Lemma 1.1 Let $B=\{z:|z-a|<r\}$, $E=\{z:|z-a|>r\}$, then

$$\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\,(f(a+re^{i\theta}))\frac{re^{i\theta}+(z-a)}{re^{i\theta}-(z-a)}\,d\theta=\begin{cases} f(z)-i\,\mathrm{Im}\,(f(a)) & \text{for (i)}\\ -f(z)+i\,\mathrm{Im}\,(f(\infty)) & \text{for (ii)}\\ -\bar f\left(\bar a+\dfrac{r^2}{z-a}\right)-i\,\mathrm{Im}\,(f(a)) & \text{for (iii)}\\ \bar f\left(\bar a+\dfrac{r^2}{z-a}\right)+i\,\mathrm{Im}\,(f(\infty)) & \text{for (iv)}\end{cases}$$

here (i): $f$ is analytic in $B$ and continuous in $\overline{B}$, $z\in B$;
(ii): $f$ is analytic in $E$ and continuous in $\overline{E}$, $z\in E$;
(iii): $f$ is analytic in $B$ and continuous in $\overline{B}$, $z\in E$;
(iv): $f$ is analytic in $E$ and continuous in $\overline{E}$, $z\in B$.

By the Schwarz basic theorem of Dirichlet's problem we obtain

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Lemma 1.2 Let U $(\theta)$ be integrable with period $2\pi$ and continuous at $\theta=\theta_0$ . Then

$$\lim_{z\to\rho e^{i\theta_0}} \frac{1}{2\pi}\int_0^{2\pi} U(\theta)\,\mathrm{Re}\left(\frac{\rho e^{i\theta}+z}{\rho e^{i\theta}-z}\right)d\theta=\pm U(\theta_0),$$

the sign of the right hand is positive when z tends to $\rho e^{i\theta_0}$ inside $|z|=\rho$ non-tangentially and negative when outside.

Villat § formula (see [6], [7]) Let $f(z)=u(z)+iv(z)$ be analytic in $q<|z|<1$ and continuous in $q\leqslant|z|\leqslant 1$ . Then

$$f(z)=\sum_{m=1}^{2}\frac{1}{2\pi}\int_0^{2\pi} u(\xi_m)\,K_m(z,\xi_m)\,d\theta-C+iD,\quad q<|z|<1 \tag{1.1}$$

where $\xi_1=e^{i\theta}$, $\xi_2=qe^{i\theta}$, and

$$K_1(z,\xi_1)=-\frac{2\omega}{\pi i}\zeta\left(\frac{\omega}{\pi i}\log\frac{z}{\xi_1}\right)-\frac{2\omega}{\pi^2}\log\frac{z}{\xi_1}$$

$$=\frac{\xi_1+z}{\xi_1-z}+2\sum_{k=1}^{\infty}\frac{q^{2k}}{1-q^{2k}}\left[\left(\frac{z}{\xi_1}\right)^k-\left(\frac{\xi_1}{z}\right)^k\right] \tag{1.2}$$

$$K_2(z,\xi_2)=-K_1(z,\xi_2); \tag{1.3}$$

here $\zeta(u)$ is the Weierstrass function with real and imaginary periods $2\omega$ and $2\omega'$ satisfying $\frac{\omega'}{i\omega}=\frac{1}{\pi}\log q$; C, D are real constants with

$$C+iD=\frac{1}{2\pi i}\int_{|z|=\rho}\frac{f(z)}{z}dz,\quad q\leqslant\rho\leqslant 1 \tag{1.4}$$

The original proof given by Villat is very long [14]. After Villat some different proofs have been given, for instance, by G. M. Golusin [5]. By rewriting (1.2) and (1.3) as the following

$$K_1(z,\xi_1)=\frac{\xi_1+z}{\xi_1-z}+\sum_{k=1}^{\infty}\left(\frac{\xi_1+q^{2k}z}{\xi_1-q^{2k}z}+\frac{\xi_1+q^{-2k}z}{\xi_1-q^{-2k}z}\right),\quad \xi_1=e^{i\theta}; \tag{1.5}$$

$$K_2(z,\xi_2)=-\frac{\xi_2+z}{\xi_2-z}-\sum_{k=1}^{\infty}\left(\frac{\xi_2+q^{2k}z}{\xi_2-q^{2k}z}+\frac{\xi_2+q^{-2k}z}{\xi_2-q^{-2k}z}\right),\quad \xi_2=qe^{i\theta}. \tag{1.6}$$

we give two proofs of (1.1) which may be simplest.

Proof I Let $f=f_1+f_2$ where $f_1$ is the sum of all nonnegative powers of the Laurent expansion of $f$ in $q<|z|<1$ and $f_2$ is that of all negative powers. By the termwise integration, (1.1) follws from (1.5), (1.6) and Lemma 1.1.

Proof II Note that $\mathrm{Re}\,(K_1(z,\xi_1))=0$ on $|z|=1$ when $z\neq\xi_1$, and $=1$ on $|z|=q$; $\mathrm{Re}\,(K_2(z,\xi_2))=1$ on $|z|=1$, and $=0$ on $|z|=q$ when $z\neq\xi_2$, Using Lemma 1.2 we compute the non-tangential limits of the real part of the right hand of (1.1) as the following

$$\lim_{z\to e^{i\varphi}} \mathrm{Re}\{\text{ the right hand of (2.1)}\}$$

$$= \lim_{z\to e^{i\varphi}} \frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\,\mathrm{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)d\theta$$

$$+\frac{1}{2\pi}\int_0^{2\pi} u(e^{i\theta})\lim_{z\to e^{i\varphi}}\mathrm{Re}\left(K_1(z,\xi_1)-\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)d\theta$$

$$+\frac{1}{2\pi}\int_0^{2\pi} u(qe^{i\theta})\lim_{z\to e^{i\varphi}}\mathrm{Re}\,(K_2,\xi_2))d\theta-C$$

$$= u(e^{i\varphi})+\frac{1}{2\pi}\int_0^{2\pi} u(qe^{i\theta})d\theta-C$$

$$= u(e^{i\varphi})$$

and similarly,

$$\lim_{z\to qe^{i\varphi}} \mathrm{Re}\{\text{the right hand of (2.1)}\}=u(qe^{i\varphi}).$$

It shows that the right hand of (1.1) which is analytic in $q<|z|<1$ is of same real part with $f(z)$ on the boundary of the annulus, and then (1.1) is true (see [1]).

