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A Mathematical Introduction to Bifurcation of Periodic Progressive Water Waves

HISASHI OKAMOTO

Department of Pure and Applied Sciences,
Univ. of Tokyo,
Komaba, Meguro-ku, Tokyo 153 Japan

§1. Introduction. This paper is written in order to provide a self-contained introduction to the structure of the set of periodic progressive water waves of incompressible inviscid fluid. The water wave problem has attracted quite a large number of both physicists and mathematicians since Stokes' pioneering paper [**41,42**] appeared in 1847. Among many aspects of the problem, we consider only periodic progressive waves, by which we mean waves travelling with a constant speed with no change of shape. Even for this restricted problem, quite a lot of works appeared. Nevertheless there remain many open questions to be answered. The contents of this paper are, except for a few propositions, already known but we collected known results so as to clarify the open problems.

Our attention is restricted to 2-dimensional irrotational flow of incompressible inviscid fluid. The problem is to find the configuration of the wave and the fluid flow beneath the wave. Therefore, it is a free boundary problem. We start with a classical description of the problem in the next section.

§2. Formulation of the problem. In this section we derive differential equations and explain their physical meaning. We take a coordinate system (x, y) moving with the wave with the same speed. The x-coordinate is taken horizontally to the right and y-coordinate is taken vertically upward. In this moving frame, the wave profile is at rest and there is an underlying flow travelling in the opposite direction. We assume that the flow is two dimensional and irrotational. We neglect viscosity and compressibility. Throughout this paper we assume that the wave profile is periodic in x with a period, say L, and is symmetric with respect to some vertical line, unless otherwise stated. We take the line as y-axis. For the moment we consider a flow of infinite depth with a smooth free boundary. In physics literatures, this is called a deep water. By this assumption, we have only to consider a flow in

$$\{(x,y); \quad -L/2 < x < L/2, \quad -\infty < y < h(x) \ \}.$$

The following description of the motion can be found in text books ([**11,30,32,40**]), so our exposition will be brief. The symmetry assumption imposes the following symmetry on the velocity vector (u, v) (see Fig. 1):

$$u(x,y) = u(-x,y), \qquad v(x,y) = -v(-x,y). \tag{2.1}$$

The periodicity and (2.1) implies that

$$v = 0 \qquad \text{on} \quad x = \pm L/2. \tag{2.2}$$

We now introduce a velocity potential and a stream function. These are real valued functions defined by

$$u = \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}, \qquad v = \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}. \tag{2.3}$$

By the definition, the function $U(x,y) + iV(x,y)$ is an analytic function of $z = x + iy$. Therefore, the problem is to find an even function $y = h(x)$ ($-L/2 < x < L/2$) and an analytic function $f(z) = U + iV$ in

$$\Omega_h \equiv \{z = x + iy \in \mathbb{C}\ ; \quad -L/2 < x < L/2, \quad -\infty < y < h(x)\},$$

which satisfy the following boundary conditions: The condition (2.2) is interpreted as

$$U = \text{constant} \qquad \text{on} \quad x = \pm L/2. \tag{2.4}$$

The free boundary $y = h(x)$ must be a stream line, which forces V to being constant on $y = h(x)$. By the definition (2.3), additive constants for U and V are undetermined. Accordingly, we normalize V so that

$$V = 0 \qquad \text{on} \quad y = h(x). \tag{2.5}$$

Determination of the constant in (2.4) is less trivial. By the symmetry assumption (2.1), it is natural to take U as an odd function in x. Thereby we put

$$U = \pm\alpha \qquad \text{on} \quad x = \pm L/2 \quad \text{(respectively)}.$$

The constant α is determined by the flux of the flow through a vertical line. We define c by $cL = 2\alpha$ and call it the propagation speed. Thus the boundary condition (2.4) is rewritten as

$$U = \pm cL/2 \qquad \text{on} \quad x = \pm L/2 \quad \text{respectively}. \tag{2.6}$$

In order to determine the free boundary, we need one more condition on $y = h(x)$. This is supplied by the Bernoulli theorem, which imposes

$$\frac{1}{2}\left|\frac{df}{dz}\right|^2 + gy + \frac{T}{m}K = \text{constant} \qquad \text{on} \quad y = h(x), \tag{2.7}$$

where g, m and T are constant called the gravity constant, the mass density, and the surface tension coefficient, respectively. K is the curvature of the free boundary and is given by

$$K = -\left(\frac{h_x}{\sqrt{1+h_x^2}}\right)_x,$$

where the subscript implies the differentiation. We finally impose a boundary condition at infinity: the flow tends to be uniform as y approach $-\infty$. Consequently we have

$$\frac{df}{dz} = u - iv \to c \qquad \text{as} \quad y \to -\infty. \tag{2.8}$$

Summing up, the problem is to find an even function $y = h(x) \quad (|x| < L/2)$, and an analytic function $f(z) \quad (z \in \Omega_h)$ satisfying (2.5-8).

One of the mathematical difficulties of the problem above is that a part of the boundary, i.e., $y = h(x)$ is not specified in advance. An idea was invented by Stokes to overcome this difficulty. His idea is to regard z as a function of f rather than regarding f as a function of z. Since $f = U + iV$ lies in

$$D \equiv \{(U,V); \quad -\frac{cL}{2} < U < \frac{cL}{2}, \quad -\infty < V < 0\},$$

the problem is transformed to a one defined in this fixed domain. Since the periodicity condition is imposed on $U = \pm cL/2$, the two vertical boundaries of D should be identified. By this reason, it is more natural to consider a circular domain which is considered in Levi-Civita [**19**]: we introduce independent variable

$$\zeta = \exp\left(-\frac{2\pi i f}{cL}\right) \tag{2.9}$$

and dependent variable

$$\omega = i \log\left(\frac{1}{c}\frac{df}{dz}\right). \tag{2.10}$$

By the relation

$$\zeta \longleftrightarrow f \longleftrightarrow z \longleftrightarrow \omega,$$

we regard ω as a function of ζ. Note that ζ runs in a unit disk cut along the negative real axis when f runs in D. The periodicity makes ω continuous up to the negative real axis, hence analytic in $0 < |\zeta| < 1$. Note also that infinity $y = -\infty$ corresponds to $\zeta = 0$. By (2.8), ω tends to zero as $\zeta \to 0$. Therefore, ω is analytically continued to a disk $|\zeta| < 1$ and satisfies $\omega(0) = 0$. Let θ and τ denote, respectively, the real and the imaginary part of ω. Let (ρ, σ) be the polar coordinates for ζ, i.e., $\zeta = \rho e^{i\sigma}$. We now rewrite (2.5) by means of $\theta(\rho,\sigma)$ and $\tau(\rho,\sigma)$:

LEMMA 2.1. *$\theta + i\tau$ satisfies*

$$e^{2\tau}\frac{\partial \tau}{\partial \sigma} - pe^{-\tau}\sin\theta + q\frac{\partial}{\partial\sigma}\left(e^{\tau}\frac{\partial\theta}{\partial\sigma}\right) = 0 \quad on \quad \rho = 1, \tag{2.11}$$

where $p = gL/(2\pi c^2), q = 2\pi T/(mc^2 L)$.

