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Some Examples of Moduli of Singularities

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There are many observations of singularities on quartic surfaces. In general, for the defining equation which has many parameters, its moduli is not easy.

We introduced the definition of normal forms of quartic curves in 2-dimensional projective space, and found all the forms that will be normal form.

The final purpose of this work is to find out the 3-dimensional analogues.

In this short work, we will give some examples of moduli of singularities on the quartic surfaces by using computer algebra. These are useful to make a required property of moduli restricted whose topological type of singularity.

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1. Introduction

For the readers who are not familiar with algebraic curves, we explain some technical terms and theorems briefly. Let $\mathbf{P}^2$ be a

2-dimensional complex projective space with the coordinate [x,y,z] and let $f_n(x,y,z)$ be a homogeneous polynomial of degree n in $\mathbf{P}^2$. We consider the set V_n: $V_n:=\{(x,y,z)|\ f_n(x,y,z)=0\}$.

We call V_n complex projective plane curves of degree n. For n=4, we call V_4 complex projective plane quartic curves. (see [2])

DEFINITION 1. *Let $f(z_0,\cdots,z_n)$ be a polynomial in $\mathbf{C}^{n+1}$ and let V be an analytic set such that $V=\{(z_0,\cdots,z_n)|f(z_0,\cdots,z_n)=0\}$. Then a point $(z_0,\cdots,z_n)$ in $\mathbf{C}^{n+1}$ is a singular point if $f(z_0,\cdots,z_n)=0$ and $\partial f(z_0,\cdots,z_n)/\partial z_i=0$, $i=0,\cdots,n$.* *(see [3])*

THEOREM 2. *Let $f(z_0,z_1,z_2)$ be a polynomial in $\mathbf{C}^3$ and let V be an anlytic set such that $V=\{(z_0,z_1,z_2)|f(z_0,z_1,z_2)=0\}$ which has an isolated singular point at the origin . Then, for any i $(i=0,1,2)$,*

(i) There exists an integer a_i so that $a_i\geq 2$, and f has a monomial $z_i^{a_i}$.

or

(ii) There exists an integer $a_i\geq 1$ and $j(i\neq j)$ and f has a monomial $z_i^{a_i}z_j$. *(see [3])*

The most important single invariant of a curve is its genus. There are several ways of defining it, all equivalent. For a curve X in projective space, we have the arithmetic genus $p_a(X)$, defined as $1-P_X(0)$, where P_X is the Hilbert polynomial of X. On the other hand, we have the geometric genus $p_g(X)$, defined as $\dim_k\Gamma(X,\omega_X)$, where ω_X is the canonical sheaf.

If X is a curve, then $p_a(X)=p_g(X)=\dim_k(X,\mathcal{O}_X)$, so we call this simply the genus of X, and denote it by g.

For fixed g one would like to endow the set $\mathfrak{M}_g$ of all curves of genus g up to isomorphism with an algebraic structure, in which case we call $\mathfrak{M}_g$ the variety of moduli of curves of genus g.

Let g=3. Then the hyperelliptic curves form an irreducible subvariety of dimension 5 of $\mathfrak{M}_3$. The nonhyperelliptic curves of genus 3 are the nonsingular plane quartic curves. Since the embedding is canonical, two of them are isomorphic as abstract curves if and only if they differ by an automorphism of P^2. The family of all these curves is parametrized by an open set $U \subset P^N$ with N=14, because a form of degree 4 has 15 cofficients. So there is a morphism $U \to \mathfrak{M}_3$, whose fibres are images of the group PGL(2) which has dimension 8. Since any individual curve has only finitely many automorphisms, the fibres have dimension =8, and so the image of U has dimension 14 - 8 = 6. So we confirm that $\mathfrak{M}_3$ has dimension 6. (see [2])