## 2. SCHWARZ 'S FORMUIA, POISSON 'S FORMUIA AND POISSON-JENSEN FORMULA IN MULTIPLY CONNECTED DOMAINS

Villat § formula is a generalized form of Schwarz § formula in annuli. It is easy to give Schwarz § formula of analytic functions in n-connected circular domains by considering geometric behaviour of integral kernels of Schwarz § formula and Villat § formula. The method in Proof II of (1.1) applies to the general case. (see [15], [17], [18])

Let $R_z$ denote an n-connected domain in z-plane bounded by circles

$$C_j: |z-a_j|=r_j, \qquad j=1,2,\cdots,n.$$

For $\xi_j\in C_j$, let $K_j(z;\xi_j)$ be the conformal mapping of $R_z$ onto the right half plane cut by $n-1$ straight segments parallal to the imaginary axis, $C_j$ to the imaginary axis, which is analytic on $\overline{R}_z$ except at the simple pole $\xi_j$ with the following expansion around the point

$z=\xi_j$

$$K_j(z;\xi_j)=\pm\frac{\xi_j+z-2a_j}{\xi_j-z}+\sum_{K=1}^{\infty}b_{jk}\left(\frac{\xi_j-z}{\xi_j+z-2a_j}\right)^K \tag{2.1}$$

the sign of the right hand is positive when $R_z$ lies inside $C_j$ and negative when outside, $b_{jk}$ is determined by $R_z$ and $\xi_j$. It is easy to prove the existence and the unicity of the mapping functions $K_j(z;\xi_j)$, $j=1,2,\cdots n$. (see [3], [15])

Theorem 2.1 Let $f(z)=u(z)+iv(z)$ be analytic in $R_z$ and continuous in $\overline{R}_z$. Then in $R_z$ we have the Schwarz representation

$$f(z)=\sum_{j=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}u(\xi_j)K_j(z;\xi_j)\,d\theta-C+iD \tag{2.2}$$

where $\xi_j=a_j+r_je^{i\theta}$, C and D are real constants, and

$$C=\mu_1=\mu_2=\cdots=\mu_n \tag{2.3}$$

here

$$\mu_j=\sum_{m=1}^{n}\alpha_m\beta_{mj}, \tag{2.4}$$

$$\alpha_m=\frac{1}{2\pi}\int_0^{2\pi}u(\xi_m)\,d\theta, \tag{2.5}$$

$$\beta_{mj}=\begin{cases}0 & \text{if } j=m\\ \operatorname{Re}(K_m(\xi_j;\xi_m)) & \text{if } j\neq m.\end{cases} \tag{2.6}$$

Proof By using the method in Proof II of (1.1), it follows that on the circle $z=a_j+r_je^{i\theta}$, $0\leqslant\theta\leqslant 2\pi$, the real part of the analytic function

$$\sum_{j=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}u(z_j)K_j(z;\xi_j)\,d\theta \tag{2.7}$$

is $u(\xi_j)+\mu_j$, $j=1,2,\cdots,n$.

Let $\omega_j(z)$ be the harmonic measure of $C_j$ at the point z with respect to the domain $R_z$, and $\varphi_j(z)$ be an analytic function in $R_z$ with $\omega_j(z)$ as its real part. Then in $R_z$ we have

$$f(z)=\sum_{j=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}u(\xi_j)K_j(z;\xi_j)\,d\theta-\sum_{j=1}^{n}\mu_j\varphi_j(z)+i\alpha \tag{2.8}$$

where $\alpha$ is a real constant.

Because $f(z)$ and (2.7) are single-valued in $R_z$, then

$$\psi(z)=\sum_{j=1}^{n}\mu_j\varphi_j(z)-i\alpha$$

is also single-valued in $R_z$ and then is analytic in $\overline{R_z}$. The real part of $\psi(z)$ on $C_j$ is the constant $\mu_j$, that is, the image of $C_j$ lies on a line $d_j$, $j=1, 2, \cdots, n$. Arbitrarily give a point $\zeta_0$ which does not lie on any $d_j$, to apply the argument principle to $\psi(z)-\zeta_0$. We obtain $\psi(z)\neq\zeta_0$ in $R_z$. And then $\psi(z)=\text{const}$, thus we have $\mu_1=\mu_2=\cdots=\mu_n$. Therefore, (2.2) follows from (2.8).

Theorem 2.2 Let $u(z)$ be harmonic in $R_z$ and continuous in $\overline{R_z}$. Then the conjugate harmonic function is single-valued in $R_z$ if and only if $\mu_j$ is independent of the lower index j where $\mu_j$ is determined by (2.4)–(2.6).

Proof If the conjugate harmonic function of $u(z)$ is single-valued then, by Theorem 2.1, the condition (2.3) holds.

Inversely, assume that (2.3) holds. Using again the method in Proof II of (1.1), for any given $\xi_0\in c_{j_0}$, we consider the non-tangential limit of the harmonic function

$$\sum_{j=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}u(\xi_j)\mathrm{Re}(K_j(z;\xi_j))d\theta \tag{2.9}$$

as z tends to $\xi_0$ in $R_z$. Note that by Lemma 1.2,

$$\lim_{z\to\xi_0}\frac{1}{2\pi}\int_0^{2\pi}u(\xi_{j_0})\,\mathrm{Re}\left(\frac{\xi_{j_0}+z-2a_{j_0}}{\xi_{j_0}-z}\right)d\theta=\pm u(\xi_0) \tag{2.10}$$

the sign of the right hand is positive when $R_z$ lies inside $C_{j_0}$ and negative when outside, and by (2.4)–(2.6)

$$\begin{aligned}&\lim_{z\to\xi_0}\sum_{\substack{j=1\\ j\neq j_0}}\frac{1}{2\pi}\int_0^{2\pi}u(\xi_j)\mathrm{Re}(K_j(z;\xi_j))d\theta\\ &=\sum_{\substack{j=1\\ j\neq j_0}}\frac{1}{2\pi}\int_0^{2\pi}u(\xi_j)\lim_{z\to\xi_0}\mathrm{Re}(K_j(z;\xi_j))d\theta\\ &=\mu_{j_0}=C.\end{aligned} \tag{2.11}$$

