数理解析研究所講究録
第 720 巻 1990 年 8-13

Instantons and representations of an associative algebra

YOSHITAKE HASHIMOTO 橋本義武

Department of Mathematics, Faculty of Science,
University of Tokyo, Hongo Tokyo 113, Japan

In this note we show that instantons on S^4 can be identified with some representations of an associative algebra.

Let A be the free algebra over $\mathbb{C}$ generated by two elements q, p.We define a new product $*$ in A as follows:

$$f_1 * f_2 = f_1(pq - qp)f_2, \quad f_1, f_2 \in A.$$

Then $(A, *)$ is an associative algebra (with no unit), which is an extention of the Weyl algebra $A/(pq - qp - 1)$. We consider finite dimensional representations of $(A, *)$. Let W be the complex vector space of dimension l, and h be a linear map from A to End W. Then h induces a linear map

$$\widetilde{h}\colon A \otimes W \to A^* \otimes W$$

defined by

$$\langle \widetilde{h}(f_1 \otimes w), f_2 \rangle = h(f_2 f_1)w, \quad f_1, f_2 \in A, \ w \in W.$$

We denote by $H(l, k)$ the set of all algebra homomorphisms $h\colon (A, *) \to$ End W such that the rank of $\widetilde{h}$ is k.

Let P be the principal $SU(l)$ bundle over $S^4 = \mathbb{R}^4 \cup \infty$ with $c_2 = k$, and $\widetilde{M}(SU(l), k)$ be the framed moduli space for anti-self-dual (ASD) connections on P: { ASD connections on P } $/\mathcal{G}_\infty$, where $\mathcal{G}_\infty$ stands for the group of all gauge transformations on P fixing the points in the fiber over ∞. $\widetilde{M}(SU(l), k)$ is a $4kl$-dimensional smooth manifold.

Our main result is the following:

THEOREM 1. *The framed moduli space* $\widetilde{M}(SU(l), k)$ *is diffeomorphic to* $H(l, k)$.

§1. Some remarks on a theorem of Donaldson.

Let $X = \mathrm{Mat}(k, k; \mathbb{C}) \times \mathrm{Mat}(k, k; \mathbb{C}) \times \mathrm{Mat}(l, k; \mathbb{C}) \times \mathrm{Mat}(k, l; \mathbb{C})$. We define the action of $G = GL(k, \mathbb{C})$ on X as follows:

$$p \cdot (\alpha_1, \alpha_2, a, b) = (p\alpha_1 p^{-1}, p\alpha_2 p^{-1}, ap^{-1}, pb)$$

for $p \in G$, $(\alpha_1, \alpha_2, a, b) \in X$. We call a point x in X *stable* when the map $G \ni p \mapsto p \cdot x \in X$ is proper. We denote by X^s the set of all stable points in X. Let

$$\omega(\alpha_1, \alpha_2, a, b) = \mathrm{tr}(d\alpha_1 \wedge d\alpha_2 + db \wedge da),$$
$$\mu = \alpha_1\alpha_2 - \alpha_2\alpha_1 + ba.$$

We can show by easy computation that

$$\omega(p\alpha_1 p^{-1}, p\alpha_2 p^{-1}, ap^{-1}, pb) = \omega(\alpha_1, \alpha_2, a, b) + \mathrm{tr}(p^{-1}dp \wedge d\mu) + \mathrm{tr}(p^{-1}dp \wedge p^{-1}dp \cdot \mu).$$

(This is suggested to the author by H. Nakajima from the viewpoint of hyperkähler structure.)

THEOREM (DONALDSON [1]). *The framed moduli space $\widetilde{M}(SU(l), k)$ is diffeomorphic to $G \backslash \mu^{-1}(0) \cap X^s$.*

So we deduce from geometric invariant theory [4] that $\widetilde{M}(SU(l), k)$ is an open dense nonsingular subset of an affine algebraic variety.