2. Normal Form

We alredy know the classification of complex projective plane cubic curves (n=3). We can list the types of complex projective plane cubic curves: Nodal curve, Cuspdal curve, Conic and Chord, Conic and tangent, Three general lines, Three concurrent lines, Multiple and single lines, Triple line and Nonsingular elliptic curve. (see [1])

The complete classification of complex projective plane quartic curves is unknown. And so-called "normal form" defining equations were not unique. For example, the defining equation of nonsingular elliptic curve in Weierstrass normal form is as follows:

$$y^2z=x^3+pxz^2+qz^3,\ 4p^3+27q^2\neq0.$$

And the defining equation of nonsingular elliptic curve in Hesse normal form is as follows: $x^3+y^3+z^3+3\lambda xyz=0,\ \lambda^3+1\neq0.$

$$4p^3/(4p^3+27q^2)=(8\lambda-\lambda^4)^3/64(1+\lambda^3)^3 \quad (\text{ j-invariant }).$$

This fact is well known. (see [5])

From 1, a variety of moduli of nonsingular plane quartic curve in P^2 has dimension 6. The relations of parameters(coefficients) determine a structure of moduli space. The structure of moduli space is important to mathematics. The need for a unique normal form may be questioned. However, what is the normal form? We define the normal form for the homogeneous polynomials in projective space to be unique. To begin with, we give a following order to the monomials of homogeneous polynomial $f=\sum a_i X^{K_i}$.

DEFINITION 3. *For the exponents $K_i=k_{i_1},\cdots,k_{i_n}$ and $K_j=k_{j_1},\cdots,k_{j_n}$ ($i\neq j$), X^{K_i} is greater than X^{K_j} if there exists an integer $s(1\leq s\leq n)$ such that $k_{i_\mu}=k_{j_\mu}$ for $\mu=1,\cdots,s-1$ and $k_{i_s}>k_{j_s}$.*
(Lexicographic linear order)

Next, we carry out the following manipulations in turn from the maximal X^{K_n} ($K_n=m,0,0,\cdots,0$) to the minimal $X^{K_{n'}}$ ($K_{n'}=0,0,\cdots,0,m$) for the homogeneous polynomial $x_n^m+a_1x_1^m+a_2x_1^{m-1}x_2+a_3x_1^{m-1}x_3+\cdots$.

MANIPULATION 4. *We try a monomial X^{K_i} to eliminate by suitable linear transformation. Then if we can make the monomial X^{K_i} to* *eliminate without generating the monomial* X^{K_j} *which is greater than* X^{K_i}*, we do so. Otherwise, we don't use the linear transformation and go to next manipulation.*

MANIPULATION 5. *If we can make the coefficient of the monomial* X^{K_i} *equal to 1 by the magnification of the coordinates without generating new dimension of coefficient of monomial* X^{K_j} *which is greater than* X^{K_i}*, we do so. Otherwise, there is nothing to be done.*

Then we obtain the following definition.

DEFINITION 6. *We call the results the normal forms of homogeneous polynomials of degree m in (n-1)-dimensional complex projective space.*

We consider it natural that the normal form should be easy to write and remember; that is, the normal form should have the fewest monomials, and each monomial should be simple. The normal forms defined in DEFINITION 6 meet the above conditions.

We obtain a new result for the normal form of nonsingular plane quartic curve (quartic surface which has only one singularity at [0,0,0,1] in P^3) in this paper.

3. Computation

We consider the normal form of nonsingular plane quartic curve by using REDUCE. Let $f(x,y,z)$ be a homogeneous polynomial of degree 4 in $\mathbf{C}^3$. Then $f(0,0,0)=\frac{\partial f(0,0,0)}{\partial x}=\frac{\partial f(0,0,0)}{\partial y}=\frac{\partial f(0,0,0)}{\partial z}=0$. Hence, the analytic set defined by $f(x,y,z)$ has a singular point at the origin in $\mathbf{C}^3$. The analytic set is a nonsingular quartic curve in $\mathbf{P}^2$ if it has only isolated singular point at the origin in $\mathbf{C}^3$.

