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The Maass Zeta Function Attached to Positive Definite Quadratic Forms

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§0. Introduction

Let m, n be positive integers with $m \geq n$. Put $\Gamma = SL(n,\mathbb{Z})$ and take a lattice L in the space $M(m,n;\mathbb{R})$ of m by n real matrices stable under the action of Γ from the right. Denote by L' the set of matrices of rank n in L. In a series of papers ([M1],[M2],[M5]) Maass made precise investigations of the following zeta functions:

$$\xi(Q,\varphi,L;s) = \sum_{x\in L'/\Gamma} \frac{Q(x)\varphi({}^t xx)}{\det({}^t xx)^{s+d/2n}}$$

where $Q(x)$ is a homogeneous polynomial function of even degree d on V invariant under the action of $SL(n)$ from the right and $\varphi(Y)$ is an automorphic form of homogeneous degree 0 on the space of positive definite symmetric matrices of size n with respect to the arithmetic subgroup Γ. According to his results, the zeta functions can be extended to meromorphic functions in the whole complex plane and satisfy a certain functional equation; however Maass' functional equation involves certain derivatives of $\varphi({}^t xx)\det({}^t xx)^{-s}$, which have not been calculated explicitly unless $Q(x)$ is harmonic.

The aim of this paper is to present an approach to the Maass zeta functions based on the theory of prehomogeneous vector spaces and to calculate an explicit formula of the functional equation. In the present paper we restrict our attention to the case where the automorphic form φ is a constant function. The general case will be treated in the subsequent paper [S4].

Put $G = SO(m) \times GL(n)$. The group G acts linearly on the space $V = M(m,n)$ of m by n matrices via

$$x \longmapsto kxg^{-1} \qquad (k \in SO(n),\ g \in GL(n),\ x \in V).$$

Then (G,V) is a prehomogeneous vector space with the singular set

$$S = \{x \in V;\ \det({}^t xx) = 0\}$$

and the Maass zeta functions can be viewed as zeta functions associated with this prehomogeneous vector space.

For simplicity we now assume that $m \geq 2n$. Let $R = \mathbb{C}[M(m,n)]^{SL(n)}$ be the ring of $SL(n)$-invariant polynomial functions on $V = M(m,n)$. To get an explicit functional equation of the Maass zeta function, it is necessary to decompose the ring R into direct sum of simple G-modules. The simple components of R are parametrized by elements in the set

$$\Lambda = \left\{(\lambda_0,\lambda) \in \mathbb{Z} \times \mathbb{Z}^{\kappa};\ \begin{matrix} \lambda_0 \equiv \lambda_1 \equiv \dots \equiv \lambda_n \pmod 2, \\ \lambda_0 \geq \dots \geq \lambda_n \geq \lambda_{n+1} = \dots = \lambda_\kappa = 0 \end{matrix}\right\}$$

if $m > 2n$, and

$$\Lambda = \left\{(\lambda_0,\lambda) \in \mathbb{Z} \times \mathbb{Z}^{\kappa};\ \begin{matrix} \lambda_0 \equiv \lambda_1 \equiv \dots \equiv \lambda_n \pmod 2, \\ \lambda_0 \geq \dots \geq \lambda_{n-1} \geq |\lambda_n| \end{matrix}\right\}$$

if $m = 2n$. We denote by $\mathfrak{R}_{\lambda_0,\lambda}$ the simple component corresponding to $(\lambda_0,\lambda) \in \Lambda$. Then our main theorem is the following:

Theorem. *Let* $Q(x)$ *be a polynomial in* $\mathfrak{R}_{\lambda_0,\lambda}$ $((\lambda_0,\lambda) \in \Lambda)$ *and* $\varphi_0(Y)$ *be a constant function. Then*

(i) $\xi(Q,\varphi_0,L;s)$ *has an analytic continuation to a meromorphic function of* s *in* $\mathbb{C}$ *and the function*

$$\prod_{i=1}^{n} \left(s + \frac{\lambda_i - i - 1}{2}\right)\left(s - \frac{\lambda_i + m - i + 1}{2}\right) \cdot \xi(Q,\varphi_0,L;s)$$

is an entire function.

(ii) *Put*

$$\xi(Q,\varphi_0,L;s) = v(L)^{1/2}\, \pi^{-ns} \prod_{i=1}^{n} \Gamma\left(s + \frac{\lambda_i - i + 1}{2}\right) \xi(Q,\varphi_0,L;s),$$

where $v(L) = \int_{V_{\mathbb{R}}/L} dx$. *Then the following functional equation holds:*

$$\xi(Q,\varphi_0,L^*;m/2-s) = \exp\left(\frac{\pi\sqrt{-1}}{2} \sum_{i=1}^{n} \lambda_i\right) \xi(Q,\varphi_0,L;s).$$

As mentioned above, the Maass zeta function can be viewed as a zeta function associated with the prehomogeneous vector space $(SO(m)\times GL(n), M(m,n))$. However to controle $SL(n)$-invariant functions appearing as coefficients of the zeta function, we need precise information on the prehomogeneous vector space $(B(m)\times GL(n), M(m,n))$, where $B(m)$ is the Borel subgroup of the special orthogonal group $SO(m)$. In Section 1 the structure of $(B(m)\times GL(n), M(m,n))$ is examined. The decomposition theorem of the ring R is due to Hoppe [H]. We give in this section a simple proof of Hoppe's decomposition theorem and make a correction to Hoppe's result in the case $m = 2n$.

Recall that the following are the facts lying behind the validity of functional equations of zeta functions associated with prehomogeneous vector spaces (cf. [SS], [S1]):

(1) local functional equations satisfied by complex powers of relative invariants,

(2) integral representation of zeta functions as a kind of Mellin transform of Theta series.

Since the general theory in [SS] and [S1] can be applied to the Maass zeta function only when both Q and φ are constant functions, it is necessary for our purpose to generalize these two facts to the Maass zeta function $\xi(Q,\varphi_0,L;s)$. In Section 2 we give an integral representation of the Maass zeta function. We prove in Section 3 a generalization of local functional equations (Theorem 3.3) and give a proof of the main theorem (Theorem 3.1), assuming a formula for generalized b-functions (Proposition 3.4). Section 4 is devoted to a calculation of the local functional equation and the b-functions of $(\mathrm{B}(m)\times\mathrm{GL}(n),\mathrm{M}(m,n))$, which plays a key role in determining the explicit form of the functional equation of the Maass zeta function.

Notation. We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of rational integers, the field of real numbers and the field of complex numbers, respectively. For an affine algebraic variety X defined over a field k, X_k stands for the set of k-rational points of X. The space of rapidly decreasing functions on a real vector space V is denoted by $\mathcal{S}(V)$. The space of compactly supported C^∞-functions on a C^∞-manifold M is denoted by $C_0^\infty(M)$. We denote by 1_n and $0^{(m,n)}$ the identity matrix of size n and the m by n zero matrix, respectively. The superscript (m,n) of a matrix $A = A^{(m,n)}$ indicates that the matrix A is of m rows and n columns. We write simply $A^{(m)}$ for $A^{(m,m)}$. For a real number a, we put $\mathrm{sgn}(a) = a/|a|$.