PROOF: On the free boundary, we have $y = h(x)$ and $f = U$. This shows that $\frac{dz}{df} = \frac{\partial x}{\partial U} + i\frac{\partial y}{\partial U}$. On the other hand, by $df/dz = ce^{\tau - i\theta}$, we have

$$\sigma = \frac{-2\pi U}{cL}, \quad \frac{\partial x}{\partial U} = \frac{e^{-\tau}}{c}\cos\theta, \quad \frac{\partial y}{\partial U} = \frac{e^{-\tau}}{c}\sin\theta, \quad \frac{dh}{dx}(x) = \tan\theta \tag{2.12}$$

and

$$\left|\frac{df}{dz}\right|^2 = c^2 e^{2\tau}, \quad \frac{\partial}{\partial x} = \frac{\partial\sigma}{\partial x}\frac{\partial}{\partial\sigma} = -\frac{2\pi e^{\tau}}{L\cos\theta}\frac{\partial}{\partial\sigma}$$

By these formulas we easily get to (2.11). ∎

We thus obtain the following reformulation:

Find a function $\omega = \omega(\zeta)$ which is continuously differentiable (in real sense) on $|\zeta| \le 1$, is analytic in $|\zeta| < 1$ and satisfy (2.11) and $\omega(0) = 0$.

Levi-Civita [**19**] considered the problem in this formulation when the surface tension is neglected ($q = 0$).

Remark. Levi-Civita took the y-axis vertically downward, while we took upward. Consequently the relation of ζ and f in (2.9) differs from his by the sign.

Remark. The symmetry assumption on $h(x)$ and $dh/dx = \tan\theta$ imply that θ is odd in σ and τ is even. Although our derivation depends on the symmetry assumption, the above formulation of Levi-Civita type do not require the oddness of θ in advance. Therefore we can include nonsymmetric solutions, if they may exist. It seems to the author that many physicist believed the validity of the symmetry assumption and it is implicitly assumed as if it were rigorously proved ([**50**]). Garabedian [**12**] proved that a wave which has one and only one crest and trough in one wave length and its profile is monotone between crest and trough must be symmetric. Zufiria [**55-57**] give a strong numerical evidence that nonsymmetric water waves bifurcate from symmetric ones. His nonsymmetric waves have six peaks in one wave length. Consequently there is no contradiction between Garabedian's theorem and Zufiria's computation. It, however, remains as a mathematical open question whether we can prove rigorously the existence of a nonsymmetric wave.

Since an analytic function is completely determined by its boundary value, a further reduction of the equation (2.11) is possible. In fact we can write (2.11) only by $\theta(1,\sigma)$ $(0 \le \sigma < 2\pi)$. To this end, we define a Hilbert transform on S^1:

$$H\left(\sum_{n=1}^{\infty}\left(a_n \sin n\sigma + b_n \cos\sigma\right)\right) = \sum_{n=1}^{\infty}\left(- a_n \cos n\sigma + b_n \sin\sigma\right).$$

Then we have $\tau(1,\sigma) = H(\theta^*)$, where $\theta^*(\sigma) = \theta(1,\sigma)$. The equation (2.11) is now written as

$$e^{2H\theta^*}\frac{dH\theta^*}{d\sigma} - pe^{-H\theta^*}\sin\theta^* + q\frac{d}{d\sigma}\left(e^{H\theta^*}\frac{d\theta^*}{d\sigma}\right) = 0 \quad (0 \le \sigma < 2\pi). \tag{2.13}$$

Accordingly the problem is to find a 2π-periodic continuous function θ^* satisfying (2.13).

In the subsequent sections we show the structure of the set of solutions to (2.13). Our first observation is that $\theta^* \equiv 0$ satisfies (2.13) for all $(p,q) \in [0,\infty) \times [0,\infty)$. $\theta^* \equiv 0$ implies $\omega \equiv 0$, hence $df/dz \equiv c$ by (2.10). This means that the flow is uniform: $(u,v) \equiv (c,0)$. Also, the last equality of (2.12) implies that the wave profile of the solution $\theta^* \equiv 0$ is completely flat. By this reason, we call $\theta \equiv 0$ a trivial solution. It is now well known that there are bifurcation points along the branch of this trivial solution. This issue will be discussed in the next section.

Thus far we have considered only flows of infinite depth. The influence of the depth on the shape of the wave has been considered in many papers. The work of Levi-Civita [**19**] was generalized by Struik [**43**] to the case of finite depth. For the case of finite depth, see [**33**] and references therein.

§3. **Primary bifurcation from the trivial flow.** In this section we consider (2.13) and determine the values of (p,q) at which solutions other than $\theta^* = 0$ bifurcate. For each positive integer m, we define Banach space X^m by

$$X^m = \left\{ f \in H^m(S^1); \int_0^{2\pi} f(\sigma)d\sigma = 0 \right\},$$

where $H^m(S^1)$ is a usual Sobolev space on the unit circle S^1. Namely, $f \in H^m(S^1)$ if and only if f and its derivatives of order $\le m$ are square summable. For each $u \in X^2$, we put

$$F(p,q;u) = e^{2Hu}\frac{dHu}{d\sigma} - pe^{-Hu}\sin u + q\frac{d}{d\sigma}\left(e^{Hu}\frac{du}{d\sigma}\right). \tag{3.1}$$

Note that $Hu \in X^m$ if $u \in X^m$ and that $Hu \in C^1(S^1)$ if $u \in X^2$. This fact makes the right hand side of (3.1) square summable for $u \in X^2$.

LEMMA 3.1. *If $u \in X^2$, then $F(p,q;u) \in X^0$. F is a C^∞-mapping from $\mathbf{R}^2 \times X^2$ to X^0.*

PROOF: By the remark mentioned just above, we have only to prove that

$$\int_0^{2\pi} F(p,q;u)d\sigma = 0.$$

Since

$$F(p,q;u) = \frac{d}{d\sigma}\left(\frac{1}{2}e^{2Hu} + qe^{Hu}\frac{du}{d\sigma}\right) - pe^{-Hu}\sin u,$$

it is sufficient to prove

$$\int_0^{2\pi} e^{-Hu}\sin u d\sigma = 0. \tag{3.2}$$

Note that

$$e^{-Hu}\sin u = \operatorname{Im}\left[e^{i(u+iHu)}\right].$$

For $X^2 \ni u = \sum_{n=1}^{\infty}(a_n \sin n\sigma + b_n \cos n\sigma)$, we define

$$\omega(\rho,\sigma) = \sum_{n=1}^{\infty}(b_n - ia_n)\zeta^n,$$

which is an analytic function of $\zeta = \rho e^{i\sigma}$. It holds that $\omega(1,\sigma) = u(\sigma) + iHu(\sigma)$. By $\omega(0) = 0$, we have

$$\int_{\rho=1} \frac{e^{i\omega}}{\zeta}d\zeta = 2\pi i.$$

Taking the real part, we obtain (3.2). Smoothness is proved in an elementary way. ∎

We thus have a mapping F from $\mathbf{R}^2 \times X^2$ into X^0 and what we should do is to find zeros of F other than $\{(p,q;0); 0 \le p < \infty, 0 \le q < \infty\}$. Note that the physical meaning of p,q requires that $p,q \in [0,\infty)$. The mapping F, however, has a well-defined mathematical meaning for $p,q \in \mathbf{R}$.