The quartic form $f(x,y,z)$ in $\mathbf{P}^2$ takes the following form

$$a_1x^4+(a_2y+a_3z)x^3+(a_4y^2+a_5yz+a_6z^2)x^2+(a_7y^3+a_8y^2z+a_9yz^2+a_{10}z^3)x$$
$$+a_{11}y^4+a_{12}y^3z+a_{13}y^2z^2+a_{14}yz^3+z^4=0.$$

Step 1

Replacing z by $z'+\lambda x$ where λ is a solution of $\lambda^4+a_{10}\lambda^3+a_6\lambda^2+a_3\lambda+a_1=0$, we reduce the form to $g_2x^3y+g_3x^3z+g_4x^2y^2+g_5x^2yz+g_6x^2z^2+g_7xy^3+g_8xy^2z$ $+g_9xyz^2+g_{10}xz^3+a_{11}y^4+a_{12}y^3z+a_{13}y^2z^2+a_{14}yz^3+z^4=0$.

$g_3\neq0 \rightarrow$ Step 2, $g_3=0$ and $g_2\neq0 \rightarrow$ Step 3, $g_3=g_2=0 \rightarrow$ Step 4

Step 2

$a_1x^3y+x^3z+a_2x^2y^2+a_3x^2yz+a_4x^2z^2+a_5xy^3+a_6xy^2z+a_7xyz^2+a_8xz^3+a_9y^4+a_{10}y^3z$ $+a_{11}y^2z^2+a_{12}yz^3+a_{13}z^4:=0$ [By a magnification of the x- y- and z-coordinates we can reduce the form to $a_1x^3y+x^3z+a_2x^2y^2+a_3x^2yz+a_4x^2z^2$ $+a_5xy^3+a_6xy^2z+a_7xyz^2+a_8xz^3+a_9y^4+a_{10}y^3z+a_{11}y^2z^2+a_{12}yz^3+a_{13}z^4:=0$].

Replacing z by $z'-a_1y$, we reduce the form to $x^3z+g_4x^2y^2+g_5x^2yz+a_4x^2z^2$ $+g_7xy^3+g_8xy^2z+g_9xyz^2+a_{10}xz^3+g_{11}y^4+g_{12}y^3z+g_{13}y^2z^2+g_{14}yz^3+a_{13}z^4=0$.

$g_4\neq0 \rightarrow$ Step 5, $g_4=0$ and $g_7\neq0 \rightarrow$ Step 6, $g_4=g_7=0$ and $g_{11}\neq0 \rightarrow$ Step 7, $g_4=g_7=g_{11}=0 \rightarrow$ Step 8.

Step 3

$x^3y+a_1x^2y^2+a_2x^2yz+a_3x^2z^2+a_4xy^3+a_5xy^2z+a_6xyz^2+a_7xz^3+a_8y^4+a_9y^3z+a_{10}y^2z^2$

$+a_{11}yz^3+a_{12}z^4:=0$.

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $x^3z+a_3x^2y^2+a_2x^2yz+a_1x^2z^2+a_7xy^3+a_6xy^2z+a_5xyz^2+a_4xz^3+a_{12}y^4+a_{11}y^3z+a_{10}y^2z^2+a_9yz^3+a_8z^4=0$.

$a_3\neq0 \to$ Step 5, $a_3=0$ and $a_7\neq0 \to$ Step 6, $a_3=a_7=0$ and $a_{12}\neq0 \to$ Step 7, $a_3=a_7=a_{12}=0 \to$ Step 8.

Step 4

In this case, the form is as follows: $f_2(y,z)x^2+f_3(y,z)x+f_4(y,z)=0$ where f_i denotes a homogeneous polynomial of degree i $(2\leq i\leq4)$.

By THEOREM 2 the above form gives a singular curve in P^2.