§1. Structure of certain prehomogeneous vector space.

Let m, n be positive integers with $m > n$. For simplicity we assume that $m/2 \geq n$.*) We put $\kappa = [m/2]$ and $\delta = 0$ or 1 according as m is even or odd. For a nondegenerate symmetric matrix $Y^{(m)}$, let $G = SO(Y) \times GL(m)$ and $V = M(m,n)$. We consider the representation ρ of G on V defined by

$$\rho(h,g)x = hxg^{-1} \quad (h \in SO(Y),\ g \in GL(n),\ x \in M(m,n)).$$

Proposition 1.1 ([SK;Section 5, Proposition 23]). *The triple* (G,ρ,V) *is a regular prehomogeneous vector space with singular set*

$$S = \{x \in V;\ \det({}^t xYx) = 0\}.$$

*) In the final version of the present paper, this restriction will be removed.

The prehomogeneous vector space (G,ρ,V) is defined over the field $\mathbb{Q}(y_{ij};1\le i\le j\le m)$ generated by the entries of Y. In this section we consider (G,ρ,V) as a prehomogeneous vector space defined over $\mathbb{C}$ and it is convenient to take the matrix

$$J = \begin{pmatrix} & & 1_\kappa \\ & 1_\delta & \\ 1_\kappa & & \end{pmatrix}$$

as Y.

Put

$$\mathrm{B}(m) = \left\{ \begin{pmatrix} A & * & * \\ 0 & 1_\delta & * \\ 0 & 0 & {}^tA^{-1} \end{pmatrix} \in \mathrm{SO}(J);\ A \in \mathrm{Trig}(\kappa)\right\},$$

where $\mathrm{Trig}(\kappa)$ is the group of nondegenerate upper triangular matrices of size κ. Then the group $\mathrm{B}(m)$ is a Borel subgroup of $\mathrm{SO}(J)$. Every element b of $\mathrm{B}(m)$ can be written as

$$b = b_1(A)b_2(v)b_3(B),$$

$$b_1(A) = \begin{pmatrix} A & & \\ & 1_\delta & \\ & & {}^tA^{-1} \end{pmatrix} \quad (A \in \mathrm{Trig}(\kappa)),$$

$$b_2(v) = \begin{pmatrix} 1_\kappa & v & -2^{-1}v\,{}^tv \\ & 1_\delta & -{}^tv \\ & & 1_\kappa \end{pmatrix} \quad (v \in \mathbb{C}^\kappa),$$

$$b_3(B) = \begin{pmatrix} 1_\kappa & 0 & B \\ & 1_\delta & 0 \\ & & 1_\kappa \end{pmatrix} \quad (B \in \mathrm{M}(\kappa),\ {}^tB = -B).$$

Also put $P = B(m) \times GL(n)$. We denote the representation of P on V obtained from ρ by restricting it to P by the same symbol ρ.

Proposition 1.2. *The triple* (P,ρ,V) *is a regular prehomogeneous vector space.*

Proof. Consider the point

$$x_0 = \begin{pmatrix} 1_n \\ 0^{(\kappa-n+\delta,n)} \\ 1_n \\ 0^{(\kappa-n,n)} \end{pmatrix} \in V.$$

Then, by an elementary calculation, it is easy to see that every element of the isotropy subgroup P_{x_0} of P at x_0 is of the form

$$(1.1) \quad (b_1(\begin{pmatrix} U^{(n)} & 0 \\ 0 & A^{(\kappa-n)} \end{pmatrix})b_2(\begin{pmatrix} 0^{(n,1)} \\ v^{(\kappa-n,1)} \end{pmatrix})b_3(\begin{pmatrix} 0^{(n)} & 0 \\ 0 & B^{(\kappa-n)} \end{pmatrix}),U)$$

where $U = \begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$. Hence $P_{x_0} \simeq \{\pm 1\}^n \times B(2(\kappa-n)+\delta)$ and $\dim P - \dim P_{x_0} = \dim V$. This shows that the triple (P,ρ,V) is a prehomogeneous vector space (cf. [SK; Section 2, Proposition 2]). The regularity of (P,ρ,V) follows directly from the regularily of (G,ρ,V). □

Now we shall determine the singular set and relative invariants of (P,ρ,V).

For a symmetric matrix T we denote by $d_i(T)$ the i-th principal minor, namely the determinant of upper left i by i block of T. Using the block decomposition

$$x = \begin{pmatrix} x_1 \\ x_2 \\ y \\ x_3 \\ x_4 \end{pmatrix} \begin{matrix} \} & n \\ \} & \kappa-n \\ \} & \delta \\ \} & n \\ \} & \kappa-n \end{matrix} \in \mathrm{V}, \qquad (x_1 \text{ has } n \text{ columns})$$

we define rational functions $P_0(x), \cdots, P_n(x)$ by

$$P_0(x) = \det({}^t xJx),$$

$$P_i(x) = P_0(x) \cdot d_i(x_3({}^t xJx)^{-1}\,{}^t x_3) \qquad (1 \le i \le n-2),$$

$$P_{n-1}(x) = \begin{cases} P_0(x) \cdot d_{n-1}(x_3({}^t xJx)^{-1}\,{}^t x_3) & \text{if } m > 2n, \\ \dfrac{P_0(x)}{P_n(x)} \cdot d_{n-1}(x_3({}^t xJx)^{-1}\,{}^t x_3) & \text{if } m = 2n, \end{cases}$$

$$P_n(x) = \det x_3.$$

Then it is easy to check the first part of the following proposition:

Proposition 1.3. (i) *The functions* $P_0(x), \cdots, P_n(x)$ *are relative invariants of* $(\mathrm{P}, \rho, \mathrm{V})$ *and the rational characters* $\chi_0, \cdots, \chi_n$ *corresponding to* $P_0(x), \cdots, P_n(x)$, *respectively, are given by*

$$\chi_0(b,g) = \det(g)^{-2},$$

$$\chi_i(b,g) = \det(g)^{-2} \cdot (a_1 \cdots a_i)^{-2} \quad (1 \le i \le n-2)$$

$$\chi_{n-1}(b,g) = \begin{cases} \det(g)^{-2}\cdot(a_1\cdots a_{n-1})^{-2} & \text{if } m > 2n, \\ \det(g)^{-1}\cdot(a_1\cdots a_{n-1})^{-1}\cdot a_n & \text{if } m = 2n, \end{cases}$$

$$\chi_n(b,g) = \det(g)^{-1}\cdot(a_1\cdots a_n)^{-1}$$

for $b = b_1(A)b_2(v)b_3(B) \in \mathrm{B}(m)$ $\left(A = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_\kappa \end{pmatrix}\right)$ *and* $g \in \mathrm{GL}(n)$.

(ii) *They are irreducible polynomials and the singular set* S_P *of* $(\mathrm{P},\rho,\mathrm{V})$ *is given by*

$$S_P = \bigcup_{i=0}^{n} \{x \in \mathrm{V};\ P_i(x) = 0\}.$$

To prove the second part of the lemma, we need some preliminaries.