Since F depends on two parameters p and q, the complete description of the set of zeros is not an easy task except for those in a small neighborhood of the trivial solution $u = 0$. As we show below, it is a routine to prove the existence of the bifurcating branch from the trivial solution $u = 0$. On the other hand, it requires formidable calculation to clarify the global structure of the solution set. In this section we consider the local structure of the bifurcation from trivial solution. By this, we mean that we show the solution set in a small neighborhood of the trivial solution.

LEMMA 3.2. *The Fréchet derivative of F at $u = 0$ is given by*

$$D_uF(p,q;0)w = \frac{d}{d\sigma}Hw - pw + q\frac{d^2w}{d\sigma^2}. \tag{3.3}$$

PROOF: Easy from (3.1). ∎

COROLLARY 3.2. *$D_uF(p,q;0)$ fails to be isomorphic if and only if (p,q) satisfies*

$$n - p - n^2q = 0, \tag{3.4}$$

for some positive integer n.

For a fixed n, (3.4) defines a line in the (p,q)-plane. We put

$$S_n = \{(p,q,r); 0 \le p < \infty, 0 \le q < \infty, (3.4)\}.$$

We notice that S_n may intersects S_m with $m \neq n$ (see Fig. 2). Classical results by [**19,31,41,51**] are included in the following

THEOREM 3.1. *If*

$$(p_0,q_0) \in S_n \setminus \bigcup_{m\neq n} S_m, \tag{3.5}$$

then $(p_0,q_0;0)$ is a bifurcation point.

PROOF: We use a closed subspace Y^m of X^m:

$$Y^m = \left\{ f \in X^m; \int_0^{2\pi} f(\sigma)\cos k\sigma d\sigma = 0 \quad (k \in \mathbb{N}) \right\}. \tag{3.6}$$

Then it is easy to verify that F is a smooth mapping from $\mathbb{R}^2 \times Y^2$ into Y^0. We also observe that the null space of $D_uF(p,q;0)$ is 1-dimensional and spanned by $\sin n\sigma$. We can then use a theorem which ensures the existence of the bifurcation from simple eigenvalue. For instance, we can use a theorem in Crandall and Rabinowitz [**9**]. ∎

Remark. In this way the proof of existence can be given in a simple way. Ours are much simpler than the one in Reeder and Shinbrot [**36**].

We call those points satisfying (3.5) simple bifurcation points of mode n. The theorem above guarantees the existence of branch of nontrivial solutions from simple bifurcation points. On the other hand, there are intersections of S_n and S_m for different m and n (see Fig. 2). We call such a point a double bifurcation point of mode (m, n). We show in a later section that the double bifurcation point is actually a bifurcation point. This fact was first proved by [**47**,15]. We give simpler proof of this in [**33**] by exploiting O(2)-symmetry.

Before we consider a global diagram, we focus on some special cases where one of the parameters vanishes. In the case that $q = 0$, the solutions are called gravity waves. In the case that $p = 0$, the solutions are called pure capillary waves. In the general case, they are called capillary-gravity waves or gravity-capillary waves.

We first consider the pure capillary waves. It is rather surprising that there are solutions which have explicit expressions in terms of elementary functions or elliptic functions. Crapper [**10**] gave the formula for the case of infinite depth and Kinnersley [**17**] for the case of finite depth. This problem will be considered in the next section.

§4. **Pure capillary waves.** In this section we consider the problem of finding solutions to $F(0, q; u) = 0$. Since

$$F(0, q; u) = \frac{d}{d\sigma}\left(\frac{1}{2}e^{2Hu} + qe^{Hu}\frac{du}{d\sigma}\right),$$

the equation is equivalent to

$$\frac{1}{2}e^{2Hu} + qe^{Hu}\frac{du}{d\sigma} = b,$$

where b is a constant. Therefore u is a solution, if it satisfies

$$q\frac{du}{d\sigma} = -\sinh(Hu). \tag{4.1}$$

As we show below, this equation (4.1) has a family of solutions written in terms of elementary functions. In particular, we can globally see the set of solutions in this special case.

THEOREM 4.1 (CRAPPER [**10**]). *Define an analytic function* ω *by*

$$\omega = \theta + i\tau = 2i\log\frac{1 + A\zeta}{1 - A\zeta}, \tag{4.2}$$

where A is a real parameter satisfying $-1 < A < 1$. Then $u = \theta(1,\sigma)$ satisfies (4.1).

PROOF: For $A \in (-1,1)$, (4.2) defines an analytic function in the unit disk. We have

$$\tau(1,\sigma) = 2\log\left|\frac{1+Ae^{i\sigma}}{1-Ae^{i\sigma}}\right| = \log\frac{1+A^2+2A\cos\sigma}{1+A^2-2A\cos\sigma} \tag{4.3}$$

$$= 4\left(A\cos\sigma + \frac{A^3}{3}\cos 3\sigma + \frac{A^5}{5}\cos 5\sigma + \cdots\right)$$

and

$$\theta = -2\arctan\left(\frac{2A\sin\sigma}{1-A^2}\right) = -4\left(A\sin\sigma + \frac{A^3}{3}\sin 3\sigma + \frac{A^5}{5}\sin 5\sigma + \cdots\right). \tag{4.4}$$

It also holds that

$$e^{\tau(1,\sigma)} = \frac{1+A^2+2A\cos\sigma}{1+A^2-2A\cos\sigma} = \frac{1+3A^2}{1-A^2} + \frac{4(1+A^2)}{1-A^2}\sum_{n=1}^{\infty} A^n \cos n\sigma,$$

which leads to

$$\sinh\tau = \frac{4(1+A^2)}{1-A^2}\left(A\cos\sigma + A^3\cos 3\sigma + \cdots\right).$$

It is now easy to check (4.1) with $q = (1+A^2)/(1-A^2)$. Therefore we are done if we choose

$$A = \sqrt{(q-1)/(q+1)}. \tag{4.5}$$

∎

We call the solutions in Theorem 4.1 Crapper's waves. By (4.2,5), Crapper's wave becomes trivial at $q=1$. In other words, Crapper's wave bifurcate at $q=1$ from the trivial solution. (4.5) also shows that the bifurcation takes place supercritically in q. We take $\tau(1,\sigma)$ as a representative of the amplitude of Crapper's solutions. Then we have

$$q = \cosh\left(\frac{1}{2}\tau(1,0)\right),$$

which follows from (4.3). Therefore the family (4.1) constitutes a pitchfork lying in $1 \le q < \infty$. In this sense we obtain a global branch of solutions. On the other hand, (3.4) shows that $(q, u) = (1/n, 0)$, $(n = 2, 3, \cdots)$ are bifurcation points, too. The branches emanating from these points are given by

$$q = \frac{1}{n}\frac{1+A^2}{1-A^2}, \qquad u(\sigma) = \theta(n\sigma), \quad n = 2, 3, \cdots,$$

where θ is given by (4.4). Thus we have Fig. 3 as a global bifurcation diagram. A question arises:

Is there a solution other than Crapper's waves ?