Step 5

$x^3z+x^2y^2+a_1x^2yz+a_2x^2z^2+a_3xy^3+a_4xy^2z+a_5xyz^2+a_6xz^3+a_7y^4+a_8y^3z+a_9y^2z^2+a_{10}yz^3+a_{11}z^4:=0$. Here, we try to eliminate the monomial x^2y^2.

Changing the coordinate so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}\alpha&\beta&1\\\gamma&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $g_1x^4+g_2x^3y+g_3x^3z+g_4x^2y^2+g_5x^2yz+g_6x^2z^2+g_7xy^3+g_8xy^2z+g_9xyz^2+xz^3+g_{11}y^4+g_{12}y^3z+y^2z^2=0$.

Now, $g_1=a_1\alpha^2\gamma+a_{10}\gamma+a_{11}+a_2\alpha^2+a_3\alpha\gamma^3+a_4\alpha\gamma^2+a_5\alpha\gamma+a_6\alpha+a_7\gamma^4+a_8\gamma^3+a_9\gamma^2+\alpha^3+\alpha^2\gamma^2$,

$g_2=a_1\alpha^2+2a_1\alpha\beta\gamma+a_{10}+2a_2\alpha\beta+3a_2\alpha\gamma^2+a_3\beta\gamma^3+2a_4\alpha\gamma+a_4\beta\gamma^2+a_5\alpha+a_5\beta\gamma+a_6\beta+4a_7\gamma^3+3a_8\gamma^2+2a_9\gamma+3\alpha^2\beta+2\alpha^2\gamma+2\alpha\beta\gamma^2$.

$g_4=2a_1\alpha\beta+a_1\beta^2\gamma+a_2\beta^2+3a_3\alpha\gamma+3a_3\beta\gamma^2+a_4\alpha+2a_4\beta\gamma+a_5\beta+6a_7\gamma^2+3a_8\gamma+a_9+\alpha^2+3\alpha\beta^2+4\alpha\beta\gamma+\beta^2\gamma^2$.

g_1 is a polynomial with two variables (α and γ), g_2 and g_4 are polynomials with three variables (α, β and γ). We reduce the forms to

$$g_4=s_2\beta^2+s_1\beta+s_0,\quad g_2=t_1\beta+t_0$$

where $s_2=3\alpha+\gamma^2+a_1\gamma+a_2$, $s_1=4\alpha\gamma+2a_1\alpha+3a_3\gamma^2+2a_4\gamma+a_5$,

$s_0=\alpha^2+3a_3\alpha\gamma+\alpha\gamma+6a_7\gamma^2+3a_8\gamma+a_9$,

$t_1=3\alpha^2+2\alpha\gamma^2+2a_1\alpha\gamma+2a_2\alpha+a_3\gamma^3+a_4\gamma^2+a_5\gamma+a_6$,

$t_0=2\alpha^2\gamma+a_1\alpha^2+3a_3\alpha\gamma^2+2a_4\alpha\gamma+a_5\alpha+4a_7\gamma^3+3a_8\gamma^2+2a_9\gamma+a_{10}$.

We use Sylvester's elimination method(see [4]). We eliminate β for g_2 and g_4 , obtain the resultant. Let $R_1(g_2,g_4)$ be the resultant. Then $R_1(g_2,g_4)$ is a polynomial of degree 6 for the variable α. $R_1(g_2,g_4)=9\alpha^6+\sum_{i=0}^{5} c_i\alpha^i\gamma^j$. Next, we eliminate α for s_2 and t_1, obtain the resultant. Let $R_2(s_2,t_1)$ be the resultant. Then $R_2(s_2,t_1)$ is a polynomial of degree 4 for the variable γ. $R_2(s_2,t_1)=-3\gamma^4+\sum_{i=0}^{3} c_i'\gamma^i$. Lastly, we eliminate α for g_1 and $R_1(g_2,g_4)$, obtain the resultant. Let $R_3(g_1,R_1(g_2,g_4))$ be the resultant. Then $R_3(g_1,R_1(g_2,g_4))$ is a polynomial of degree 24 for the variable γ. $R_3(g_1,R_1(g_2,g_4))=\sum_{i=0}^{24} c_i''\gamma^i$. $R_2(s_2,t_1)$ and $R_3(g_1,R_1(g_2,g_4))$ are polynomials with a variable γ. Let γ_0 be a solution of $R_3(g_1,R_1(g_2,g_4))=0$. Then, if $R_2(s_2,t_1)(\gamma_0)\neq 0$, the simultaneous system of algebraic equations ($g_1=g_2=g_4=0$) has a common root for the three variables (α,β,γ). We think about the existence of solutions of the simultaneous system of algebraic equations.

