数理解析研究所講究録
第 714 巻 1990 年 133-136

ON CERTAIN P-VALENTLY STARLIKENESS CONDITIONS FOR ANALYTIC FUNCTIONS

MAMORU NUNOKAWA 群馬大・教育 布川 護

SHIGEYOSHI OWA 近畿大・理工 尾和 重義

HITOSHI SAITOH 群馬高専 斎藤 斉

1. Introduction.

Let $A(p)$ denote the class of functions

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n$$

which are analytic in the open unit disk $E = \{z: |z| < 1\}$. A function $f(z) \varepsilon A(p)$ is called p-valently starlike with respect to the origin if and only if

$$\mathrm{Re}\, \frac{zf'(z)}{f(z)} > 0 \qquad \text{in } E.$$

We denote by $S(p)$ the subclass of $A(p)$ consisting of functions which are p-valently starlike in E. Krzyz [1] showed by a counterexample that if $f(z) \varepsilon A(1)$, the condition $\mathrm{Re}\, f'(z) > 0$ in E does not imply $f(z) \varepsilon S(1)$. In [3], Mocanu proved the following theorem.

Theorem. If $f(z) \varepsilon A(1)$ and

$$|\arg f'(z)| < \alpha_o \frac{\pi}{2} = 0.968\ldots \qquad \text{in } E,$$

where $\alpha_o = 0.6165\ldots$ is the unique root of the equation

$$2 \tan^{-1}(1 - \alpha) + \pi(1 - 2\alpha) = 0,$$

then $f(z) \varepsilon S(1)$.

Definition. Let $F(z)$ be analytic and univalent in E and suppose that $F(E) = D$. If $f(z)$ is analytic in E, $f(0) = F(0)$, and $f(E)$ D, then we say that $f(z)$ is subordinate to $F(z)$ in E, and we write

$$f(z) \prec F(z) .$$

2. Preliminaries.

We shall use the following lemma to prove our results.

Lemma 1. Let $\beta^* = 1.218\ldots$ be the solution of $\pi\beta = 3\pi/2 - \tan^{-1}\beta$ and let $\alpha = \alpha(\beta) = \beta + (2/\pi)\tan^{-1}\beta$, for $0 < \beta < \beta^*$. If $p(z)$ is analytic in E, with $p(0) = 1$, then

$$p(z) + zp'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \implies p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta}.$$

We owe this lemma to [2, Theorem 5].

3. Main theorem.

Theorem 1. Let $p \geqq 2$. If $f(z) \varepsilon A(p)$ and

(1) $$|\arg f^{(p)}(z)| < \frac{\pi}{2}\alpha_1 = 1.249\ldots \quad \text{in E,}$$

where $\alpha_1 = \frac{1}{2} + \frac{2}{\pi}\tan^{-1}\frac{1}{2} = 0.795\ldots$, then $f(z)\varepsilon S(p)$ or $f(z)$ is p-valently starlike in E.

Proof. If we put

$$p(z) = \frac{f^{(p-1)}(z)}{p!z},$$

then we have

$$p(z) + zp'(z) = \frac{1}{p!} f^{(p)}(z)$$

and $p(0) = 1$.

From the assumption (1), we have

$$|\arg(p(z) + zp'(z))| = |\arg f^{(p)}(z)| < \frac{\pi}{2}\alpha_1 \quad \text{in E.}$$

Then, from Lemma 1, we have

(2) $$\left|\arg \frac{f^{(p-1)}(z)}{p!z}\right| = \left|\arg \frac{f^{(p-1)}(z)}{z}\right| < \frac{\pi}{4} \quad \text{in E.}$$

On the other hand, we easily have

(3) $$\frac{f^{(p-2)}(z)}{z^2} = \frac{1}{z^2}\int_0^z f^{(p-1)}(t)\,dt$$

$$= \frac{1}{r^2}\int_0^r \frac{f^{(p-1)}(t)}{t}\,\rho\,d\rho$$

where $z = re^{i\theta}$, $0 < r < 1$, $t = \rho e^{i\theta}$ and $0 \leqq \rho \leqq r$.