Lemma 1.4. $P_0(x),\ldots, P_n(x)$ *are polynomial functions.*

The proof is based on the simple fact that, for a square matrix $A = (a_{ij})$, every entry of $\det(A)A^{-1}$ is a polynomial of a_{ij}.

For $\lambda = (\lambda_1,\ldots,\lambda_\kappa) \in \mathbb{Z}^\kappa$, we define a rational character χ_λ of $\mathrm{B}(m)$ by

$$\chi_\lambda(b_1(A)b_2(v)b_2(B)) = a_1^{-\lambda_1}\cdots a_\kappa^{-\lambda_\kappa}.$$

Then any rational character χ of $\mathrm{P} = \mathrm{B}(m) \times \mathrm{GL}(n)$ is of the form

$$\chi(h,g) = \chi_{\lambda_0,\lambda}(h,g) = \chi_\lambda(h)\cdot \det(g)^{-\lambda_0}$$

for some $\lambda \in \mathbb{Z}^\kappa$ and $\lambda_0 \in \mathbb{Z}$. Let $X_\rho(\mathrm{P})$ be the multiplicative group of rational characters of P corresponding to some relative invariants of $(\mathrm{P},\rho,\mathrm{V})$. By (1.1) and [SK; Section 4, Proposition 19] , we have

$$X_\rho(\mathrm{P}) = \left\{\chi_{\lambda_0,\lambda};\ \begin{matrix}\lambda_0 \equiv \lambda_1 \equiv \dots \equiv \lambda_n \pmod 2 \\ \lambda_{n+1} = \dots = \lambda_\kappa = 0\end{matrix}\right\}.$$

We denote by $P_{\lambda_0,\lambda}$ the relative invariant corresponding to $\chi_{\lambda_0,\lambda} \in X_\rho(\mathrm{P})$, namely the rational function satisfying

$$P_{\lambda_0,\lambda}(\rho(b,g)x) = \chi_{\lambda_0,\lambda}(b,g)P_{\lambda_0,\lambda}(x) \qquad ((b,g) \in \mathrm{P}).$$

Recall that $P_{\lambda_0,\lambda}$ is determined by (λ_0,λ) uniquely up to nonzero constant multiple ([SK;Section 4, Proposition 3]). Put

$$X_\rho(\mathrm{P})^+ = \{\chi_{\lambda_0,\lambda} \in X_\rho(\mathrm{P});\ P_{\lambda_0,\lambda} \text{ is a polynomial}\}.$$

Let R be the ring of polynomial functions on V invariant under the action of $\mathrm{SL}(n)$ from the right:

$$R = \{Q(x) \in \mathbb{C}[\mathrm{M}(m,n)];\ Q(xg) = Q(x)\ (g \in \mathrm{SL}(n))\}.$$

We consider the ring R as a left G-module via

$$((h,g)\cdot P)(x) = P(\rho(h,g)^{-1}x) \quad (h \in \mathrm{SO}(J),\ g \in \mathrm{GL}(n)).$$

Then a relatively P-invariant polynomial function is nothing but the highest weight vector of a rational representaion of G contained in R. The highest weight corresponding to

$P_{\lambda_0,\lambda}$ is $\chi_{\lambda_0,\lambda}^{-1}$. It is known that the character $\chi_{\lambda_0,\lambda}^{-1}$ of P is a highest weight of some rational representation of G if and only if

$$(1.2)\quad \begin{cases} \lambda_1 \geq \dots \geq \lambda_\kappa \geq 0 & \text{when } m \text{ is even,} \\ \lambda_1 \geq \dots \geq \lambda_{\kappa-1} \geq |\lambda_\kappa| & \text{when } m \text{ is odd.} \end{cases}$$

Therefore we obtain the inclusion relation

$$X_\rho(P)^+ \subset \{\chi_{\lambda_0,\lambda} \in X_\rho(P);\ \lambda \text{ satisfies (1.2)}\}.$$

Lemma 1.5. (i) *Put*

$$\Lambda = \left\{(\lambda_0,\lambda) \in \mathbb{Z} \times \mathbb{Z}^\kappa;\ \begin{matrix} \lambda_0 \equiv \lambda_1 \equiv \dots \equiv \lambda_n \pmod 2, \\ \lambda_0 \geq \dots \geq \lambda_n \geq \lambda_{n+1} = \dots = \lambda_\kappa = 0 \end{matrix}\right\}$$

or

$$= \left\{(\lambda_0,\lambda) \in \mathbb{Z} \times \mathbb{Z}^\kappa;\ \begin{matrix} \lambda_0 \equiv \lambda_1 \equiv \dots \equiv \lambda_n \pmod 2, \\ \lambda_0 \geq \dots \geq \lambda_{n-1} \geq |\lambda_n| \end{matrix}\right\}$$

according as $m > 2n$ *or* $m = 2n$ $(= 2\kappa)$. *Then*

$$X_\rho(P)^+ = \{\chi_{\lambda_0,\lambda};\ (\lambda_0,\lambda) \in \Lambda\}.$$

Proof. Let $P_{\lambda_0,\lambda}(x)$ be a polynomial relative invariant. Then λ satisfies the condition (1.2) and we obtain

$$\chi_{\lambda_0,\lambda} = \prod_{i=0}^{n-1} \chi_i^{(\lambda_i-\lambda_{i+1})/2} \times \begin{cases} \chi_n^{\lambda_n} & (m > 2n), \\ \chi_n^{(\lambda_{n-1}+\lambda_n)/2} & (m = 2n). \end{cases}$$

This implies that there exists a nonzero constant c such that

$$P_{\lambda_0,\lambda} = c \prod_{i=0}^{n-1} P_i^{(\lambda_i-\lambda_{i+1})/2} \times \begin{cases} P_n^{\lambda_n} & (m > 2n), \\ P_n^{(\lambda_{n-1}+\lambda_n)/2} & (m = 2n). \end{cases}$$

Note that the exponents of $P_1,\ldots,P_n$ are non-negative integers. Assume that $\lambda_0 < \lambda_1$. Then, since P_0 is irreducible, P_0 divides some P_i $(1 \leq i \leq n)$. This is impossible. Hence $\lambda_0 \geq \lambda_1$. This shows the inclusion relation

$$X_\rho(\mathrm{P})^+ \subset \{\chi_{\lambda_0,\lambda};\ (\lambda_0,\lambda) \in \Lambda\}.$$

The opposite inclusion relation follows immediately from the above expression of $P_{\lambda_0,\lambda}$ as a product of $P_0,\ldots,P_n$. □

Now we can complete the proof of Proposition 1.3.