Therefore, the non-tangential limit of (2.9) is $u(\xi_0)+C$, and so $u(z)$ is the real part of an analytic function, that is,

$$u(z)=\mathrm{Re}\left\{\sum_{i=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}u(\xi_j)K_j(z;\xi_j)d\theta-C\right\} \tag{3.12}$$

Theorem 2.3 Let $u(z)$ be harmonic in $R_z$ and continuous in $\overline{R_z}$. Then in $R_z$ we have the Poisson representation

$$u(z)=\sum_{j=1}^{n}\frac{1}{2\pi}\int_{0}^{2\pi}u(\xi_j)\mathrm{Re}\,(K_j(z;\xi_j))d\theta-\sum_{j=1}^{n}\mu_j\omega_j(z) \tag{3.13}$$

where $\mu_j$ is defined by (2.4)-(2.6), $\omega_j(z)$ is the harmonic measure of $C_j$.

Proof  Let

$$U(z)=u(z)-\sum_{j=1}^{n}\alpha_j\omega_j(z) \tag{2.14}$$

which is harmonic in $R_z$ and its integral mean value on every $C_j$ is 0. By Theorem 2.2 and (2.12) we have

$$U(z)=\sum_{j=1}^{n}\frac{1}{2\pi}\int_{0}^{2\pi}u(\xi_j)\mathrm{Re}\,(K_j(z;\xi_j))d\theta-\sum_{j=1}^{n}\frac{1}{2\pi}\int_{0}^{2\pi}\alpha_j\mathrm{Re}\,(K_j(z;\xi_j))d\theta. \tag{2.15}$$

Let $\beta_{jj}'=1$, $\beta_{jk}'=\beta_{jk}$ if $j\neq k$, then

$$\frac{1}{2\pi}\int_{0}^{2\pi}\mathrm{Re}\,(K_j(z;\xi_j))d\theta=\sum_{k=1}^{n}\beta_{jk}'\omega_K(z).$$

Therefore,

$$\begin{aligned}
&\sum_{j=1}^{n}\frac{1}{2\pi}\int_{0}^{2\pi}\alpha_j\mathrm{Re}\,(K_j(z;\xi_j))d\theta\\
&=\sum_{j=1}^{n}\alpha_j\sum_{k=1}^{n}\beta_{jk}'\omega_K(z)\\
&=\sum_{K=1}^{n}\omega_K(z)\sum_{j=1}^{n}\alpha_j\beta_{jk}'\\
&=\sum_{k=1}^{n}\omega_k(z)(\alpha_k+\mu_k)
\end{aligned} \tag{2.16}$$

and then (2.13) follows from (2.14)-(2.16).

Along the same way, we obtain the Schwarz basic theorem in n-connected domains and an integral representation of the solution of the Dirichlet problem. (see [17])

Theorem 2.4 Let $u(\zeta)$ be a real valued function defined on the boundary of $R_Z$ and integrable as a function of $\theta$ on every $C_j$. Then

$$U(z)=\sum_{j=1}^{n}\frac{1}{2\pi}\int_{0}^{2\pi}u(\xi_j)\mathrm{Re}\,(K_j(z;\xi_j))d\theta-\sum_{j=1}^{n}\mu_j\omega_j(z) \tag{2.17}$$

is harmonic in $R_z$ where $\xi_j = a_j + r_j e^{i\theta}$. If $u(\zeta)$ is continuous at boundary point $\zeta_0$ then the non-tangential limit of $U(z)$ at $\zeta_0$ is $u(\zeta_0)$.

Theorem 2.5 Let $\Omega$ be an n-connected domain, the boundary of $\Omega$ be locally connected and every boundary component be not a single point. Let $z = f(w)$ map $\Omega$ onto an n-connected circular domain $R_Z$. Then the solution of Dirichlet's problem in $\Omega$ with a continuous boundary value function $u(\zeta)$ is

$$U(w) = \sum_{j=1}^{n} \frac{1}{2\pi} \int_0^{2\pi} u(f^{-1}(\xi_j)) \mathrm{Re}(K_j(f(w);\xi_j)) d\theta - \sum_{j=1}^{n} \mu_j \omega_j(f(w)). \qquad (2.18)$$

where $\mu_j$ is defined by (8.4) – (8.6) but $u'(\xi_j)$ is replaced by $u(f^{-1}(\xi_j))$.

Finaly, we give the Poisson-Jensen formular in n-connected domains (see [18]).

Theorem 2.6 Let $f(z)$ be meromorphic in $\overline{R_Z}$ to have zeros at $a_1, a_2, \cdots, a_m$ and poles at $b_1, b_2, \cdots, b_p$ in $R_Z$, and have no zeros and poles on the boundary. Then for $z_0 \in R_Z$ to be distinct from the zeros and the poles we have

$$\log|f(z_0)| = \sum_{j=1}^{n} \frac{1}{2\pi} \int_0^{2\pi} \log|f(\xi_j)| \mathrm{Re}(K_j(z_0;\xi_j)) d\theta + \sum_{i=1}^{m} \log|S(z_0, a_i)| - \sum_{k=1}^{p} \log|S(z_0, b_K)| - K \qquad (2.19)$$

where

$$K = \sum_{i=1}^{m} \log|S(\xi_j, a_i)| - \sum_{k=1}^{p} \log|S(\xi_j, b_k)| + \mu_j, \qquad (2.20)$$

$S(z, \alpha)$ is the conformal mapping to map $R_Z$ onto the unit disk cut by $n-1$ concentric circular arcs, $\alpha$ to 0, $C_n$ to the unit circle; $\mu_j$ is defined by (2.4) – (2.6) but $u'(\xi_j)$ is replaced by $\log|f(\xi_j)|$.