We can not say anything definite but the following proposition.

PROPOSITION 4.1. *In the case of $p = 0$, there is no secondary bifurcation from the branches of Crapper's waves.*

The proof of this proposition will be presented elsewhere. This proposition indicates that solutions other than Crapper's waves are, if they exists, separated from the trivial solutions and Crapper's solutions.

Since (4.2) yields that

$$\frac{dz}{df} = c^{-1}\left(\frac{1 - A\zeta}{1 + A\zeta}\right)^2,$$

we have the following parametrization of the free boundary:

$$\frac{x}{L} = \alpha - \frac{2}{\pi}\frac{A\sin 2\pi\alpha}{1 + A^2 + 2A\cos 2\pi\alpha} \qquad (0 \le \alpha < 1)$$

$$\frac{y}{L} = -\frac{2}{\pi} + \frac{2}{\pi}\frac{1 + A\cos 2\pi\alpha}{1 + A^2 + 2A\cos 2\pi\alpha} \qquad (0 \le \alpha < 1).$$

By this formula we can draw figures of the free boundaries (see Fig. 4). In these figures, the values of A are shown. Figures for positive A are obtained by shifting the figures of $-A$ by half a wave length. While $|A|$ is small, the wave profiles look like sinusoidal curves. They, however, form particular shapes for large $|A|$ and, y is not a single valued function of x for $|A| > 0.414215\cdots$. If $|A| > 0.454670\cdots$, then the free boundary has a self-intersection and becomes physically meaningless. Note, however, that (4.4) is well-defined for all $A \in (-1, 1)$ and is a mathematical solution to (4.1).

In the case of finite depth, the solutions corresponding to Crapper's waves are given by elliptic functions (Kinnersley **[17]**).

§5. Gravity waves of infinite depth. In this section we consider gravity waves of infinite depth. Hence we put $q = 0$. We remark that $F(p,0;\theta) = 0$ is equivalent to

$$\frac{d}{d\sigma} H\theta = pe^{-3H\theta}\sin\theta. \tag{5.1}$$

In this section we restrict our attention to symmetric waves, whence $\theta(\sigma)$ is an odd function of σ. We note that a harmonic function $\Phi(\rho,\sigma)$ in the unit disk which is odd in σ satisfies

$$\begin{aligned}\Phi(1,\sigma) &= -\frac{1}{\pi}\int_0^{2\pi}\frac{\partial\Phi}{\partial\rho}(1,\gamma)\log\left|\sin\frac{\sigma-\gamma}{2}\right|d\gamma \\ &= \frac{1}{\pi}\int_0^{\pi}\frac{\partial\Phi}{\partial\rho}(1,\gamma)\log\left|\frac{\sin\frac{\sigma+\gamma}{2}}{\sin\frac{\sigma-\gamma}{2}}\right|d\gamma\end{aligned}$$

We thereby put

$$K(\sigma,\gamma) = \frac{1}{\pi}\log\left|\frac{\sin\frac{\sigma+\gamma}{2}}{\sin\frac{\sigma-\gamma}{2}}\right|. \tag{5.2}$$

This is a positive function of $0 < \sigma,\gamma < \pi, \sigma \neq \gamma$. The equation (5.1) is now written as

$$\theta(\sigma) = p\int_0^{\pi} K(\sigma,\gamma)e^{-3H\theta(\gamma)}\sin\theta(\gamma)d\gamma.$$

For all $m \in \mathbb{N}$ we define a space Z^m of functions on $[0,\pi]$ such that

$$Z^m = \left\{ f\ ;\quad \int_0^{\pi} f(\sigma)\cos k\sigma d\sigma = 0 \qquad (\text{ for all } k \in \mathbb{N}\ ) \right\}.$$

Note that $\theta \mapsto e^{-3H\theta}\sin\theta$ is a smooth mapping from Z^2 into $H^2(0,\pi)$. We define a linear operator T by

$$Tf = \int_0^{\pi} K(\sigma,\gamma)f(\gamma)d\gamma.$$

Since

$$K(\sigma,\gamma) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\sigma\sin n\gamma}{n}, \tag{5.3}$$

the operator T sends an element of $H^m(0,\pi)$ boundedly into Z^{m+1}. We define G by

$$G(\theta) = T\Big(e^{-3H\theta}\sin\theta\Big)$$

and solve $\theta = pG(\theta)$ in Z^2.

PROPOSITION 5.1. *G is a smooth mapping from Z^2 into itself satisfying*

$$G(\theta)(0) = G(\theta)(\pi) = 0 \qquad (\theta \in Z^2\ ).$$

It is a compact operator in Z^2. If θ satisfies $0 \le \theta(\sigma) \le \pi/2$ for all $\sigma \in [0,\pi]$ then G satisfies

$$G(\theta)(\sigma) > 0 \qquad (\sigma \in (0,\pi)).$$

The Fréchet derivative at $\theta = 0$ is given by

$$G_\theta(0)\eta = T\eta. \tag{5.4}$$

Since the proof is easy, we omit the proof.

COROLLARY 5.1. *$I - pG_\theta(0)$ is isomorphic if and only if $p \notin \mathbb{N}$.*

PROOF: By the compactness of G this operator is isomorphic if and only if it has trivial null space. The conclusion follows from (5.3,4). ∎

We now consider the existence of nontrivial solutions. Stokes' theorem, which was later proved rigorously by Levi-Civita [**19**] and Nekrasov [**31**], is included in the following Theorem 5.1. Before that, we need:

LEMMA 5.1. *It holds for all $\theta \in Z^2$ that*

$$G(\overline{\theta})(\sigma) = -G(\theta)(-\sigma),$$

where $\overline{\theta}(\gamma) = -\theta(-\gamma)$.

The proof is easy.

THEOREM 5.1. *$(p,\theta) = (n,0)$ is a bifurcation point for all $n \in \mathbb{N}$. The bifurcation occurs subcritically in p.*

PROOF: The existence part is included in Theorem 3.1. To show subcriticality, we need some computation, which is omitted here.

We now turn to the following question:

Where does the solution branch go ?

Amazingly Stokes [**42**] already gave an " answer " to this question. He reasoned as follows: climbing up the branch, the crest becomes higher and the trough deeper. Finally the crest forms a corner of angle $2\pi/3$ (see Fig. 5) and the solution is no longer smooth. This is called the Stokes conjecture by several authors ([**1,2,3,46**]). His conjecture includes that $0 \leq |\theta| \leq \pi/6, \theta(0) = \pi/6, \theta(\pi) = 0$ and that the wave profile between two consecutive crests are concave. Concerning this conjecture, the following theorem due to Krasovskii is the first mathematical result which contains global nature.

THEOREM 5.2 (KRASOVSKII [**18**]). *For all $\eta \in [0, \pi/6)$, there are $p > 0$ and a $\theta \in Y^2$ such that*

$$\max_{0 \leq \sigma \leq \pi} \theta(\sigma) = \eta, \qquad \theta(\sigma) > 0 \qquad (\sigma \in (0, \pi))$$

satisfying $\theta = pG(\theta)$.