Proof of Proposition 1.3 (ii). Let $Q(x)$ be a prime divisor of $P_i(x)$. Then it is also a relative invariant (cf. [SK; Section 4, Proposition 5]). As is shown in the proof of Lemma 1.5, $Q(x)$ is a product of $P_0(x),\cdots,P_n(x)$. This can occur only when $P_i(x)$ is irreducible. An elementary calculation shows that

$$\mathrm{V}' = \mathrm{V} - \bigcup_{i=0}^{n} \{x \in \mathrm{V};\ P_i(x) = 0\}$$

is a single P-orbit. This proves Proposition 1.3 (ii). □

Let $R_{\lambda_0,\lambda}$ be the subspace of R spanned by

$\{(h,g)P_{\lambda_0,\lambda};\ (h,g) \in \mathbb{G}\}$. Every polynomial in $R_{\lambda_0,\lambda}$ is homogeneous of degree $\lambda_0 n$. Put

$$\Lambda^* = \left\{\lambda \in \mathbb{Z}^\kappa;\ \begin{array}{c}\lambda_1 \equiv \lambda_2 \equiv \dots \equiv \lambda_n \pmod 2, \\ \lambda_1 \geq \dots \geq \lambda_n \geq \lambda_{n+1} = \dots = \lambda_\kappa = 0\end{array}\right\} \quad (m > 2n),$$

$$= \left\{\lambda \in \mathbb{Z}^\kappa;\ \begin{array}{c}\lambda_1 \equiv \lambda_2 \equiv \dots \equiv \lambda_n \pmod 2, \\ \lambda_1 \geq \dots \geq \lambda_{n-1} \geq |\lambda_n|\end{array}\right\} \quad (m = 2n).$$

For $\lambda = (\lambda_1,\dots,\lambda_\kappa) \in \Lambda^*$, put $R_\lambda = R_{\lambda_1,\lambda}$. Then

$$R_{\lambda_0,\lambda} = P_0(x)^{(\lambda_0-\lambda_1)/2} R_\lambda \qquad ((\lambda_0,\lambda) \in \Lambda).$$

By the relation between relatively P-invariant polynomials and highest weight vectors of simple $\mathbb{G}$-modules contained in R, Lemma 1.5 can be translated into the following Proposition:

Proposition 1.6. *The decomposition of* R *into direct sum of simple* $\mathbb{G}$*-modules is given by*

$$R = \bigoplus_{(\lambda_0,\lambda)\in\Lambda} R_{\lambda,\lambda_0} = \bigoplus_{\ell=0}^{\infty} \bigoplus_{\lambda\in\Lambda^*} P_0(x)^\ell R_\lambda.$$

Let $P_0(\frac{\partial}{\partial x})$ be the differential operator with constant coefficients satisfying

$$P_0(\tfrac{\partial}{\partial x})\exp(\mathrm{tr}({}^t yJx)) = P_0(y)\exp(\mathrm{tr}({}^t yJx)).$$

Proposition 1.7. *The space* $\bigoplus_{\lambda\in\Lambda^*} R_\lambda$ *is characterized by the differential equation* $P_0(\frac{\partial}{\partial \chi})Q(x) = 0$, namely,

$$\bigoplus_{\lambda\in\Lambda^*} R_\lambda = \{Q(x) \in R;\ P_0(\tfrac{\partial}{\partial x})Q(x) = 0\}.$$

Proof. The proposition is an immediate consequence of the formula for the b-function of (P,ρ,V), which will be proved in Section 4 (Theorem 4.2).

Remark. Propositions 1.6 and 1.7 are due to Hoppe ([H;Satz 7, Korollar 7.1], see also [M6]). However Hoppe's result contains a slight inaccuracy, which results from that he missed the fact that, when $m = 2n$, $P_0(x)f_{n-1}(x)$ can be divided by $P_n(x)$.

Take a $W \in \mathrm{GL}(m,\mathbb{C})$ such that $J = {}^tWW$. For $\lambda \in \Lambda^*$, put

$$\mathcal{R}_\lambda = \{Q(W^{-1}x);\ Q(x) \in R_\lambda\}.$$

Then

$$R = \bigoplus_{\ell=0}^{\infty} \bigoplus_{\lambda\in\Lambda^*} (\det {}^txx)^\ell\, \mathcal{R}_\lambda \tag{1.3}$$

gives a decomposition of R into simple $\mathbf{SO}(m)$-modules. Put

$$K = SO(m) = \{k \in \mathrm{GL}(m)_{\mathbb{R}};\ {}^tkk = 1_m\}$$

and

$$K_0 = \left\{\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in SO(m);\ k_1 \in SO(n),\ k_2 \in SO(m-n)\right\}.$$

Let $C(K/K_0)$ (resp. $L^2(K/K_0)$) be the space of continuous functions (resp. measurable functions square integrable with respect to the normalized K-invariant measure) on K/K_0.

Define a mapping $\alpha:\mathfrak{R}_\lambda \longrightarrow C(K/K_0)$ by

(1.4) $\alpha(Q)(k) = Q(kx_0), \quad x_0 = \begin{pmatrix} 1_n \\ 0 \end{pmatrix}.$

We denote the image $\alpha(\mathfrak{R}_\lambda)$ by H_λ. Since $\mathfrak{R}_\lambda$ is a simple $SO(m)$-module, the mapping $\alpha:\mathfrak{R}_\lambda \longrightarrow H_\lambda$ is an isomorphism and H_λ gives an irreducible unitary subrepresentation of K of $L^2(K/K_0)$.

Proposition 1.8. *The irreducible decomposition of the regular representation of* K *on* K/K_0 *is given by*

$$L^2(K/K_0) = \bigoplus_{\lambda\in\Lambda^*} H_\lambda.$$

§2. Integral representations of the Maass zeta functions

2.1. Put

$$K = SO(m) = \{k \in GL(m)_{\mathbb{R}};\ {}^t kk = 1_m\}.$$

We consider the $\mathbb{R}$-structure of (G,ρ,V) such that

$$G = G_{\mathbb{R}} = K \times GL(n,\mathbb{R}) \quad \text{and} \quad V_{\mathbb{R}} = M(m,n;\mathbb{R}).$$

Put

$$GL(n,\mathbb{R})^+ = \{g \in GL(n,\mathbb{R});\ \det(g) > 0\}$$

and $$G^+ = G_{\mathbb{R}}^+ = K \times GL(n,\mathbb{R})^+.$$

Put $\Gamma = SL(n,\mathbb{Z})$ and let L be a lattice in $V_{\mathbb{R}}$ stable under the Γ-action from the right. Set

$$V' = V_{\mathbb{R}} - S_{\mathbb{R}} = \{x \in V_{\mathbb{R}};\ \operatorname{rank} x = n\}$$

and $$L' = L \cap V'.$$

The set V' is a single G^+-orbit.

For a homogeneous polynomial $Q(x)$ in R of degree d, the Maass zeta function is defined by the Dirichlet series

$$\xi(Q,L;s) = \sum_{x\in L'/\Gamma} Q(x)(\det {}^t xx)^{-s-d/2n},$$

which is absolutely convergent for $\mathrm{Re}\ s > m/2$ (see Corollary to Proposition 2.3).