## 3. DIFFERENTIABILITY WITH RESPECT TO THEPAPAMETER OF ANALYTIC FUNCTION FAMILY CONTAINING ONE PARAMETRIC VARIABLE

Suppose that $G(t)$, $a \leqslant t \leqslant b$, is a domain family in z-plane, and function $f(z,t)$ is defined in $G(t)$. Let $t_0 \in [a,b]$ be a fixed value. $f(z,t)$ is called uniformly continuous for t at $t = t_0$ with respect to $E \subset G(t_0)$ if there exists an $\eta > 0$ such that

$$E \subset G(t) \qquad \text{for} \quad |t - t_0| < \eta, \ t \in [a,b]$$

and if for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$, $\delta < \eta$, such that

$$|f(z,t) - f(z,t_0)| < \varepsilon \qquad \text{for} \quad z \in E, |t - t_0| < \delta, t \in [a,b].$$

If for any $z \in G(t_0)$ there is a neighbourhood E of z satisfying the condition, then $f(z,t)$ is called locally uniformly continuous for t at $t = t_0$. The uniform differentiability and the locally uniform differentiability or $f(z,t)$ can be defined similarly.

Now let the n-connected domain family $G(t)$ be given, $a \leqslant t \leqslant b$, and satisfy the following presuppositions:

1) $0, \infty \notin G(t)$;

2) the boundary $\Gamma(t)$ of $G(t)$ consists of n disjoint closed Jordan curves $z=\Omega_m(\theta, t)$, $\theta \in [0, 2\pi]$, $m=1, 2, \cdots, n$;

3) the function $\Omega_m(\theta, t)$ is uniformly differentiable for t at $t=t_0$ with respect to the interval $[0, 2\pi]$, where $t_0 \in [a, b]$ is a fixed value;

4) the n curves of $\Gamma(t_0)$ are analytic.

We investigate the univalent analytic function family $w=F(z, t)$ defined in $G(t)$, whose image domain family $B(t)$, $a \leqslant t \leqslant b$, satisfying the following presuppositions;

1) $0, \infty \notin B(t)$;

2) the boundary of $B(t)$ consists of n analytic Jordan curves $w=\sigma_m(\theta, t)$, $\theta \in [0, 2\pi]$, $m=1, 2, \cdots, n$;

3) the function $\sigma_m(\theta, t)$ is uniformly differentiable for t at $t=t_0$ with respect to the interval $[0, 2\pi]$;

4) $B(t_0)$ is an n-connected circular domain $R_w$.

Then We have

Theorem 3.1 Let $F(z, t)$, $G(t)$ and $B(t)$ satisfy the presuppositions. Then $F(z, t)$ is locally uniformly differentiable for t at $t=t_0$ with respect to $G(t_0)$ and so is the inverse function $z=\Phi(w, t)$ to $B(t_0)$, furthermore.

$$\frac{\partial \Phi(w, t)}{\partial t}\Big|_{t=t_0} = -w \frac{\partial \Phi(w, t_0)}{\partial w} \left\{ \sum_{m=1}^{n} \frac{1}{2\pi} \int_0^{2\pi} L_m(\theta) K_m(w; \xi_m) d\theta - C + iD \right\} \tag{3.1'}$$

where $\xi_m$, $K_m(w, \xi_m)$ are determined as that in Section 2 but the domain here is $R_w$; C, D are real constants, The value C is given by (2.3)–(2.6) but to substitute $L_m(\theta)$ for $u(\xi_m)$, and

$$L_m(\theta) = \mathrm{Re}\left\{ \frac{\partial}{\partial t}\left[ \log|\sigma_m(\theta, t)| - \frac{\Omega_m(\theta, t)}{\xi_m \dfrac{\partial \Phi(\xi_m; t_0)}{\partial \xi_m}} \right] \right\}_{t=t_0}. \tag{3.2}$$

Proof We may assume $F(z, t_0) \equiv z$. Set

$$\begin{aligned} f(\zeta(t), t) &= \frac{1}{t-t_0} \log|F(\zeta(t), t)/\zeta(t)|, \quad t \neq t_0, \ \zeta(t) \in \Gamma(t) \\ f(\zeta(t_0), t_0) &= \left\{ \frac{\partial}{\partial t} \log|F(\zeta(t), t)/\zeta(t)| \right\}_{t=t_0}, \quad \zeta(t_0) \in \Gamma(t_0) \end{aligned} \tag{3.3}$$

By the assumption of the theorem, there exists the solution $u(z, t)$ of Dirichlet's problem in $G(t)$ with the boundary value $f(\zeta(t), t)$. Obviously, for $t \neq t_0$ we have

$$u(z,t)=\frac{1}{t-t_0}\log|F(z,t)/z|,\quad z\in G(t). \tag{3.4}$$

As Theorem 8 of [8] , it follows that $u(z,t)$ is locally uniformly continuous for $t$ at $t=t_0$ with respect to $G(t_0)$. And so is the real part of the function

$$\varphi(z,t)=\log F(z,t)/z=P(z,t)+iQ(z,t). \tag{3.5}$$

$Q(z,t)$ has the same property by the same reason. Therefore, $\varphi(z,t)$ is locally uniformly differentiable for $t$ at $t=t_0$ respect with to $G(t_0)$ and then, by the Weierstrass theorem, the derivative function is analytic in $G(t_0)$, It is easy to see that the function (3.5) is continuous in $\overline{G(t_0)}$ except at most several boundary points.

Differentiating the equality

$$F(z,t)=ze^{\varphi(z,t)} \tag{3.6}$$

with respect to $t$ at $t=t_0$, applying (2.2) to (3.5) and then removing the assumption of $F(z,t_0)\equiv z$ we obtain that

$$\frac{\partial F(z,t)}{\partial t}\Big|_{t=t_0}=F(z,t_0)\left\{\sum_{m=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\partial}{\partial t}\log\left|\frac{\sigma_m(\theta,t)}{F(\Omega_m(\theta,t),t_0)}\right|\right]_{t=t_0}\cdot K_m(F(z,t_0);\xi_m)d\theta-C+iD\right\}. \tag{3.7}$$

As Theorem 5 of [8] ,it follows that $\Phi(w,t)$ is locally uniformly differentiable for $t$ at $t=t_0$ with respect to $B(t_0)$ and

$$\frac{\partial\Phi(w,t)}{\partial t}\Big|_{t=t_0}=-\frac{\partial\Phi(w,t_0)}{\partial w}\cdot\frac{\partial F(\Phi(w,t_0),t)}{\partial t}\Big|_{t=t_0} \tag{3.8}$$

Then we obtain (3.1) from (3.7) and (3.8).