For other mathematical results of global nature, see [**16,46**]. For numerical computations of branches of gravity waves, see [**8,44,45,48,49,55-57**].

§6. Stokes' highest wave.

In this section we consider what is called the highest wave. The discovery of this wave is due to Stokes and he made some conjectures on it, some of which are not even now proved. Let us assume the following physical hypothesis, which is well confirmed by numerical experiment:

As we trace the branch of the gravity waves, their crest become sharper and eventually form a corner of positive angle (see Fig. 4).

If we admit this, then the angle of the corner must be $2\pi/3$. This is understood by the following argument due to Stoke [**42**] and Michell [**29**]: If the crest form an angle, then the corner point must be a stagnation point, otherwise, the fluid particles of both sides of the corner have non-zero speed tangent to the free boundary and cannot be continuous at the corner. Let us choose the origin at the corner (= crest). Since the corner is a stagnation point, we have

$$\frac{df}{dz} \sim az^n \qquad \text{near the crest,}$$

where a, n are constants and $n > 0$. Let the angle between the free boundary to the right of the corner with the x-axis be $-\alpha$ (see Fig. 5). Then the angle between the free boundary to the left of the corner with the positive x-axis is $\pi + \alpha$. What we should prove is that $\alpha = \pi/6$. Since

$$\frac{a^2|z|^{2n}}{2} - g|z|\sin\alpha = \text{constant}$$

should holds asymptotically near the origin, the constant n must be $1/2$. When the argument of z changes from $\pi+\alpha$ to $2\pi-\alpha$ near the origin, then df/dz decreases by 2α (see Fig. 5). On the other hand, $df/dz \sim az^{1/2}$ implies that it increases by $\operatorname{sgn} a(\pi-2\alpha)/2$. This implies that $2\alpha=(\pi-2\alpha)/2$ must holds. Therefore $\alpha=\pi/6$.

We now show that the highest wave satisfies the following integral equation:

$$\theta(\sigma)=\int_0^\pi K(\sigma,\gamma)\frac{\sin\theta(\gamma)}{3\int_0^\gamma \sin\theta(\xi)d\xi}d\gamma. \tag{6.1}$$

PROOF: Since

$$e^{3H\theta}\frac{d}{d\sigma}H\theta=p\sin\theta, \tag{6.2}$$

it holds that

$$\frac{1}{3}e^{3H\theta}=c_0+p\int_0^\sigma \sin\theta(\xi)d\xi,$$

where c_0 is a constant. Putting $\sigma=0$, we have $c_0=(1/3)\exp(3H\theta(0))$. If we write by d the value of $|df/dz|$ at the crest, then $c_0=(1/3)(d/c)^3$ by (2.10). Hence we obtain by (6.2)

$$\frac{dH\theta}{d\sigma}=\frac{p\sin\theta(\sigma)}{(d/c)^3+3p\int_0^\sigma \sin\theta(\xi)d\xi}. \tag{6.3}$$

At $\sigma=0$, $df/dz=0$, which implies $d=0$. This leads to

$$\frac{dH\theta}{d\sigma}=\frac{\sin\theta(\sigma)}{3\int_0^\sigma \sin\theta(\xi)d\xi}.$$

By this equality we obtain (6.1). ∎

Remark 6.1. In (6.1) the denominator vanishes at $\gamma=0$. However, this apparent singularity is canceled by $K(\sigma,0)\equiv 0$.

The existence of the solution to (6.1) was first proved by Toland [**46**]. It has the following properties ([**3,46**]):

$$\theta\in C([0,\pi])\cup C^\infty((0,\pi]),\qquad \lim_{\sigma\to 0}\theta(\sigma)=\pi/6. \tag{6.4}$$

As is seen, (6.3) is a singular perturbation problem with small parameter d/c. Keady and Norbury [**16**] proved that for all d with $p > (d/c)^3 > 0$, (6.3) has at least one solution. Toland [**46**] proved that as $(d/c)^3 \to 0$, these solutions have a subsequence which is convergent and that the limit function satisfies (6.1). The property (6.4) was proved by [**3,46**].

Theorems in [**16,18**] requires some deep results in mathematics and clarify some of the global bifurcation diagram. Nonetheless, there are many questions remained. First, how the wave of extreme form is connected with the solutions of almost extreme form ? In other words, how the solution to (6.1) are related to smooth solutions to (5.1). Since the arguments in [**46**] uses approximate solutions and a subsequence of them, the connection is not so clear. Second, [**8,49**] give numerical evidence that there are secondary bifurcations in the branches of mode 2 and mode 3. [**55-57**] shows further that there are tertiary bifurcations. Can we rigorously prove this ? There is a very rich structure in the bifurcation of gravity waves. On the other hand, the author's knowledge is too limited to explain further and we only quote [**20-27**].

§7. Stokes expansion.

In this section we present the original formulation by Stokes. The formulation is not so convenient as (2.13) from the mathematical point of view. It is, however, suitable for numerical computations.

Recall that z is an analytic function of f satisfying $dz/df \to c$ as $V \to -\infty$. Consequently it allows the following expansion:

$$z = \frac{f}{c} + \sum_{n=1}^{\infty} \frac{iL(A_n + iB_n)}{2n\pi} \exp\left(-\frac{2n\pi i f}{cL}\right) + \frac{iL\alpha_0}{2\pi}.$$

Here A_n, B_n and α_0 are real constants. On the free boundary, $f = U$ is real. Therefore we obtain the following parametrization of the free boundary:

$$x = \frac{U}{c} + \sum_{n=1}^{\infty} \left[\frac{LA_n}{2n\pi} \sin\left(\frac{2n\pi U}{cL}\right) - \frac{LB_n}{2n\pi} \cos\left(\frac{2n\pi U}{cL}\right)\right],$$

$$y = \sum_{n=1}^{\infty} \left[\frac{LA_n}{2n\pi} \cos\left(\frac{2n\pi U}{cL}\right) + \frac{LB_n}{2n\pi} \sin\left(\frac{2n\pi U}{cL}\right)\right] + \frac{L\alpha_0}{2\pi}.$$

We derive a nondimensional form of this expression as follows: we put $\xi = 2\pi U/cL$ and $X(\xi) = 2\pi x/L, Y(\xi) = 2\pi y/L$. It then holds that

$$X(\xi) = \xi + \sum_{n=1}^{\infty} \left[\frac{A_n}{n} \sin(n\xi) - \frac{B_n}{n} \cos n\xi\right] \tag{7.1}$$

$$Y(\xi) = \sum_{n=1}^{\infty} \left[\frac{A_n}{n} \cos(n\xi) + \frac{B_n}{n} \sin n\xi\right] + \alpha_0. \tag{7.2}$$