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&\alpha&\beta\\0&1&\gamma\\0&0&1\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $x^3z+x^2y^2+g_1x^2yz+g_2x^2z^2+g_3xy^3+g_4xy^2z+g_5xyz^2+g_6xz^3+g_7y^4+g_8y^3z+g_9y^2z^2+g_{10}yz^3+g_{11}z^4=0$. Here, we set as follows:

$$\alpha=\frac{-a_3}{2},\quad \beta=\frac{4a_1^2-16a_2-9a_3^2}{48},\quad \gamma=\frac{-2a_1+3a_3}{4}.$$

Then we reduce the form to $x^3z+x^2y^2+h_4xy^2z+h_5xyz^2+h_6xz^3+h_7y^4+h_8y^3z$

$+h_9y^2z^2+h_{10}yz^3+h_{11}z^4=0$. $x^3z+x^2y^2+a_1xy^2z+a_2xyz^2+a_3xz^3+a_4y^4+a_5y^3z+a_6y^2z^2+a_7yz^3+a_8z^4:=0$. Then the above $g_1, g_2, g_4, s_2, s_1, s_0, t_1$ and t_0 are as follows:

$$g_1=a_4\gamma^4+a_5\gamma^3+a_6\gamma^2+a_7\gamma+a_8+\alpha^3+\alpha^2\gamma^2+a_1\alpha\gamma^2+a_2\alpha\gamma+a_3\alpha ,$$
$$g_2=a_7+2a_1\alpha\gamma+a_1\beta\gamma^2+a_2\alpha+a_2\beta\gamma+a_3\beta+4a_4\gamma^3+3a_5\gamma^2+2a_4\gamma+3\alpha^2\beta+2\alpha^2\gamma+2\alpha\beta\gamma^2,$$
$$g_3=a_1\alpha+2a_1\beta\gamma+a_2\beta+6a_4\gamma^2+3a_5\gamma+a_6+\alpha^2+3\alpha\beta^2+4\alpha\beta\gamma+\beta^2\gamma^2,$$
$$s_2=3\alpha+\gamma^2, \quad s_1=4\alpha\gamma+2a_1\gamma+a_2 , \quad s_0=\alpha^2+\alpha\gamma+6a_4\gamma^2+3a_5\gamma+a_6 ,$$
$$t_1=3\alpha^2+2\alpha\gamma^2+a_1\gamma^2+a_2\gamma+a_3 , \quad t_0=2\alpha^2\gamma+2a_1\alpha\gamma+a_2\alpha+4a_4\gamma^3+3a_5\gamma^2+2a_6\gamma+a_7 .$$