We also consider the local zeta function

$$\Phi(Q,f;s) = \int_{V'} (\det {}^t xx)^{s-d/2n-m/2}\, Q(x)f(x)\, dx$$

$$(f \in \mathcal{S}(V_{\mathbb{R}}),\ s \in \mathbb{C}),$$

where dx is the standard Euclidean measure on $V_{\mathbb{R}} = M(m,n;\mathbb{R})$. The integral $\Phi(Q,f;s)$ is absolutely convergent for $\mathrm{Re}\ s > 0$ and has an analytic continuation to a meromorphic function of s in $\mathbb{C}$.

Let π be an irreducible unitary representation of the compact Lie group K. Denote by H_π the representation space of π equipped with hermitian inner product $\langle\ ,\ \rangle$.

In order to obtain an integral representation of the Maass zeta function, we introduce the following $\mathrm{End}(H_\pi)$-valued integral:

$$Z_\pi(f,L;s) = \int_{G^+/\Gamma} \det(g)^{-2s}\, \pi(k) \sum_{x\in L'} f(\rho(k,g)x)\, dk\, dg,$$

$$(f \in \mathcal{S}(V_{\mathbb{R}}))$$

where dg is a Haar measure on $GL(n,\mathbb{R})^+$ and dk is the Haar measure on K so normalized that the total volume is equal to 1. If π is the trivial representation of K, then $Z_\pi(f,L;s)$ gives an integral representaion of the zeta function considered by Koecher [K] and is absolutely convergent for Re $s > m/2$. This implies the following lemma:

Lemma 2.1. *The integral* $Z_\pi(f,L;s)$ *is absolutely convergent for* Re $s > m/2$.

Put

$$K_0 = \left\{\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in K;\ k_1 \in SO(n),\ k_2 \in SO(m-n)\right\}.$$

and

$$H_{\pi,0} = \{v \in H_\pi;\ \pi(k)v = v \quad (k \in K_0)\}.$$

By the irreducibility of π, we have $\dim H_{\pi,0} \le 1$. When $\dim H_{\pi,0} = 1$, the representation π is called *of class* 1 (with respect to K_0). The projection pr of H_π onto $H_{\pi,0}$ is given by the integral

$$pr = \int_{K_0} \pi(k_0)\, dk_0,$$

where dk_0 is the normalized Haarmeasure on K_0.

Put $x_0 = \begin{pmatrix} 1_n \\ 0^{(m-n,n)} \end{pmatrix}$. Then we can find a $(k_x,g_x) \in G^+$ such that $\rho(k_x,g_x)x_0 = x$ for any $x \in V' = V_{\mathbb{R}}-S_{\mathbb{R}}$, since $V' = \{x \in V_{\mathbb{R}};\ \text{rank}\ x = n\}$ is a single G^+-orbit. We define an $\text{End}(H_\pi)$-valued function φ_π on V' by setting

$$\varphi_\pi(x) = \pi(k_x)\circ pr \qquad (x \in V').$$

Since the coset k_xK_0 is uniquely determined by x, the function φ_π does not depend on the chioce of k_x.

Lemma 2.2. *Assume that* Re $s > m/2$.

(i) *The integral* $Z_\pi(f,L:;s)$ *vanishes unless* π *is of class* 1.

(ii) *If* π *is of class* 1, *then*

$$Z_\pi(f,L;s) = \left\{\int_{V'}(\det {}^t xx)^{s-m/2}\, \varphi_\pi(x)\, f(x)\, dx\right\}$$

$$\times \left\{\sum_{x\in L'/\Gamma} \varphi_\pi(x)^* (\det {}^t xx)^{-s}\right\}.$$

where $\varphi_\pi(x)^*$ *is the adjoint operator of* $\varphi_\pi(x)$.

Proof. Note that, for Re $s > m/2$, the integral $Z_\pi(f,L;s)$ is absolutely convergent and the following calculation is justified by the Fubini theorem.

For an $x = \rho(k_x,g_x)x_0 \in V'$, the isotropy subgroup G_x^+ of G^+ at x is given by

$$G_x^+ = \{(k_x\begin{pmatrix} k_1 & 0 \\ 0 & k_2\end{pmatrix}k_x^{-1}, g_x k_1 g_x^{-1});\ k_1 \in SO(n),\ k_2 \in SO(m-n)\}$$

$$\simeq K_0.$$

We normalize the Haar measure $d\mu_x$ on G_x^+ by $\int_{G_x^+} d\mu_x = 1$. The measure $\omega(x) = (\det {}^t xx)^{-m/2}\, dx$ on V' is G^+-invariant. We can normalize a Haar measure dg on $\mathrm{GL}(n,\mathbb{R})^+$ so that the following integral formula holds:

$$\int_{G^+} F(k,g)\, dk\, dg = \int_{V'} \omega(y) \int_{G_x^+} F((k_y,g_y)h)\, d\mu_x(h)$$

$$(F \in \mathrm{L}^1(G^+)).$$

By this formula, we obtain

$$Z_\pi(f,L;s) = \sum_{x\in L'/\Gamma} \int_{V'} \left(\frac{\det {}^t yy}{\det {}^t xx}\right)^s f(y)\ \omega(y) \int_{K_0} \pi(k_y k_0 k_x^{-1})\ dk_0$$

$$= \sum_{x\in L'/\Gamma} \int_{V'} \left(\frac{\det {}^t yy}{\det {}^t xx}\right)^s f(y)\ \pi(k_y)\circ pr\circ\pi(k_x^{-1})\ \omega(y)$$

$$= \int_{V'} (\det {}^t yy)^s f(y)\ \pi(k_y)\circ pr\ \omega(y)$$

$$\times \sum_{x\in L'/\Gamma} (\det {}^t xx)^{-s}\ pr\circ\pi(k_x^{-1}).$$

Since π is unitary, we have $pr\circ\pi(k_x^{-1}) = (\pi(k_x)\circ pr)^*$. Hence we get

$$Z_\pi(f,L;s) = \left\{\int_{V'} (\det {}^t yy)^s \varphi_\pi(y)\ f(y)\ \omega(y)\right\}$$

$$\times \left\{\sum_{x\in L'/\Gamma} \varphi_\pi(x)^* (\det {}^t xx)^{-s}\right\}.$$

If π is not of class 1, then pr is the 0-map and hence $Z_\pi(f,L;s) = 0$. □

2.2. An irreducible unitary representation π of K is contained in the regular representation of K on $L^2(K/K_0)$ if and only if π is of class 1 with respect to K_0 and then the multiplicity of π is equal to 1. For π of class 1, take a unit vector v_0 in $H_{\pi,0}$. Then the mapping

$$q: H_\pi \longrightarrow L^2(K/K_0)$$

$$v \longmapsto q(v;k) = \langle v, \pi(k)^{-1} v_0\rangle$$

gives an embedding of H_π and the image $q(H_\pi)$ coincides with H_λ for some $\lambda \in \Lambda^*$ (cf. Proposition 1.8). In this case we write $\pi = \pi_\lambda$.