When $n=2$, the theorem is just a result of P. P. Kufarev and N. B. Semuchina [9] (with a little improvement ). Similarly we can prove that

<u>Theorem 3.2</u> In Theorem 3.1, if we assume that the boundary of $G(t)$ and $B(t)$ are some continuous curves ($a\leqslant t\leqslant b$, $t\neq t_0$) and remove the condition 4) required by $B(t)$, then the functions $\Phi(w,t)$ and $F(a,t)$ are still locally uniformly differentiable for the parameter $t$ at $t=t_0$. (See [16] )

## 4. VARIATION THEOREM AND PARAMETRIC REPRESENTATION THEOREM

Let $R_w^{(\varepsilon)}$ denote a subregion of an n-connected region $R_w$ with circle boundary. The complement set $R_w\setminus R_w^{(\varepsilon)}$ are n semi-closed annuli $Q_k$. The distance betweem the two circles of each annulus is $\varepsilon$.

Let $\psi_K(w,t)$, $k=1,2,\cdots,n$, denote n nuivalent functions containing parameter t defined in n annuli $Q_K$ respectively. The boundary curves of the image domain of $Q_K$ are denoted by $\Gamma_{r_k}(t)$ and $\Gamma^{(k)}(t)$, Suppose that the imege regions of these annuli do not intersect each other and that $G^{(\varepsilon)}(t)\subset G(t)$ where $G(t)$ is the n-connected region bounded by n curves $\Gamma_{r_k}(t)$ corresponding to the boundary circles of $R_w$ and $G^{(\varepsilon)}(t)$ denote the n-connected region bounded by the n curves $\Gamma^{(k)}(t)$ corresponding to the n boundary circles of $R_w^{(\varepsilon)}$.

Now we establish the variation theorem and the parametric representation theorem for univalent functions in n-connected regions which generalize the results of [4], [5] and [9]-[12].

Throrem 4.1 Suppose that the function $f(w)$ is anslytic and univalent in $R_w$ and that when $T>0$ is sufficiently small, the function $\psi_k(w,t)$ has the following expansion in $Q_K$ for $t\in[0,T]$

$$\psi_k(k,t)=f(w)+tg_K(w)+o(t),\ k=1,2,\cdots,n \tag{4.1}$$

where $g_K(w)$ is well-defined in $\overline{Q}_K$. Next suppose that $w=F(w,t)$ maps $G(t)$ onto $R_w(t)$ one-to-one and conformally, $R_w(0)=R_w$, and that the centers $a_j(t)$ and radii $r_j(t)$, $j=1,2,\cdots,n$, of the n boundary circles $C_j(t)$ of $R_w(t)$ are differentiable for t at $t=0$. Let $\Phi(w,t)$ be the inverse function of $F(z,t)$. Then the following expansion holds in $R_w(t)$

$$\Phi(w,t)=f(w)+twf'(w)P(w)+o(t) \tag{4.2}$$

where $o(t)$ is uniform with respest to every closed set of $R_w$, and

$$P(w)=\sum_{j=1}^{n}\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_0^{2\pi}B_j(\xi_j)K_j(w;\xi_j)d\theta-C+iD, \tag{4.3}$$

$$B_j(\xi_j)=\mathrm{Re}\left(\frac{g_j(\xi_j)}{\xi_j f'(\xi_j)}\right)-\left[\frac{\partial}{\partial t}\log|a_j(t)+r_j(t)e^{j\theta}|\right]_{t=0} \tag{4.4}$$

$K_j(w;\xi_j)$ is defined as that in section 2 but here the region is $R_w^{(\varepsilon)}$, $\xi_j$ is a variable point on the j-th boundary circie of $R_w^{(\varepsilon)}$ with $\arg(\xi_j-a_j)=\theta$. C, D are real constants; the value of C is given by (2.3)-(2.6) but to substitute $B_m(\xi_m)$ for $u(\xi_m)$.

Proof It follows fromTheorem 3.2 that $F(z,t)$ is unifromly differentiable for t at $t=0$ on every closed subset of $G(0)$.

Denote the image region of $G^{(\varepsilon)}(t)$ under the mapping $F(z,t)$ by $B^{(\varepsilon)}(t)$. Applying Theorem 3.1 to $G^{(\sigma)}(t)$, $B^{(\varepsilon)}(t)$ and $F(z,t)$, we obtain that $\Phi(w,t)$ is locally uniformly differentiable for t at $t=0$ with respect to $R_w^{(\varepsilon)}$ and that the following equality holds

$$\left.\frac{\partial\Phi(w,t)}{\partial t}\right|_{t=0}=-wf'(w)\left\{\sum_{m=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\partial}{\partial t}\log\left|\frac{F(\Omega_m(\theta,t;\varepsilon),t)}{f^{-1}(\Omega_m(\theta,t;\varepsilon))}\right|\right]_{t=0}K_m(w;\xi_m)d\theta-C+iD\right\} \tag{4.5}$$

where $\Omega_m(\theta, t; \varepsilon)$ is the parametric representation of $\Gamma^{(m)}(t)$, Let $\varepsilon \to 0$ in (4.5), we obtain that

$$\frac{\partial \Phi(w,t)}{\partial t}\Big|_{t=0} = w f'(w) p(w) \qquad (4,6)$$

Hence in $R_w(t)$

$$\Phi(w,t) = \Phi(w,0) + t\frac{\partial \Phi(w,t)}{\partial t}\Big|_{t=0} + o(t)$$
$$= f(w) + t w f'(w) p(w) + o(t) \quad .$$

and we have also proved that $o(t)$ is locally uniform in $R_w$.

When $n=2$, Theorem 4.1 is just the result of [9].