For any solution of $R_1(g_2,g_4)=0$, if $a_4\neq0$ or $a_5\neq0$, $g_1=0$ has solutions for the variable γ. Hence, if $a_4\neq0$ or $a_5\neq0$, $R_3(g_1,R_1(g_2,g_4))=0$ has solutions for the variable γ. If $a_4=a_5=0$, by THEOREM 2 this form gives a singular curve in P^2. And for any solution of $R_3(g_1,R_1(g_2,g_4))=0$, if $R_2(s_2,t_1)=0$ and $s_2\neq0$, the simultaneous system of algebraic equations ($g_1=g_2=g_4=0$) has a common root for the three variables (α,β,γ). If $R_2(s_2,t_1)=0$ and $s_2=0$ for any solution of $R_3(g_1,R_1(g_2,g_4))=0$ (i.e. the common root of $g_1=0$ and $R_1(g_2,g_4)=0$ is only $\alpha=\frac{-\gamma^2}{3}$ for any solution of $R_3(g_1,R_1(g_2,g_4))=0$), $\alpha:=\frac{-\gamma^2}{3}$, then $g_1=0$, $t_1=0$ and

$$t_0=\frac{12a_4\gamma^3+a_2\gamma^2+9a_5\gamma^2+2a_3\gamma+6a_6\gamma+3a_7}{3} .$$

$f(x,y,z):=x^3z+x^2y^2+a_1xy^2z+a_2xyz^2+a_3xz^3+a_4y^4+a_5y^3z+a_6y^2z^2+a_7yz^3+a_8z^4$.

$4f(x,y,z)=\frac{\partial f(x,y,z)}{\partial x}x + \frac{\partial f(x,y,z)}{\partial y}y + \frac{\partial f(x,y,z)}{\partial z}z$. And $g_1=f(\alpha,\gamma,1)$, $t_1=\frac{\partial f(\alpha,\gamma,1)}{\partial x}$, $t_0=\frac{\partial f(\alpha,\gamma,1)}{\partial y}$. Hence, for $R_2(s_2,t_1)=0$ and $\alpha=\frac{-\gamma^2}{3}$, $t_0\neq0$. We eliminate α for g_2 and g_4 , obtain the resultant. Let $R_4(g_2,g_4)$ be the resultant. For $R_2(s_2,t_1)=0$ and $\alpha=\frac{-\gamma^2}{3}$, $R_4(g_2,g_4)=27t_0\beta^5+\sum_{i=0}^{4} c_i'''\beta^i$.

Therefore, if $R_2(s_2,t_1)=0$ for any solution of $R_3(g_1,R_1(g_2,g_4))=0$, there exist the common root of $g_1=0$ and $R_1(g_2,g_4)=0$ such that $s_2\neq0$.

Hence, for any parameter $(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10},a_{11})$, we can eliminate the monomial x^2y^2. We consider it in Step 6,7,8.

EXAMPLE. For $a_1=a_2=a_3=a_4=a_5=a_6=a_8=a_9=a_{10}=0$ and $a_7=a_{11}=1$,
$R_2(s_2,t_1)=-3\gamma^4$,
$R_3(g_1,R_1(g_2,g_4))=25\gamma^{24}-24\gamma^{22}-846\gamma^{20}+13472\gamma^{18}+22983\gamma^{16}+74160\gamma^{14}$
$+77500\gamma^{12}+216864\gamma^{10}+22599\gamma^8+158760\gamma^6-21870\gamma^4+729$.

Step 6
$x^3z+a_1x^2yz+a_2x^2z^2+xy^3+a_3xy^2z+a_4xyz^2+a_5xz^3+a_6y^4+a_7y^3z+a_8y^2z^2+a_9yz^3$
$+a_{10}z^4:=0$.

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&\alpha&\beta\\0&1&\gamma\\0&0&1\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $x^3z+g_1x^2yz+g_2x^2z^2+xy^3+g_3xy^2z+g_4xyz^2+g_5xz^3+g_6y^4+g_7y^3z+a_8y^2z^2+a_9yz^3$
$+a_{10}z^4=0$. Here, we set as follows:

$$\alpha=\frac{-a_1}{3},\quad \beta=\frac{-a_1{}^3+3a_1a_3-9a_2}{27},\quad \gamma=\frac{a_1{}^2-3a_3}{9}.$$

Then we reduce the form to $x^3z+xy^3+a_1xyz^2+a_2xz^3+a_3y^4+a_4y^3z+a_5y^2z^2$
$+a_6yz^3+a_7z^4=0$, $(a_2,a_6,a_7)\neq(0,0,0)$.
The dimension of parameter-space of this form is equal to 6.