Next we seek a criterion for the stability in this case. Let $A^m \in \mathrm{Mat}(2^m l, k; \mathbb{C})$ be the matrix which is the column of matrices $a\alpha_{i_1} \cdots \alpha_{i_m}$, $i_j = 0, 1$, and $B^m \in \mathrm{Mat}(k, 2^m l; \mathbb{C})$ be the matrix which is the row of matrices $\alpha_{i_1} \cdots \alpha_{i_m} b$, *i. e.*

$$A^0 = a,\ A^1 = \begin{pmatrix} a\alpha_1 \\ a\alpha_2 \end{pmatrix},\ A^2 = \begin{pmatrix} a\alpha_1\alpha_1 \\ a\alpha_2\alpha_1 \\ a\alpha_1\alpha_2 \\ a\alpha_2\alpha_2 \end{pmatrix}, \ldots, A^m = \begin{pmatrix} A^{m-1}\alpha_1 \\ A^{m-1}\alpha_2 \end{pmatrix},$$
$$B^0 = b,\ B^1 = (\,\alpha_1 b \quad \alpha_2 b\,),\ B^2 = (\,\alpha_1\alpha_1 b \quad \alpha_1\alpha_2 b \quad \alpha_2\alpha_1 b \quad \alpha_2\alpha_2 b\,),$$
$$\ldots, B^m = (\,\alpha_1 B^{m-1} \quad \alpha_2 B^{m-1}\,).$$

We set

$$A_m = \begin{pmatrix} A^0 \\ \vdots \\ A^m \end{pmatrix}, \quad B_m = (\,B^0 \quad \ldots \quad B^m\,).$$

LEMMA 2. *The point $x = (\alpha_1, \alpha_2, a, b) \in X$ is stable if and only if* rank $A_{k-1}B_{k-1} = k$.

LEMMA 2'. *The point $x = (\alpha_1, \alpha_2, a, b) \in X$ is stable if and only if* rank $A_m B_n = k$ *for some m, n.*

PROOF: We can test the stability of a point by the following:

HILBERT CRITERION ([1,4]). *The point $x \in X$ is stable for G if and only if for all $g \in G$ and integers $(w_1, \dots, w_k) \neq (0, \dots, 0)$:*

$$g \begin{pmatrix} t^{w_1} & & \\ & \ddots & \\ & & t^{w_k} \end{pmatrix} g^{-1} \cdot x \to \infty \qquad \text{as} \quad t \to \infty.$$

CLAIM: If rank A_{m+1} = rank A_m, then rank $A_{m'}$ = rank A_m for all $m' \geq m$. Similarly, if rank B_{m+1} = rank B_m, then rank $B_{m'}$ = rank B_m for all $m' \geq m$.

PROOF: Assume that rank A_{m+1} = rank A_m. Then the row vectors in A^{m+1} can be written by the linear combinations of the row vectors in A_m. So the row vectors in $A^{m+2} = \begin{pmatrix} A^{m+1}\alpha_1 \\ A^{m+1}\alpha_2 \end{pmatrix}$ are the linear conbinations of the row vectors in $A_m\alpha_1$, $A_m\alpha_2$, which are the row vectors in A_{m+1}. So rank A_{m+2} = rank A_{m+1}. The claim follows by induction. ∎

Now we go back to the proof of Lemma 2, 2'. First we assume that rank $A_{k-1} = k' < k$. If $k = 1$, then $a = 0$ and

$$t^{-1} \cdot (\alpha_1, \alpha_2, a, b) = (\alpha_1, \alpha_2, 0, t^{-1}b) \to (\alpha_1, \alpha_2, 0, 0) \quad \text{as} \quad t \to \infty.$$

This implies that $(\alpha_1, \alpha_2, a, b)$ is not stable.