Theorem 4.2 For any given n-connected region B in z-plane whose each boundary component consists of Jordan arcs of finite unmber and a pair of complex numbers $z_0$ and $w_0$, $z_0 \in B$, $0, \infty \notin B$, there exists an n-connected region family $R_w(t)$ with circle boundary whose centers $a_j(t)$ and radii $r_j(t)$, $j=1,2,\cdots,n$, are 2n differentiable functions of parameter t, not all constant, $w_0 \in R_w(t)$, $0 \leqslant t \leqslant t_0$, such that the limit function

$$f(w) = \lim_{t \to t_0} \Phi(w,t) \qquad (4.7)$$

is a univalent and conformal mapping of a region with circle boundary onto B, $f(w_0) = z_0$, where $\Phi(w,t)$ is univalent and conformal in $R_w(t)$, $w_0$ to $z_0$, and satisfying the following relation in $R_w(t)$

$$\frac{\partial \Phi}{\partial t} = w\frac{\partial \Phi}{\partial w}\sum_{j=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}[K_j(w;\xi_j) - K_j(w_0;\xi_j)]d\psi_j(\theta;t) \quad (0 \leqslant t \leqslant t_0) \qquad (4.8)$$

here $K_j(w;\xi_j)$ is defined as in section 2 but the region here is $R_w(t)$, $\xi_j = a_j(t) + r_j(t)e^{i\theta}$,

$$\psi_j(\theta;t) = \lim_{\substack{\varepsilon \to 0 \\ \varepsilon > 0}} \int_0^{\theta}\left[\frac{\partial}{\partial t}\log\left|\frac{F(\Phi(\eta_j,t),T)}{\eta_j}\right|\right]_{T=t} d\theta \qquad (4.9)$$

$\eta_j$ is a variable point on the j-th boundary circle of $R_w^{(\varepsilon)}(t)$ with $\arg(\eta_j - a_j(t)) = \theta$; $F(z,t)$ is the inverse function of $\Phi(w,t)$.

Proot (a) It is easy to prove that for the region B there exists an n-connected region family $G(t)$, $0 \leqslant t \leqslant t_0$, satisfying the following conditions:

1) $z_0 \in G(t)$, $0, \infty \notin G(t)$;

2) for any two values $t_1, t_2 \in [0, t_0)$, $t_1 < t_2$ implies $\overline{G(t_1)} \subset G(t_2)$ (or assume that the contra-relation $\overline{G(t_2)} \subset G(t_1)$ is always true);

3) $G(t)$ tends to B as $t \to t_0$;

4) the connected components of the boundary $\Gamma(t)$ of $G(t)$ consist of Jordan arcs of

finite number, whose parametric equations are $z=\Omega_m(\theta,t)$, $\theta\in[0,2\pi]$, $m=1,2,\cdots,n$;

5) the function $\Omega_m(\theta,t)$ is uniformly differentiable for t at every $t\in(0,t_0)$ with respect to $[0,2\pi]$.

(b) For the given region family $G(t)$, $0\leqslant t\leqslant t_0$, satisfying the conditions 1) – 5) listed above, it is not difficult to prove that there exists a corresponding function family $F(z,t)$, $0\leqslant t\leqslant t_0$, which map $G(t)$ onto n-connected regions $R_w(t)$ with circle boundary one – to – one and conformally, $z_0$ to $w_0$, $R_w(t)$ tends to a region with circle boundary as $t\to t_0$ and $a_j(t)$, $r_j(t)$ are differentiable, $j=1,2,\cdots n$.

Suppose that $\varepsilon>0$ is sufficiently small. Let $G^{(\varepsilon)}(t)$ be the image region of $R_w^{(\varepsilon)}(t)$ under the inverse mapping $z=\Phi(w,t)$. It is easy to see from Theorem 3.1 that the partial derivativer of the function $\Phi(w,t)$ with respect to the parameter t exist everywhere in $R_w^{(\varepsilon)}(t)$ and the following formula holds

$$\frac{\partial\Phi}{\partial t}=-w\frac{\partial\Phi}{\partial w}\left\{\sum_{j=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\partial}{\partial t}\log\left|\frac{F^{-1}(\Omega_j(\theta,t;\varepsilon))}{F(\Omega_j(\theta,t;\varepsilon),T)}\right|\right]_{T=t}\cdot K_j(w;\eta_j)d\theta-C_\varepsilon+iD_\varepsilon\right\} \tag{4.10}$$

where $C_\varepsilon$ and $D_\varepsilon$ are real constants, $K_j(w;\eta_j)$ is determined by $R_w^{(\varepsilon)}(t)$ and is defined as before, $\Omega_j(\theta,t;\varepsilon)=\Phi(\eta_j,t)$.

Using the condition $z_0=\Phi(w_0,t)$ and introducing the function

$$\psi_j(\theta;t;\varepsilon)=\int_0^{\theta}\left[\frac{\partial}{\partial t}\log\left|\frac{F(\Phi(\eta_j,t),T)}{\eta_j}\right|\right]_{T=t}d\theta \tag{4.11}$$

we can rewrite (4.10) as

$$\frac{\partial\Phi}{\partial t}=w\frac{\partial\Phi}{\partial w}\left\{\sum_{j=1}^{n}\frac{1}{2\pi}\int_0^{2\pi}[K_j(w;\eta_j)-K_j(w_0;\eta_j)d\psi_j(\theta;t;\varepsilon)\right\} \tag{4.12}$$

Setting $\varepsilon\to 0$ in (4.12), we obtain (4.8) by means of a proposition established by L. Ahlfors in [2] and exchanging the order of taking limit and integrating.

Obviously, the limit function (4.7) possesses character required.

## 5. EXTREMAL PROBLEM OF DIFFERENTIABLE FUNCTIONALS

As an application, we discuss the extremal problem of a class of differentiable functionals.

Let G be an n-connected region, $M_G$ denote the set of all meromorphic functions in G, $N_G$ denots the holomorphic function family in G, and K be an univalent subfamily of

$M_G$.