Step 7
$x^3z+a_1x^2yz+a_2x^2z^2+a_3xy^2z+a_4xyz^2+a_5xz^3+y^4+a_6y^3z+a_7y^2z^2+a_8yz^3+a_9z^4:=0$.

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1&\alpha&\beta\\0&1&\gamma\\0&0&1\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $x^3z+g_1x^2yz+g_2x^2z^2+g_3xy^2z+g_4xyz^2+g_5xz^3+y^4+g_6y^3z+a_7y^2z^2+g_8yz^3+a_9z^4=0$.
Here, we set as follows:

$$\alpha=\frac{-a_1}{3},\quad \beta=\frac{2a_1{}^4-9a_1{}^2a_3+27a_1a_6-108a_2}{324},\quad \gamma=\frac{-2a_1{}^3+9a_1a_3-27a_6}{108}$$

Then we reduce the form to

$x^3z+a_1xy^2z+a_2xyz^2+a_3xz^3+y^4+a_4y^2z^2+a_5yz^3+a_6z^4=0$, $(a_3,a_5,a_6)\neq(0,0,0)$.

The dimension of parameter-space of this form is equal to 5.

Step 8

$x^3z+a_1x^2yz+a_2x^2z^2+a_3xy^2z+a_4xyz^2+a_5xz^3+a_6y^3z+a_7y^2z^2+a_8yz^3+a_9z^4:=0$.

Replacing x by $x'+\lambda y$ where λ is a solution of $\lambda^3+a_1\lambda^2+a_3\lambda+a_6=0$, we reduce the form to

$x^3z+g_1x^2yz+g_2x^2z^2+g_3xy^2z+g_4xyz^2+g_5xz^3+g_7y^2z^2+g_8yz^3+g_9z^4=0$.

Changing the coordinates so that $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}x'\\y'\\z'\end{pmatrix}$. We reduce the form to $g_3x^2yz+g_7x^2z^2+g_1xy^2z+g_4xyz^2+g_8xz^3+y^3z+a_2y^2z^2+g_5yz^3+g_9z^4=0$.

From THEOREM 2, the analytic set defined by this form is singular curve in P^2.

We obtain a following lemma.

Lemma 7 *There exists the following two types of forms as the normal form of non-singular quartic curve in* P^2 *(quartic surface which has only one singularity at [0,0,0,1] in* P^3*).*

Type **I** : $x^3z+xy^3+a_1xyz^2+a_2xz^3+a_3y^4+a_4y^3z+a_5y^2z^2+a_6yz^3+a_7z^4=0$,

where $a_1\xi\zeta+a_2\xi+a_3\zeta^4+a_4\zeta^3+a_5\zeta^2+a_6\zeta+a_7+\xi^3+\xi\zeta^3\neq0$

for all ξ,ζ *such that*

$$9\zeta^7+48a_3^2\zeta^6+3(5a_1+24a_3a_4)\zeta^5+3(3a_2+16a_3a_5+9a_4^2)\zeta^4$$
$$+(7a_1^2+24a_3a_6+36a_4a_5)\zeta^3+6(a_1a_2+3a_4a_6+2a_5^2)\zeta^2+(a_1^3+12a_5a_6)\zeta$$
$$+a_1^2a_2+3a_6^2=0,$$