From (2), we easily have

$$|\arg \frac{f^{(p-1)}(z)}{z}|z|| = |\arg \frac{f^{(p-1)}(z)}{z}| < \frac{\pi}{4} \qquad \text{in } E,$$

and the same is true of its integral mean of (3) (see e.g. [5, Lemma 1]). Therefore we have

(4) $$|\arg \frac{f^{(p-2)}(z)}{z^2}| = |\arg \frac{1}{r^2}\int_0^r \frac{f^{(p-1)}(t)}{t}\rho\, d\rho|$$

$$= |\arg \int_0^r \frac{f^{(p-1)}(t)}{t}\rho\, d\rho|$$

$$< \frac{\pi}{4} \qquad \text{in } E.$$

From (2) and (4), we have

$$|\arg \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)}| = |\arg \frac{f^{(p-1)}(z)}{z}\cdot\frac{z^2}{f^{(p-2)}(z)}|$$

$$\leqq |\arg \frac{f^{(p-1)}(z)}{z}| + |\arg \frac{f^{(p-2)}(z)}{z^2}|$$

$$< \frac{\pi}{2} \qquad \text{in } E.$$

This shows that

$$\mathrm{Re}\, \frac{zf^{(p-1)}(z)}{f^{(p-2)}(z)} > 0 \qquad \text{in } E.$$

Applying the same method as in the proof of [4, Theorem 5], we have

$$\mathrm{Re}\, \frac{zf'(z)}{f(z)} > 0 \qquad \text{in } E.$$

This shows that $f(z)$ is p-valently starlike in E.

Theorem 2. Let $p \geqq 2$. If $f(z)\varepsilon A(p)$ and

(5) $$\mathrm{Re}\, f^{(p)}(z) > 0 \qquad \text{in } E,$$

then we have

$$|\arg \frac{zf'(z)}{f(z)}| < \frac{\pi}{2} 2\alpha_2 \qquad \text{in } E$$

where $\alpha_2 = 0.638...$ is the solution of the equation

$$1 = \beta + \frac{2}{\pi}\tan^{-1}\beta .$$

Proof. Applying the same method as in the proof of Theorem 1 and from the assumption (5), we have

(6) $$\left|\arg \frac{f^{(p-1)}(z)}{z}\right| < \frac{\pi}{2}\alpha_2 \qquad \text{in } E.$$

Applying the same method as in the proof of Theorem 1 and from (6), we have

$$\left|\arg \frac{f^{(p-2)}(z)}{z^2}\right| < \frac{\pi}{2}\alpha_2 \qquad \text{in } E.$$

Repeating the same method as the above, we have

(7) $$\left|\arg \frac{f'(z)}{z^{p-1}}\right| < \frac{\pi}{2}\alpha_2 \qquad \text{in } E$$

and

(8) $$\left|\arg \frac{f(z)}{z^p}\right| < \frac{\pi}{2}\alpha_2 \qquad \text{in } E.$$

Then from (7) and (8), we have

$$\left|\arg \frac{zf'(z)}{f(z)}\right| \leqq \left|\arg \frac{f'(z)}{z^{p-1}}\right| + \left|\arg \frac{f(z)}{z^p}\right|$$

$$< \frac{\pi}{2} 2\alpha_2 \qquad \text{in } E.$$

This completes our proof.

References

[1] J. Krzyz, A counterexample concering univalent functions, Folia Soc. Scient. Lubliniensis, 2(1962), 57-58.

[2] S.S. Miller and P.T. Mocanu, Marx-Strohhacker differential subordination system, Proc. Amer. Math. Soc., 99(1987), 527-534.

[3] P.T. Mocanu, Some starlikeness conditions for analytic functions, Rev. Roumaine Math. Pures Appl., 33(1988), 117-124.

[4] M. Nunokawa, On the theory of multivalent functions, Tsukuba J. Math., Vol.11, No.2(1987), 273-286.

[5] Ch. Pommerenke, On close-to-convex analytic functions, Trans. Amer. Math. Soc., 114(1965), 176-186.