A functional $\Phi[f]$ defined in $M_G$ is called weakly differentiable with respect to K if $\Phi[f]$ does not take the value $\infty$ in K and for any $f \in K$, $h \in M_G$ the limit (functional derivative)

$$\lim_{\lambda \to 0} \frac{1}{\lambda} \{ \Phi[f+\lambda h] - \Phi[f] \} \qquad (\lambda \text{ is real}) \tag{5.1}$$

exists (finite or infinite).

A real functional $\Phi[f]$ defined in $M_G$ is called $A_K$-type if for every $f \in K$ the functional dervative is the real part of some complex functional $D_f^{(\Phi)}[h]$ in $M_G$ which does not take the value $\infty$ in $N_G$.

Let $\{L\}$ denote some n-connected region family in w-plane and every region of the family contain the point $w = w_0$ but do not contain the point $w = \infty$. Let E denote the union of all regions in $\{L\}$. $w_1, w_2, \cdots, w_m$ are m points arbitrarily taken in E but distinct from $w_0$.

Let

$$w^* = F(w; w_1, w_2, \cdots, w_m; \varepsilon, \bar{\varepsilon}) \tag{5.2}$$

denote the function satisfying the following conditions:

1) F is analytic with respect to $w, \varepsilon, \bar{\varepsilon}$ when $|\varepsilon| < \lambda_0$ for some positive number $\lambda_0$, $w \in E \setminus \{w_1, w_2, \cdots, w_m\}$;

2) $F(w_0; w_1, w_2, \cdots, w_m; \varepsilon, \bar{\varepsilon}) = z_0$ where $z_0$ is a fixed point.

3) when $|\varepsilon|$ is sufficiently small, for any region D in $\{L\}$ there existe a region $D^*$ in $\{L\}$ such that the function F maps D into $D^*$ univalently except arbitrarily small neighbourhood of those $w_K$ which lie in $\bar{D}$;

4) for sufficiently small $|\varepsilon|$ and $w \in E \setminus \{w_1, w_2, \cdots, w_m\}$ we have the following expansion

$$w^* = w + \varepsilon p_1(w; w_1, \cdots, w_m) + \bar{\varepsilon} p_2(w; w_1, \cdots, w_m) + o(|\varepsilon|) \tag{5.3}$$

where $P_1$ is a rational fraction of w only to have simple poles $w_1, w_2, \cdots, w_m$ in E; $P_2$ is analytic in E; $P_1$ and $P_2$ take the value 0 at $w = w_0$; the residue of $P_1$ at $w = w_K$ is denoted by $r_K(w_1, \cdots, w_m)$, $k = 1, 2, \cdots, m$.

Let $K_L$ denote the set of all univalent and conformal mappings which map n-connected regions with circle boundary onto regions in $\{L\}$, $z_0$ to $w_0$.

Theorem 5.1 Suppose that for any given m points $w_1, w_2, \cdots, w_m$ in E to be distinct from $w_0$ there exists a function (5.2). Then the following variation formulas hold in $K_1$:

(i) If $f \in K_1$ and the corresponding region L has m outer points $w_1', w_2', \cdots, w_m'$ in E, then the function

$$f^\Delta(z) = f(z) + \varepsilon p_1(f(z); w_1, w_2, \cdots, w_m) + \bar{\varepsilon} p_2(f(z); w_1, w_2, \cdots w_m) + o(|\varepsilon|) \tag{5.4}$$

belongs to $K_L$ where $w_K$ is an arbitrary point satisfying $|w_K - w_K'| < \rho$ for some sufficiently small number $\rho > 0$.

(ii) If $f \in K_L$ then the function

$$f^{\Delta\Delta}(z) = f(z) + \varepsilon p_1(f(z); f(z_1), \cdots, f(z_m)) + \bar{\varepsilon} p_2(f(z); f(z_1), \cdots, f(z_m))$$

$$-\frac{1}{2}\varepsilon z f'(z)\sum_{k=1}^{m}\frac{r_k(f(z_1), \cdots, f(z_m))}{z_k^2 f'(z_k)}\left\{\frac{z+z_K}{z-z_K} - \frac{z_0+z_K}{z_0-z_K}\right.$$

$$\left.+\sum_{j=2}^{n}\frac{z_k}{z_k-a_j} - \sum_{j=1}^{n}\frac{1}{2\pi i}\int_{C_j}\frac{z_k}{\xi - z_k}\left[H_j(z,\xi) - H_j(z_0,\xi)\right]\frac{d\xi}{\xi - a_j}\right\}$$

$$+\frac{1}{2}\bar{\varepsilon} z f'(z)\sum_{k=1}^{m}\frac{\overline{r_k(f(z_1), \cdots, f(z_m))}}{\overline{z_k^2 f'(z_k)}}\left\{\frac{r_1^2+\bar{z}_k^2}{r_1^2-\bar{z}_k^2} - \frac{r_1^2+\bar{z}_k z_0}{r_1^2-\bar{z}_k z_0}\right.$$

$$-\sum_{j=2}^{n}\left[\frac{r_j^2+(\bar{z}_k-\bar{a}_j)(z-a_j)}{r_j^2-(\bar{z}_k-\bar{a}_j)(z-a_j)} - \frac{r_j^2+(\bar{z}_k-\bar{a}_j)(z_0-a_j)}{r_j^2-(\bar{z}_k-\bar{a}_j)(z_0-a_j)}\right]$$

$$\left.+\sum_{j=1}^{n}\frac{1}{2\pi i}\int_{C_j}\frac{\bar{z}_k}{\bar{\xi}-\bar{z}_k}\left[H_j(z,\xi) - H_j(z_0,\xi)\right]\frac{d\xi}{\xi - a_j}\right\} + o(|\varepsilon|) \qquad (5.5)$$

belongs to $K_L$, where $z_1, z_2, \cdots, z_m$ are m arbitrary points in the region $R_z$ corresponding to f; $C_j$, $a_j$ and $r_j$ are the j-th boundary circle of $R_z$ and its center and radius respectively, the direction of the integral path $C_j$ is chosen such that the region lies on the left side of $C_j$; the function

$$H_j(z,\xi) = K_j(z;\xi) - \left(\pm\frac{\xi + z - 2a_j}{\xi - z}\right) \qquad (5.6)$$

the sign inside the bracket is positive when $j = 1$ and negative otherwise, here we suppose that $R_z$ lies inside $C_1$.