Composing the mapping q with the inverse mapping of α

defined by (1.4), we define a K-isomorphism $Q: H_\pi \longrightarrow \mathfrak{R}_\lambda$ by

$$Q(v;x) = \alpha^{-1}(q(v;k))(x) \quad (v \in H_\pi),$$

namely, $Q(v;x)$ is the polynomial in $\mathfrak{R}_\lambda$ satisfying

$$Q(v;kx_0) = \langle v, \pi(k)^{-1}v_0 \rangle.$$

Proposition 2.3. *Let* $\pi = \pi_\lambda$ $(\lambda \in \Lambda^*)$ *be an irreducible unitary representation of* $SO(m)$ *of class* 1. *When* $\mathrm{Re}\, s > m/2$, *the following identity holds for any* v, $w \in H_\pi$:

$$\langle Z_\pi(f,L;s)v,w \rangle = \xi(Q(v;\cdot),L;s)\cdot\Phi(\overline{Q(w;\cdot)},f;s).$$

Proof. For x, $y \in V'$, we can easily prove the identity

$$\langle \varphi_\pi(y)\varphi_\pi(x)^* v,w \rangle = \frac{Q(v;x)}{(\det {}^t xx)^{\lambda_1/2}} \cdot \frac{\overline{Q(w;y)}}{(\det {}^t yy)^{\lambda_1/2}}.$$

Now the lemma follows immediately from this identity and lemma 2.2. □

Remark. By the decomposition (1.3), it is sufficient for the description of analytic properties of the Maass zeta functions to consider the case where $Q(x)$ is in $\mathfrak{R}_\lambda$ for some $\lambda \in \Lambda^*$. Conversely, since the form of the functional equations of the Maass zeta functions depend on λ (see Theorem 3.1 below), it is necessary to consider the decomposition (1.3).

Corollary to Proposition 2.3. *For any homogeneous* *polynomial* $Q(x)$ *in* R, *the Maass zeta function* $\zeta(Q,L;s)$ *is absolutely convergent for* $\mathrm{Re}\, s > m/2$.

§3. Functional equations

3.1. For a lattice L in $V_{\mathbb{R}}$, let L^* be the lattice dual to L:

$$L^* = \{y \in V_{\mathbb{R}};\ \mathrm{tr}({}^t yx) \in \mathbb{Z} \quad \text{for all} \quad x \in L\}.$$

The following is the main theorem of the present paper:

Theorem 3.1. *Let* $Q(x)$ *be a polynomial in* $\mathfrak{R}_\lambda$ $(\lambda \in \Lambda^*)$. *Then*

(i) $\zeta(Q,L;s)$ *has an analytic continuation to a meromorphic function of* s *in* $\mathbb{C}$ *and the function*

$$\prod_{i=1}^{n} \left(s + \frac{\lambda_i - i - 1}{2}\right)\left(s - \frac{\lambda_i + m - i + 1}{2}\right) \cdot \zeta(Q,L;s)$$

is an entire function.

(ii) *Put*

$$\xi(Q,L;s) = v(L)^{1/2}\, \pi^{-ns} \prod_{i=1}^{n} \Gamma\left(s + \frac{\lambda_i - i + 1}{2}\right) \zeta(Q,L;s),$$

where $v(L) = \int_{V_{\mathbb{R}}/L} dx$. *Then the following functional equation holds:*

$$\xi(Q,L^*;m/2-s) = \exp\left(\frac{\pi\sqrt{-1}}{2} \sum_{i=1}^{n} \lambda_i\right) \xi(Q,L;s).$$

3.2. The proof of the theorem above is based on the functional equations satisfied by $Z_\pi(f,L;s)$ and $\Phi(Q,f;s)$. First let us ~~prove~~ consider the functional equation satisfied by $Z_\pi(f,L;s)$.

For $f \in \mathcal{S}(V_\mathbb{R})$, define the Fourier transform $\hat{f}$ of f by setting

$$\hat{f}(y) = \int_{V_\mathbb{R}} f(x) \exp(2\pi\sqrt{-1}\ \mathrm{tr}({}^t yx))\, dx.$$

The proof of the following proposition is quite similar to that of [S1; Lemma 6.1].

Proposition 3.2. *If* $f \in \mathcal{S}(V_\mathbb{R})$ *satisfies the condition*

(3.3) $f(x) = \hat{f}(x) = 0$ *for any* $x \in V_\mathbb{R}$ *such that* rank $x < n$,

then $Z_\pi(f,L;s)$ *has an analytic continuation to an entire function of* s *and satisfies the functional equation*

$$Z_\pi(\hat{f},L;s) = v(L)^{-1} Z_\pi(f,L^*;m/2-s).$$

Remark. Let f_0 be a function in $C_0^\infty(V')$. Then the functions $\det({}^t\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right))f_0(x)$ and $\det({}^t xx)\hat{f}_0(x)$ satisfy the condition (3.3) in the proposition above (cf. [SS; p.169, Additional remark 2]), [S1; Lemma 6.2]).

3.3. The local functional equation satisfied by $\Phi(Q,f;s)$ is given in the following theorem:

Theorem 3.3. *Let* $Q(x)$ *be a polynomial in* $\mathfrak{R}_\lambda$. *Then the following functional equation holds for any* $f \in \mathcal{S}(V_{\mathbb{R}})$:

$$\Phi(Q,\hat{f};s) = e^{\lambda_1 n\pi\sqrt{-1}/2}\, \pi^{-2ns+\frac{n(m-2)}{2}} \prod_{i=1}^{n} \sin \pi(s - \frac{\lambda_1+m-i-1}{2})$$

$$\times \prod_{i=1}^{n} \Gamma(s + \frac{\lambda_i-i+1}{2})\Gamma(s - \frac{\lambda_i+m-i-1}{2})\ \Phi(Q,f;m/2-s).$$

The proof of Theorem 3.3 is based on the following proposition, which will be proved in Section 4.

Proposition 3.4. *For* $Q(x)$ *in* $\mathfrak{R}_\lambda$ $(\lambda \in \Lambda^*)$, *let* $Q(\frac{\partial}{\partial x})$ *be the differential operator with constant coefficients satisfying*

$$Q(\frac{\partial}{\partial x})\exp({}^t yx) = Q(y)\exp({}^t yx).$$

Then

(i) $$Q(\frac{\partial}{\partial x})(\det {}^t xx)^s = b_\lambda(s)Q(x)(\det {}^t xx)^{s-\lambda_1},$$

where

$$b_\lambda(s) = 2^{\lambda_1 n} \prod_{i=1}^{n} \prod_{j=1}^{(\lambda_1+\lambda_i)/2} (s + \frac{i+1}{2} - j) \prod_{j=1}^{(\lambda_1-\lambda_i)/2} (s + \frac{m-i+1}{2} - j).$$

(ii) $$\det({}^t\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right))\{Q(x)(\det {}^t xx)^s\} = \beta_\lambda(s)Q(x)(\det {}^t xx)^{s-1},$$

where

$$\beta_\lambda(s) = 2^{2n} \prod_{i=1}^{n} (s + \frac{\lambda_1-\lambda_i}{2} + \frac{i-1}{2})(s + \frac{\lambda_1+\lambda_i}{2} + \frac{m-i-1}{2}).$$