The Bernoulli condition (2.5) is expressed as follows:

$$\frac{\mu}{2}\frac{1}{X'^2+Y'^2}+Y-\kappa\frac{X'Y''-X''Y'}{(X'^2+Y'^2)^{3/2}}=\text{constant}, \tag{7.3}$$

where the prime means the differentiation in ξ and two parameters μ and κ are given by

$$\mu=\frac{2\pi c^2}{gL}, \qquad \kappa=\frac{4\pi^2 T}{gL^2}. \tag{7.4}$$

Differentiating (7.3) in ξ, we obtain

$$\frac{\mu}{2}\frac{d}{d\xi}\frac{1}{X'^2+Y'^2}+Y-\kappa\frac{d}{d\xi}\frac{X'Y''-X''Y'}{(X'^2+Y'^2)^{3/2}}=0. \tag{7.5}$$

Thus we have an equation which is closed in X' and Y' only. The following expression is useful:

$$X'(\xi)=1+\sum_{n=1}^{\infty}\mathrm{Re}\Big[(A_n+iB_n)e^{-in\xi}\Big], \qquad Y'(\xi)=\sum_{n=1}^{\infty}\mathrm{Im}\Big[(A_n+iB_n)e^{-in\xi}\Big]. \tag{7.6}$$

The problem is now to seek $A_1, A_2, \cdots$ which satisfies (7.4,5). This formulation was used in [**6-8,38,56**] to compute waves numerically.

We now introduce an idea due to Longuet-Higgins [**22**,26]. For the sake of convenience we consider symmetric gravity waves. Therefore $\kappa=0$ and $B_n=0 \quad (n\geq 1)$. We now have

$$X'(\xi)=1+\sum_{n=1}^{\infty}A_n\cos(n\xi), \qquad Y'(\xi)=-\sum_{n=1}^{\infty}A_n\sin(n\xi). \tag{7.7}$$

Choosing α_0 so that the integral constant is zero, we have

$$\frac{\mu}{2}+Y(X'^2+Y'^2)=0. \tag{7.8}$$

Putting $A_0=1$, we have $X'=\sum_{n=0}^{\infty}A_n\cos(n\xi)$ by (7.7). The equation (7.8) gives a cubic equation in A_n. We write this as follows:

$$X'^2+Y'^2=P_0+2P_1\cos\xi+2P_2\cos 2\xi+\cdots,$$

where $P_n = \sum_{m=0}^{\infty} A_{m+n} A_m \quad (n = 0, 1, \cdots)$. For the notational convenience, we define $H_0 = \alpha_0$ and $H_n = A_n/(2n)$ for $n = 1, 2, \cdots$. The equation (7.6) now gives

$$-\mu/2 = \left(H_0 + 2H_1 \cos\xi + 2H_2 \cos 2\xi + \cdots\right)\left(P_0 + 2P_1 \cos\xi + 2P_2 \cos 2\xi + \cdots\right).$$

This may be written as follows:

$$\begin{aligned}
&H_0 P_0 + (H_1 + H_1)P_1 + (H_2 + H_2)P_2 + \cdots = -\mu/2, \\
&H_1 P_0 + (H_0 + H_2)P_1 + (H_1 + H_3)P_2 + \cdots = 0, \\
&H_2 P_0 + (H_1 + H_3)P_1 + (H_0 + H_4)P_2 + \cdots = 0, \\
&\cdots\cdots\cdots
\end{aligned}$$

Let us define $G_n = \sum_{m=0}^{\infty} H_{|n-m|} A_m$ for $n = 0, 1, \cdots$. The equation is now written as

$$\begin{aligned}
A_0 G_0 + A_1 G_1 + A_2 G_2 + A_3 G_3 + \cdots &= -\mu/2, \\
A_0 G_1 + A_1 G_2 + A_2 G_3 + \cdots &= 0, \\
A_0 G_2 + A_1 G_3 + \cdots &= 0, \\
\cdots\cdots&
\end{aligned}$$

This system can be solved in G_n and we obtain that $G_0 = -\mu/(2A_0)$ and $G_1 = G_2 = \cdots = 0$. The result is written as follows:

$$\begin{aligned}
-\mu &= 2H_0 + A_1 A_1 + A_2 A_2/2 + A_3 A_3/3 + \cdots, \\
0 &= A_1 + 2H_0 A_1 + A_1 A_2 + A_2 A_3/2 + \cdots, \\
0 &= A_2/2 + A_1 A_1 + 2H_0 A_2 + A_1 A_3 + \cdots, \\
0 &= A_3/3 + A_2 A_1/2 + A_1 A_2 + 2H_0 A_3 + \cdots, \\
&\cdots\cdots\cdots
\end{aligned}$$

Note that these equations are quadratic in $H_0, A_1, A_2, \cdots$ whereas the original equations are cubic. In view of this system of equations, we consider a infinite matrix $M = M(x)$ whose entries $M_{i,j} \quad (i, j = 1, 2, 3, \cdots)$ are given by

$$\begin{aligned}
&M_{ii} = x_0 && i = 0, 1, \cdots \\
&M_{i,j} = x_{|i-j|}/|i-j| && \text{for} \quad i \neq j
\end{aligned}$$

Defining $\hat{x} = (1, x_1, x_2, \cdots)$ for $x = (x_0, x_1, x_2, \cdots)$, we consider now

$$M(x)\hat{x} = (-\mu, 0, 0, \cdots). \tag{7.9}$$

Hence x_0 stands for H_0 and x_n for A_n $(n = 1, 2, \cdots)$. This system of equations is written in a more concise form as follows: Let Φ be defined by

$$\Phi(\mu; x_0, x_1, \cdots) = \frac{1}{4}(x_0 + \mu)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\frac{x_n^2}{n^2} + \frac{x_0}{2}\sum_{n=1}^{\infty}\frac{x_n^2}{n} + \sum_{n=1}^{\infty} x_n \sum_{k=1}^{\infty}\frac{x_k x_{n+k}}{k(n+k)}$$

Then (7.9) is equivalent to

$$\frac{\partial \Phi}{\partial x_n} = 0 \qquad (n = 0, 1, \cdots).$$

This remarkable fact was found by [**26**].

In order to analyze this equation, we give the following spaces of sequences:

$$V^s = \left\{ (x_n)_{n=0}^{+\infty} \ ; \quad \sum_{n=1}^{+\infty} n^{2s} x_n^2 < +\infty, \ x_m \in \mathbf{R} \quad \right\}.$$

Its norm is

$$\|x\|_s = \sqrt{x_0^2 + \sum_{m=1}^{+\infty} m^{2s} x_m^2}.$$

For $x = (x_n)$ and $y = (y_n)$ in V^s, we define $z = (z_n)$ by

$$z_0 = x_0 y_0 + \sum_{m=1}^{+\infty} \frac{x_m y_m}{m},$$

$$z_n = \sum_{m=1}^{n-1} \frac{x_{n-m} y_m}{n-m} + x_0 y_n + \sum_{m=n+1}^{+\infty} \frac{x_{m-n} y_m}{m-n} \qquad (n \geq 1).$$

We define a bilinear form B by $z = B(x, y)$. It has the following nice property:

PROPOSITION 7.1. *For $s \geq 0$, B is a bounded bilinear form from $X^s \times X^s$ to X^s.*