Proof (5.4) can be proved directly from the assumption of the theorem. (5.5) can be obtain from the variation formula in Theorem 4.1 by computing and using the condition that every function in $K_L$ maps $z_0$ to $w_0$.

Starting from Theorem 5.1, we can obtain variation formulas of form (5.4) and (5.5) for many unlvalent function families, Here we will solve a general extremal problem of functionals by means of the theorem.

It is easy to prove from (5.4) that the extremal region of $A_{KL}$-type functional in $K_L$

has no outer points in E. Furthermore, we have

Theorem 5.2 Let $\Phi[f]$ be $A_{K_L}$- type functional. If $w=f(z)$ is the extremal function of the functional $\Phi[f]$ with respect of $K_L$, then for any m points $z_1, z_2, \cdots, z_m$ in the region $R_z$ corresponding to f, the following equality holds:

$$
\begin{aligned}
& D_j^{(\Phi)}[p_1(f(\zeta); r(z, \cdots, f(z_m)))] + \overline{D_j^{(\Phi)}[p_2(f(\zeta); f(z_1), \cdots, f(z_m))]} \\
&= \frac{1}{2}\sum_{k=1}^{m} \frac{r_k(f(z_1), \cdots, f(z_m))}{z_k^2 f'(z_k)} \Bigg\{ D_j^{(\Phi)}\Bigg[\zeta f'(\zeta)\Bigg(\frac{\zeta+z_k}{\zeta-z_k} - \frac{z_0+z_k}{z_0-z_k} \\
&\quad + \sum_{j=2}^{n} \frac{z_k}{z_k-a_j} - \frac{1}{2\pi i}\sum_{j=1}^{n}\int_{C_j} \frac{z_k}{\xi-z_k}(H_j(\zeta,\xi)-H_j(z_0,\xi))\frac{d\xi}{\xi-a_j}\Bigg)\Bigg] \\
&\quad + \overline{D_j^{(\Phi)}\Bigg[\zeta f'(\zeta)\Bigg(\frac{r_1^2+\bar{z}_k}{r_1^2-\bar{z}_k} - \frac{r_1^2+\bar{z}_k z_0}{r_1^2-\bar{z}_k z_0} - \sum_{j=2}^{n}\Bigg[\frac{r_j^2+(\bar{z}_k-\bar{a}_j)(\zeta-a_j)}{r_j^2-(\bar{z}_k-\bar{a}_j)(\zeta-a_j)}} \\
&\quad \overline{- \frac{r_j^2+(\bar{z}_k-\bar{a}_j)(z_0-a_j)}{r_j^2-(\bar{z}_k-\bar{a}_j)(z_0-a_j)}\Bigg] + \frac{1}{2\pi i}\sum_{j=1}^{n}\int_{C_j}\frac{\bar{z}_k}{\bar{\xi}-\bar{z}_k}(H_j(\zeta,\xi)-H_j(z_0,\xi))\frac{d\xi}{\xi-a_j}\Bigg)\Bigg]}\Bigg\}
\end{aligned}
\tag{5.7}
$$

Proof The funcctional differential equation (5.7) satisfied by the extremal functions can be derived by using the formula (5.5) and the definition of $A_{K_L}$- tpye functional.

The theorem is a generalization of a main result of G.G. Shlionski [13].

## REFERENCES


1. Ahlfors, L.V., Complex Analysis, 3rd edition, McGraw–Hill Book Company, New York, 1979.
2. ———— Bounded analytic functions, Duke Math. J., vol. 14 (1947), 1–11.
3. Golusin, G.M., Geometric Theory of Functions of a Complex Variable, Moscow, 1952.
4. ———— A variational method in conformal mapping I, Mat. Sb., vol. 19. (1946), 203–236.
5. ———— On the parametric representation of functions to be univalent in annuli, Mat. Sb., vol. 29 (71), (1951), No. 2, 469–476.
6. Komatu, Y., Theory of Conformal Mappings, Kyoritsu, Tokyo, 1943–1947.
7. Kunugi, K., Theory of Functions of a Complex Variable, Iwanami, Tokyo, 1958.
8. Kufarev, P.P., On analytic function familes containing one parametric variable, Mat. Sb., vol. 13, (1943), 87–118.
9. Kufarev, P.P. and Semuchina, N.B. On Golusin's variation method in 2–connected domains, Dokl.

Akad .Nauk SSSR ,vol .107 , (1956 ) ,No .4 ,505 – 507 .

10 . Lebedev , N . A ., On the parametric representation of functions to be analytic and univalent in annuli , Dokl .Akad .Nauk SSSR ,vol .103 , (1955 ) ,No .5 ,767 – 768 .

11 .Lowner ,K ., Untersuchungen uber Schlichte Konforme Abbildungen des Einheitskreises I ,Math .Ann ., vol 89 , (1923 ) ,103 – 121 ..

12 .Schiffer ,M ., A method of variation within the family of simple functions , Proc .London Math .Soc ., vol .44 , (1938 ) ,432 – 449 .

13 .Shlionskii ,G .R ., Extremal problems of differentiable functiionals in theory of univalent functions ,Vest . Lenin .Uni .,vol .13 , (1958 ) ,64 – 83 .

14 . Villat ,H ., Le probleme de Dirichlet dens une aire anulare , Rend .deleire .Math .di Palermo ,vol .33 , (1912 ) ,134 – 175 .

15 . Yang Weiqi ,Two simple proofs of Villat 's formula and its generalization in n- connected circular domains ,J .Beij ing Institute of Tech .,vol .1 , (1981 ) ,No .1 ,1 – 5 .

16 . ———— ,The variatiional theorem and the parametric representation theorem of conformal mappings in muliply connected domains ,Acta Math ,Sinica .vol .24 , (1981 ),No .1 ,26 – 35 .

17 . ———— ,Poisson 's formula and solution of Dirichlet problem in multiply connected domains ,J . Beijing Institute of Tech .,vol .2 , (1982 ) ,No .2 ,1 – 7 .

18 . ———— ,Poisson- Jensen formula in multiply connected domains ,Pure and Applied Math .4 (1988 ) , 37 – 40 .