$$243\xi^7+27(-64a_3^3+27a_4)\xi^6+27(32a_1a_3^2+6a_2-24a_3a_5+27a_4^2)\xi^5$$
$$+27(16a_1a_3^2a_4-64a_2a_3^3+18a_2a_4+16a_3^2a_6-24a_3a_4a_5+9a_4^3)\xi^4$$
$$+(4a_1^3+36a_1^2a_3a_4+576a_1a_2a_3^2-36a_1a_3a_6-54a_1a_4a_5+27a_2^2$$
$$-432a_2a_3a_5+486a_2a_4^2+54a_5a_6)\xi^3+3(4a_1^3a_4-32a_1^2a_3^2a_5$$
$$+96a_1a_2a_3^2a_4+12a_1a_3a_4a_6+32a_1a_3a_5^2-18a_1a_4^2a_5-192a_2^2a_3^3$$
$$+27a_2^2a_4+96a_2a_3^2a_6-144a_2a_3a_4a_5+54a_2a_4^3-12a_3a_6^2+18a_4a_5a_6$$
$$-8a_5^3)\xi^2+(16a_1^4a_3^2-16a_1^3a_3a_5+9a_1^3a_4^2+12a_1^2a_2a_3a_4+6a_1^2a_4a_6$$
$$+4a_1^2a_5^2+96a_1a_2^2a_3^2-12a_1a_2a_3a_6-18a_1a_2a_4a_5-3a_1a_6^2-72a_2^2a_3a_5$$
$$+81a_2^2a_4^2+16a_2a_5a_6)\xi+16a_1^3a_3^2a_6-32a_1^2a_2a_3^2a_5-16a_1^2a_3a_5a_6$$
$$+9a_1^2a_4^2a_6+48a_1a_2^2a_3^2a_4+12a_1a_2a_3a_4a_6+32a_1a_2a_3a_5^2-18a_1a_2a_4^2a_5$$
$$-6a_1a_4a_6^2+4a_1a_5^2a_6-64a_2^3a_3^3+48a_2^2a_3^2a_6-72a_2^2a_3a_4a_5+27a_2^2a_4^3$$
$$-12a_2a_3a_6^2+18a_2a_4a_5a_6-8a_2a_5^3+a_6^3=0,$$
$$a_1\xi+a_2+3\xi^2+\zeta^3=0 \text{ and } a_1\xi+4a_3\zeta^3+3a_4\zeta^2+2a_5\zeta+a_6+3\xi\zeta^2=0 .$$

Type **II** : $x^3z+a_1xy^2z+a_2xyz^2+a_3xz^3+y^4+a_4y^2z^2+a_5yz^3+a_6z^4=0$,

where $a_1\xi\zeta^2+a_2\xi\zeta+a_3\xi+a_4\zeta^2+a_5\zeta+a_6+\xi^3+\zeta^4\neq 0$

for all ξ, ζ *such that*

$$48\zeta^6+4(a_1^3+12a_4)\zeta^4+8(a_1^2a_2+3a_5)\zeta^3+(4a_1^2a_3+5a_1a_2^2+12a_4^2)\zeta^2$$
$$+(4a_1a_2a_3+a_2^3+12a_4a_5)\zeta+a_2^2a_3+3a_5^2=0,$$
$$432\xi^6-144a_1^2\xi^5+12(a_1^4-12a_1a_4+36a_3)\xi^4$$
$$+12a_1(2a_1^2a_4-8a_1a_3+5a_2^2)\xi^3+(4a_1^4a_3-a_1^3a_2+12a_1^2a_4^2+36a_1a_2a_5$$
$$-96a_1a_3a_4+24a_2^2a_4+144a_3^2)\xi^2+2(4a_1^3a_3a_4-a_1^2a_2^2a_4-81a_1^2a_3^2$$
$$+10a_1a_2^2a_3-2a_2^4)\xi+a_1^3a_5^2-2a_1^2a_2a_4a_5+4a_1^2a_3a_4^2+12a_1a_2a_3a_5$$
$$-16a_1a_3^2a_4-4a_2^3a_5+8a_2^2a_3a_4+16a_3^3=0,$$
$$a_1\zeta^2+a_2\zeta+a_3+3\xi^2=0 \text{ and } 2a_1\xi\zeta+a_2\xi+2a_4\zeta+a_5+4\zeta^3=0 .$$

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