PROOF: Let x, y and z be as above. It holds that

$$|z_0| \leq 2\|x\|_{-1/2}\|y\|_{-1/2}$$

and for $n \geq 1$,

$$|z_n|^2 \leq 3\left|\sum_{m=1}^{n-1} \frac{x_{n-m}y_m}{n-m}\right|^2 + 3x_0^2y_n^2 + 3\left|\sum_{m=n+1}^{\infty} \frac{x_{m-n}y_m}{m-n}\right|^2$$

$$\leq 3\Big(\sum_{m=1}^{n-1}(n-m)^{2s}x_{n-m}^2\Big)\Big(\sum_{m=1}^{n-1}\frac{y_m^2}{(n-m)^{2s+2}}\Big) + 3x_0^2y_n^2$$

$$+3\Big(\sum_{m=n+1}^{\infty}(m-n)^{2s}x_{m-n}^2\Big)\Big(\sum_{m=n+1}^{\infty}\frac{y_m^2}{(m-n)^{2s+2}}\Big)$$

$$\leq 3\|x\|_s\sum_{m=1}^{n-1}\frac{y_m^2}{(n-m)^{2s+2}} + 3x_0^2y_n^2 + 3\|x\|_s\sum_{m=n+1}^{\infty}\frac{y_m^2}{(m-n)^{2s+2}}$$

Multiplying n^{2s}, we take a sum in n. Since $s \geq 0$, it holds that

$$\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\frac{n^{2s}y_m^2}{(n-m)^{2+2s}} = \sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\frac{(m+k)^{2s}}{k^{2+2s}}y_m^2 \leq \sum_{m=0}^{\infty}c_1m^{2s}y_m^2,$$

where c_1 is a positive constant independent of y and m. It also holds that

$$\sum_{n=1}^{\infty}\sum_{m=n+1}^{\infty}\frac{n^{2s}y_m^2}{(m-n)^{2+2s}} = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{n^{2s}y_{n+k}^2}{k^{2+2s}} = \sum_{j=2}^{\infty}\sum_{k=1}^{j-1}\frac{(j-k)^{2s}}{k^{2+2s}}y_j^2 \leq \sum_{j=2}^{\infty}c_2j^{2s}y_j^2,$$

where c_2 is a positive constant independent of y and j. Making use of these inequalities, we easily obtain $\|z\|_s \leq c\|x\|_s\|y\|_s$, where c is a positive constant depending only on $s \geq 0$. ∎

Notation. For $x = (x_0, x_1, x_2, \cdots)$ we put

$$\widetilde{x} = (0, x_1, x_2, \cdots).$$

For $\mu \in \mathbf{R}$, we put

$$e_\mu = (-\mu, 0, 0, \cdots).$$

What we have to solve is $B(x, \hat{x}) = e_\mu$. Note that $B(e_\mu, \hat{e}_\mu) = e_\mu$. We thereby put

$$F(\mu, u) = B(e_\mu + u, (e_\mu + u)^\wedge) - e_\mu = B(e_\mu + u, \hat{e}_\mu + \widetilde{u}) - e_\mu$$

$$= B(u, e_{-1}) + B(e_\mu, \widetilde{u}) + B(u, \widetilde{u}).$$

By Proposition 7.1, F is a smooth mapping from X^s into X^s for nonnegative s. We easily obtain its Fréchet derivative.

PROPOSITION 7.1.

$$D_uF(\mu,0)w = B(w,e_{-1}) + B(e_\mu,w) = (w_0,(1-\mu)w_1,(1/2-\mu)w_2,(1/3-\mu)w_3,\cdots)$$

In particular, $D_uF(\mu,0)$ is an isomorphism from X^s onto X^s if and only if $\mu \neq 1$, $1/2, 1/3, \cdots$.

Proof is omitted since it is easy. Applying a theorem which guarantees bifurcation from simple eigenvalue (e.g. [9]), we see that $(1/n, 0)$ is a bifurcation point for $n = 1, 2, \cdots$.

One of the interesting points of this formulation is a possibility that we can have a priori estimates for the error for numerical computations. For instance, we define a truncated map of F by

$$F^{(N)}(\mu, u_0, u_1, \cdots, u_N) = P_N F(\mu, (u_0, u_1, \cdots, u_N, 0, \cdots)),$$

where P_N denote the projection onto the first $N+1$ components. Computing zeros of $F^{(N)}$, we can obtain numerical solutions. In the computations in the past, there seems to be no computation with a priori error estimate. If, however, we use the present formulation, we think that an a priori error estimate is possible thanks to the simple form of F.

§8. **Structure of the set of capillary-gravity waves.** In this section we consider the case where both p and q are positive. Due to the limitation of the paper, we only give a brief survey of this subject. The theory of capillary-gravity waves is qualitativle different from that of gravity waves in the following two point. First, the appearance of q makes the problem a singular perturbation problem for a small q. Hence we must be careful when we compute capillary-gravity waves for small q. This singular perturbation problem seems not to be analyzed so far. Secondly, even in the case of moderately large q, there is a problem which arises as a consequence of double eigenvalue. As we saw in §3, there are points (p, q) at which (3.4) are satisfied two distinct positive integers. Let $0 < m < n$ be the integers. Then the linearized operator $D_uF(p, q; 0)$ has a kernel spanned by the following four functions:

$$\sin m\sigma, \quad \cos m\sigma \quad, \sin n\sigma, \quad \cos n\sigma$$

Using restricted function space Y^2, we have a null space spanned by $sinm\sigma$ and $\sin n\sigma$. Thus the problem is a bifurcation from a double eigenvalue. We then use a method due to Fujii, Mimura and Nishiura. See [**33**], for details. In [**34**] normal forms are obtained near the singular points of the above type. The normal forms explains some of the computational results in [**6,7,38**]. Fig. 5 is borrowed from [**38**]. Many other interesting and new bifurcation diagrams are found in [**38**]. For small amplitude solutions, see also Pierson and Fife [**36**].

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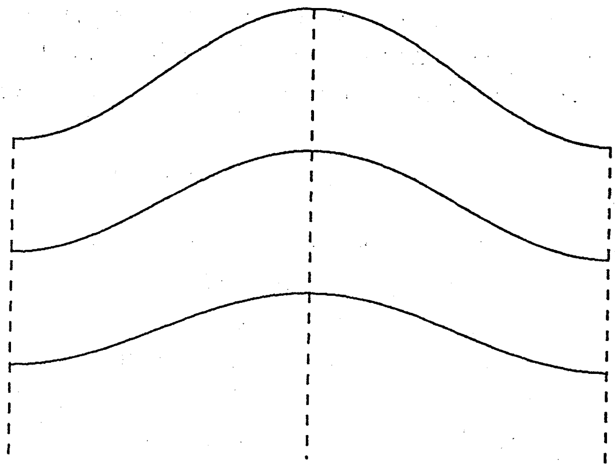