If $k > 1$, we deduce from the Claim that rank $A_{k-2} = k'$. So

$$A_{k-1}g = \begin{pmatrix} A_{k-2}g \\ A^{k-1}g \end{pmatrix} = \begin{pmatrix} A' & 0 \\ * & 0 \end{pmatrix},$$

for some $g \in G$, where the column vectors in A' are linearly independent. Particularly, $ag = (\, * \quad 0 \,)$. Since the row vectors in $A_{k-2}\alpha_1$ are the ones in A_{k-1},

$$(\, A' \quad 0 \,) g^{-1}\alpha_1 g = (\, * \quad 0 \,).$$

This implies that $g^{-1}\alpha_1 g = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$. Similarly we get $g^{-1}\alpha_2 g = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}$. So

$$\begin{pmatrix} 1_{k'} & \\ & t^{-1}1_{k-k'} \end{pmatrix} \cdot (g^{-1}\alpha_1 g, g^{-1}\alpha_2 g, ag, g^{-1}b)$$

converges as $t \to \infty$. Therefore if rank $A_{k-1} < k$, then $x = (\alpha_1, \alpha_2, a, b)$ is not stable. Similarly, if rank $B_{k-1} < k$, x is not stable.

Next we assume that $(\alpha_1, \alpha_2, a, b)$ is not stable. From the Hilbert Criterion we get some $g \in G, (w_1, \dots, w_k)$ such that

$$\begin{pmatrix} t^{w_1} & & \\ & \ddots & \\ & & t^{w_k} \end{pmatrix} \cdot (g^{-1}\alpha_1 g, g^{-1}\alpha_2 g, ag, g^{-1}b)$$

converges as $t \to \infty$. We may assume that $w_1 \geq \dots \geq w_k$. If $w_{k'} \geq 0 > w_{k'+1}$, we deduce that

$$ag = (* \quad 0), \, g^{-1}\alpha_1 g = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}, \, g^{-1}\alpha_2 g = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}.$$

This implies that $A_m g = (* \quad 0)$. Similarly, if $w_{k'} > 0 \geq w_{k'+1}$, then $g^{-1}B_n = \begin{pmatrix} 0 \\ * \end{pmatrix}$. Therefore if $(\alpha_1, \alpha_2, a, b)$ is not stable, then rank $A_m B_n < k$ for all m, n. ∎

§2 The proof of Theorem 1.

First we give the map φ from $\widetilde{M}(SU(l), k)$ to $H(l, k)$. Let

$$h(f) = \varphi(\alpha_1, \alpha_2, a, b)(f) = af(\alpha_1, \alpha_2)b$$

for $(\alpha_1, \alpha_2, a, b) \in \mu^{-1}(0) \cap X^s$. φ is G-invariant. Since $\mu(\alpha_1, \alpha_2, a, b) = 0$,

$$\begin{aligned} h(f_1 * f_2) &= h(f_1(pq - qp)f_2) \\ &= af_1(\alpha_1, \alpha_2)(\alpha_2\alpha_1 - \alpha_1\alpha_2)f_2(\alpha_1, \alpha_2)b \\ &= af_1(\alpha_1, \alpha_2)baf_2(\alpha_1, \alpha_2)b \\ &= h(f_1)h(f_2). \end{aligned}$$

We give $i \colon \mathbf{C}^k \to A^* \otimes \mathbf{C}^l$, $j \colon A \otimes \mathbf{C}^l \to \mathbf{C}^k$ by

$$\begin{aligned} \langle i(v), f \rangle &= af(\alpha_1, \alpha_2)v \\ j(f \otimes w) &= f(\alpha_1, \alpha_2)bw \end{aligned}$$

for $f \in A$, $v \in V$, $w \in W$. Then we have $\widetilde{h} = i \circ j$. Lemma 2' implies that i is injective and that j is surjective, so rank $\widetilde{h} = k$. Therefore $h \in H(l, k)$.