Proof of Theorem 3.3. First we consider the case where $\lambda = (0,\dots,0)$ and $Q(x)$ is a constant function. In this case, it is well known that the functional equation in Theorem 3.3 holds for any $f \in \mathcal{S}(V_{\mathbb{R}})$. When $Q(x) \equiv 1$ let us write simply $\Phi(f;s)$ instead of $\Phi(Q,f;s)$. Now we consider the general case. It is easy to see that

$$Q(x)\hat{f}(x) = (-2\pi\sqrt{-1})^{-\lambda_1 n}(Q(\tfrac{\partial}{\partial x})f)^{\wedge}(x).$$

Therefore, by the functional equation for $\lambda = (0,\dots,0)$, we obtain

$$\Phi(Q,\hat{f};s) = (-2\pi\sqrt{-1})^{-\lambda_1 n}\ \Phi((Q(\tfrac{\partial}{\partial x})f)^{\wedge};s-\lambda_1/2)$$

$$= (-2\pi\sqrt{-1})^{-\lambda_1 n}\ \pi^{-2n(s-\lambda_1/2)+n(m-2)/2}\ \Phi(Q(\tfrac{\partial}{\partial x})f;(m+\lambda_1)/2-s)$$

$$\times \prod_{i=1}^{n} \sin \pi(s - \frac{\lambda_1+m-i-1}{2})\ \Gamma(s - \frac{\lambda_1+i-1}{2})\Gamma(s - \frac{\lambda_1+m-i-1}{2}).$$

By integrating by parts, we have from Proposition 3.4 (i)

$$\Phi(Q(\tfrac{\partial}{\partial x})f;(m+\lambda_1)/2-s) = (-1)^{\lambda_1 n}\ b_{\lambda}(\lambda_1/2-s)\ \Phi(Q,f;m/2-s).$$

This proves the theorem. □

Remark. Local functional equations and b-functions attached to representations on polynomial rings, which are similar to Thoerem 3.3 and proposition 3.4 (i), have been previously considered in [RS] for prehomogeneous vector spaces of commutative parabolic type.

Proof of Theorem 3.1. As in the remark following

Proposition 3.2, let f_0 be a function in $C_0^{\infty}(V')$ and put $f(x) = \det({}^t\left(\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial x}\right))f_0(x)$. For a $Q(x)$ in $\mathfrak{R}_{\lambda}$, take $v, w \in H_{\pi_{\lambda}}$ such that $Q(v;x) = Q(x)$ and $\overline{Q(w;x)} = Q(x)$. Then, by Proposition 2.3 and Proposition 3.2, the function

$$\xi(Q,L;s)\Phi(Q,f;s) = \langle Z_{\pi}(f,L;s)v,w\rangle$$

is an entire function. By integrating by parts, we have from Proposition 3.4 (ii)

$$\Phi(Q,f;s) = \beta_{\lambda}(s - (\lambda_1+m)/2)\Phi(Q,f_0;s-1).$$

We can choose an f_0 so that $\Phi(Q,f_0;s-1) \neq 0$. Hence the function $\beta_{\lambda}(s - (\lambda_1+m)/2)\xi(Q,L;s)$ has an analytic continuation to an entire function. This proves the first part. By the functional equation on $Z_{\pi}(f,L;s)$ in Proposition 3.2, we obtain

$$\xi(Q,L;s)\Phi(Q,f;s) = v(L)^{-1}\xi(Q,L^*;m/2-s)\Phi(Q,f;m/2-s).$$

Hence it follows from Theorem 3.3 that

$$v(L)^{-1}\xi(Q,L^*;m/2 - s)$$
$$= e^{\lambda_1 n\pi\sqrt{-1}/2}\,\pi^{-2ns+\frac{n(m-2)}{2}}\,\xi(Q,L;s)$$
$$\times \prod_{i=1}^{n} \sin \pi(s - \frac{\lambda_1+m-i-1}{2})\Gamma(s + \frac{\lambda_i-i+1}{2})\Gamma(s - \frac{\lambda_i+m-i-1}{2}).$$

Using the formula $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$, we can easily rewrite the identity above into the form given in Theorem 3.1 (ii). □

§4. Local functional equation and the b-function of (P,ρ,V)

In this section we retain the notation used in Section 1. Consider the standard $\mathbb{R}$-structure of (P,ρ,V):

$$P_{\mathbb{R}} = B(m)_{\mathbb{R}} \times GL(n)_{\mathbb{R}}, \quad V_{\mathbb{R}} = M(m.n;\mathbb{R}).$$

We identify the vector space dual to V with V itself via the symmetric bilinear form

$$(x,y) = \mathrm{tr}({}^t yJx).$$

Then the representation ρ^* of P contragredient to ρ is given by

$$\rho^*(b,g)y = \rho(b,{}^t g^{-1})y = by\,{}^t g.$$

Then $P_0,\ldots, P_n$ are relative invariants of (P,ρ,V) and the corresponding rational characters are given by

$$\chi_0^*(h,g) = \det(g)^2,$$

$$\chi_i^*(h,g) = \det(g)^2\cdot(a_1\cdots a_i)^{-2} \quad (1 \le i \le n-2)$$

$$\chi_{n-1}^*(h,g) = \begin{cases} \det(g)^2\cdot(a_1\cdots a_{n-1})^{-2} & \text{if } m > 2n, \\ \det(g)\cdot(a_1\cdots a_{n-1})^{-1}\cdot a_n & \text{if } m = 2n, \end{cases}$$

$$\chi_n^*(h,g) = \det(g)\cdot(a_1\cdots a_n)^{-1}$$

for $h = b_1(A)b_2(v)b_2(B) \in B(m)$ $\left(A = \begin{pmatrix} a_1 & & * \\ & \ddots & \\ 0 & & a_\kappa \end{pmatrix}\right)$ and $g \in GL(n)$.

Hence

$$\chi_0 = \chi_0^{*-1},$$

$$\chi_i = \chi_0^{*-2}\cdot\chi_i^* \quad (1 \le i \le n-2),$$

(4.1)
$$\chi_{n-1} = \begin{cases} \chi_0^{*-2}\cdot\chi_{n-1}^* & \text{if } m > 2n, \\ \chi_0^{*-1}\cdot\chi_{n-1} & \text{if } m = 2n, \end{cases}$$

$$\chi_n = \chi_0^{*-1}\chi_n^*$$

and

(4.2) $$\det \rho(b,g) = \det(g)^{-m} = \chi_0^{m/2}.$$

For $\varepsilon = \pm 1$ and $\eta = (\eta_1,\dots,\eta_n) \in \{\pm 1\}^n$, put

$$V_{\varepsilon,\eta} = \left\{ x \in V_{\mathbb{R}};\ \begin{array}{l} \operatorname{sgn} P_i(x) = \eta_{i+1}\cdots\eta_n \quad (0 \le i \le n-2), \\ \operatorname{sgn} P_{n-1}(x) = \begin{cases} \eta_n & (m > 2n) \\ \varepsilon\eta_n & (m = 2n) \end{cases} \\ \operatorname{sgn} P_n(x) = \varepsilon \end{array} \right\}.$$