On the other hand, the inverse $\psi \colon H(l, k) \to \widetilde{M}(SU(l), k)$ is defined as follows. For $h' \in H(l, k)$, we set $V = \operatorname{Coim} \widetilde{h}' \cong \operatorname{Im} \widetilde{h}' \cong \mathbf{C}^k$. Let

$$\widetilde{h}' = i' \circ j', \qquad \begin{aligned} i' &\colon V \to A^* \otimes W, \\ j' &\colon A \otimes W \to V. \end{aligned}$$

For $f \in A$ we define $\langle f| \in \text{Hom}\,(V,W)$, $|f\rangle \in \text{Hom}\,(W,V)$ by

$$\langle f|(v) = \langle i'(v), f\rangle, \quad v \in V,$$
$$|f\rangle(w) = j'(f \otimes w), \quad w \in W.$$

We set $a' = \langle 1|$, $b' = |1\rangle$. The multiplications by q, p in A induce linear maps $\alpha_1', \alpha_2' \in \text{End}\ V$ respectively:

$$\alpha_1'|f\rangle = |qf\rangle, \quad \alpha_2'|f\rangle = |pf\rangle$$

for $f \in A$. If $|f\rangle = 0$, then $h(f'f) = 0$ for all $f' \in A$. So $\alpha_1', \alpha_2' \in \text{End}\ V$ are well-defined. We get

$$\psi(h') = (\alpha_1', \alpha_2', a', b') \in X$$

by fixing the basis of V,W. Since

$$\bigcap_{f\in A} \text{Ker}\ a'f(\alpha_1', \alpha_2') = \bigcap_{f\in A} \text{Ker}\ \langle f| = 0,$$
$$\sum_{f\in A} \text{Im}\ f(\alpha_1', \alpha_2')b' = \sum_{f\in A} \text{Im}\ |f\rangle = V,$$

we deduce from Lemma 2' that $\psi(h')$ is stable. Since $h'\colon (A,*) \to \text{End}\ W$ is an algebra homomorphism, we have

$$\begin{aligned}\langle f_1|\alpha_1'\alpha_2' - \alpha_2'\alpha_1' + b'a'|f_2\rangle &\\ &= h'(f_1(qp-pq)f_2) + \langle f_1|1\rangle\langle 1|f_2\rangle \\ &= -h'(f_1 * f_2) + h'(f_1)h'(f_2) \\ &= 0.\end{aligned}$$

Therefore $\psi(h') \in G\backslash\mu^{-1}(0) \cap X^s$.

If $(\alpha_1', \alpha_2', a', b') = \psi(h')$,

$$\begin{aligned}a'f(\alpha_1', \alpha_2')b' &= \langle 1|f(\alpha_1', \alpha_2')|1\rangle \\ &= \langle 1|f\rangle \\ &= h'(f).\end{aligned}$$

Hence $\varphi \circ \psi(h') = h'$.

If $h' = \varphi(\alpha_1, \alpha_2, a, b)$, we can take $i' = i$, $j' = j$ by the stability. Then

$$\langle f| = af(\alpha_1, \alpha_2), \quad |f\rangle = f(\alpha_1, \alpha_2)b.$$

This implies that

$$\langle 1| = a, \quad |1\rangle = b,$$
$$|qf\rangle = \alpha_1 f(\alpha_1, \alpha_2)b = \alpha_1|f\rangle,$$
$$|pf\rangle = \alpha_2 f(\alpha_1, \alpha_2)b = \alpha_2|f\rangle.$$

Hence $\psi \circ \varphi = \text{id}$. ∎

REFERENCES

1. S.K.Donaldson, *Instantons and geometric invariant theory*, Comm. Math. Phys. **93** (1984), 453–460.
2. M.Furuta and Y.Hashimoto, *Invariant instantons on* S^4, preprint (1988).
3. Y.Hashimoto, *Group actions on the moduli spaces for instantons over* S^4, Master thesis (in Japanese) (1987).
4. D.Mumford and J.Fogarty, "Geometric Invariant Theory, 2nd Edition," Springer-Verlag, Berlin-Heidelberg-New York, 1982.