We have

$$V_{\mathbb{R}} - S_{P,\mathbb{R}} = \bigcup_{(\varepsilon,\eta)\in\{\pm 1\}^{n+1}} V_{\varepsilon,\eta}.$$

Define the local zeta functions of (P,ρ,V) by the integrals

$$\Phi_{\varepsilon,\eta}(f;s_0,s) = \Phi_{\varepsilon,\eta}(f;s_0,s_1,\dots,s_n)$$

$$= \int_{V_{\varepsilon,\eta}} |P_0(x)|^{s_0} \prod_{i=1}^{n-1} |P_i(x)|^{(s_i-s_{i+1})/2} |P_n(x)|^{s_n^*} f(x)\, dx$$

$(f \in \mathcal{S}(V_{\mathbb{R}}),\ (s_0,s) \in \mathbb{C}^{n+1},\ (\varepsilon,\eta) \in \{\pm 1\}^{n+1})$, where $s_n^* = s_n$ or $(s_{n-1}+s_n)/2$ according as $m > 2n$ or $m = 2n$. Then $\Phi_{\varepsilon,\eta}(f;s_0,s)$ are absolutely convergent for

(4.3)
$$\operatorname{Re} s_0 > 0,\quad \operatorname{Re} s_1 \ge \dots \ge \operatorname{Re} s_{n-1} \ge \begin{cases} \operatorname{Re} s_n \ge 0 & (m > 2n), \\ |\operatorname{Re} s_n| & (m = 2n), \end{cases}$$

and have analytic continuations to meromorphic functions of

(s_0, s) in $\mathbb{C}^{n+1}$.

Define the Fourier transform $\hat{f}$ of $f \in \mathcal{S}(V_{\mathbb{R}})$ by

$$\hat{f}(y) = \int_{V_{\mathbb{R}}} f(x)\, e^{2\pi\sqrt{-1}\,\mathrm{tr}({}^t yJx)}\, dx.$$

Theorem 4.1. *The following functional equations hold for any* $f \in \mathcal{S}(V_{\mathbb{R}})$:

$$\Phi_{\varepsilon,\eta}(\hat{f}; s_0, s) = \sum_{(\varepsilon^*,\eta^*)\in\{\pm1\}^{n+1}} \Gamma_{\varepsilon,\eta}^{\varepsilon^*,\eta^*}(s_0, s)\Phi_{\varepsilon^*,\eta^*}(f; -m/2-s_0-s_1),$$

where

$$\Gamma_{\varepsilon,\eta}^{\varepsilon^*,\eta^*}(s_0, s) = 2^{-n(n+1)/2}\,\pi^{-n(2s_0+s_1)-n(m+2)/2}$$

$$\times \prod_{i=1}^{n} \Gamma\left(s_0 + \frac{s_1-s_i}{2} + \frac{i+1}{2}\right)\Gamma\left(s_0 + \frac{s_1+s_i}{2} + \frac{m-i+1}{2}\right)$$

$$\times \sum_{\substack{\nu\in\{\pm1\}^n \\ \nu_1\cdots\nu_n=\varepsilon\varepsilon^*}} \exp\left(\frac{\pi\sqrt{-1}}{2}\left\{\frac{(-1)^n}{2}\delta\varepsilon\varepsilon^* + L_{\eta,\eta^*}^{\nu}(s_0, s)\right\}\right),$$

$$L_{\eta,\eta^*}^{\nu}(s_0, s) = \frac{1}{2}\sum_{i=1}^{n} \nu_i\left(\sum_{j=1}^{i-1}\eta_j + \sum_{j=i+1}^{\nu}\eta_j^*\right)$$

$$+ \sum_{i=1}^{n}\left\{\eta_i\nu_i\left(s_0 + \frac{s_1-s_i}{2} + \frac{i+1}{2}\right) + \eta_i^*\nu_i\left(s_0 + \frac{s_1+s_i}{2} + \frac{m-i+1}{2}\right)\right.$$

and $\delta = 1$ *or* 0 *according as* m *is odd or even.*

In the special case $n = 1$, Theorem 4.1 has been proved in [S3; Theorem 3.6]. The proof of the general case is simlar and is omitted.

For $(\lambda_0,\lambda) \in \Lambda$, let $P_{\lambda_0,\lambda}(\frac{\partial}{\partial x})$ be the differential operator with constant coefficients satisfying

$$P_{\lambda_0,\lambda}(\frac{\partial}{\partial x})\exp(\mathrm{tr}({}^t xJy)) = P_{\lambda_0,\lambda}(y)\exp(\mathrm{tr}({}^t xJy)).$$

Then there exists a polynomial $b_{\lambda_0,\lambda}(s_0,s)$ such that

$$(4.2)\quad P_{\lambda_0,\lambda}(\frac{\partial}{\partial x})\{P_0(x)^{s_0}\prod_{i=1}^{n-1} P_i(x)^{(s_i-s_{i+1})/2}\cdot P_n(x)^{s_n^*}\}$$
$$= b_{\lambda_0,\lambda}(s_0,s)P_{\lambda_0,\lambda}(x)$$
$$\times P_0(x)^{s_0-\lambda_0}\prod_{i=1}^{n-1} P_i(x)^{(s_i-s_{i+1})/2}\cdot P_n(x)^{s_n^*},$$

which is called the b-function of (P,ρ,V) (see e.g. [S1;Lemma 3.1]). By Theorem 4.1 and the expression of the b-function in terms of the coefficients of the local functional equation (cf [S1;(5-8)]), we can easily calculate $b_{\lambda_0,\lambda}(s_0,s)$.

Theorem 4.3. *For* $(\lambda_0,\lambda) \in \Lambda$,

$$b_{\lambda_0,\lambda}(s_0,s) = 2^{\lambda_0 n}\prod_{i=1}^{n}\{\prod_{j=1}^{(\lambda_0+\lambda_i)/2}(s_0 + \frac{s_1-s_i}{2} + \frac{i+1}{2} - j)$$
$$\times\prod_{j=1}^{(\lambda_0-\lambda_i)/2}(s_0 + \frac{s_1+s_i}{2} + \frac{m-i+1}{2} - j)\}.$$

Proof of Proposition 3.4. Since $Q(x) \in \mathcal{R}_\lambda$ is a

K-translate of $P_{\lambda_1,\lambda}(W^{-1}x)$, we may assume that $Q(x) = P_{\lambda_1,\lambda}(W^{-1}x)$. Then Proposition 3.4 is an immediate consequence of (4.2) and we have

$$b_\lambda(s) = b_{\lambda_1,\lambda}(s,\overbrace{0,\ldots,0}^{n}),$$

$$\beta_\lambda(s) = b_{2,\mathbf{0}}(s,\lambda_1,\ldots,\lambda_n),$$

where $\mathbf{0} = (0,\ldots,0) \in \Lambda^*$. □

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