Fig. 1

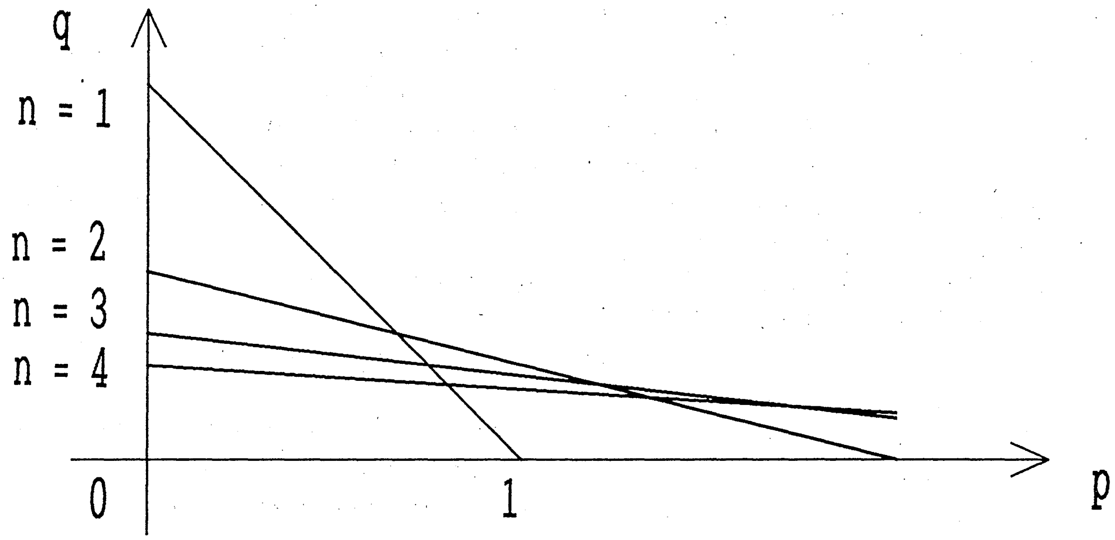

Fig. 2

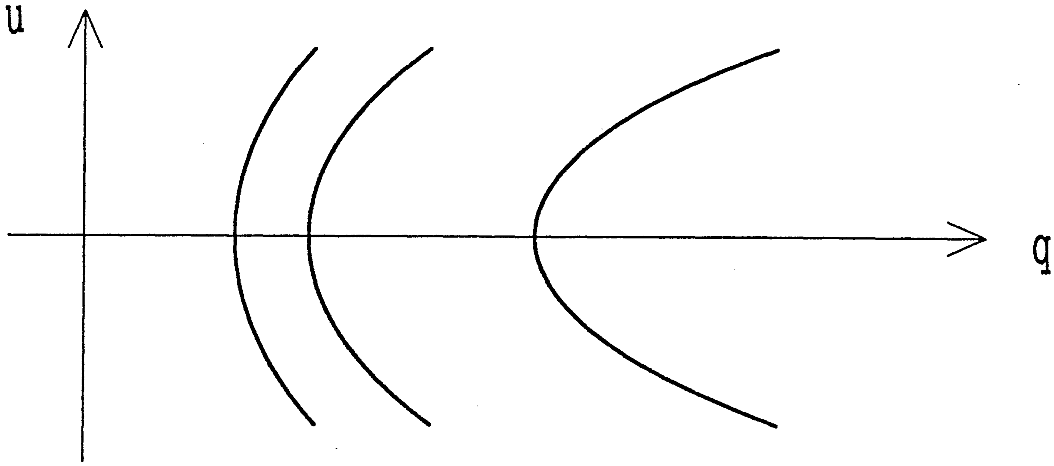

Fig. 3

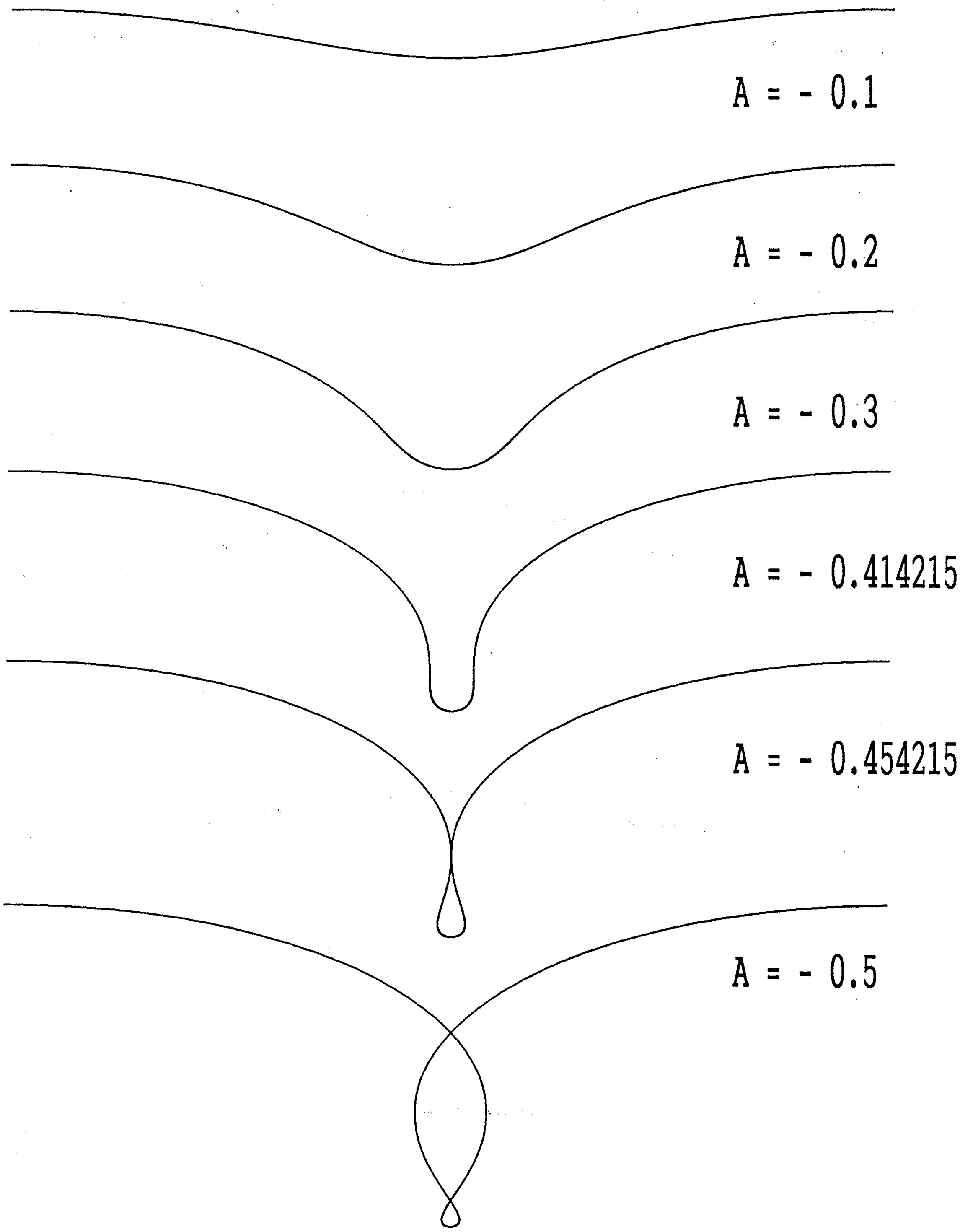

Fig. 4

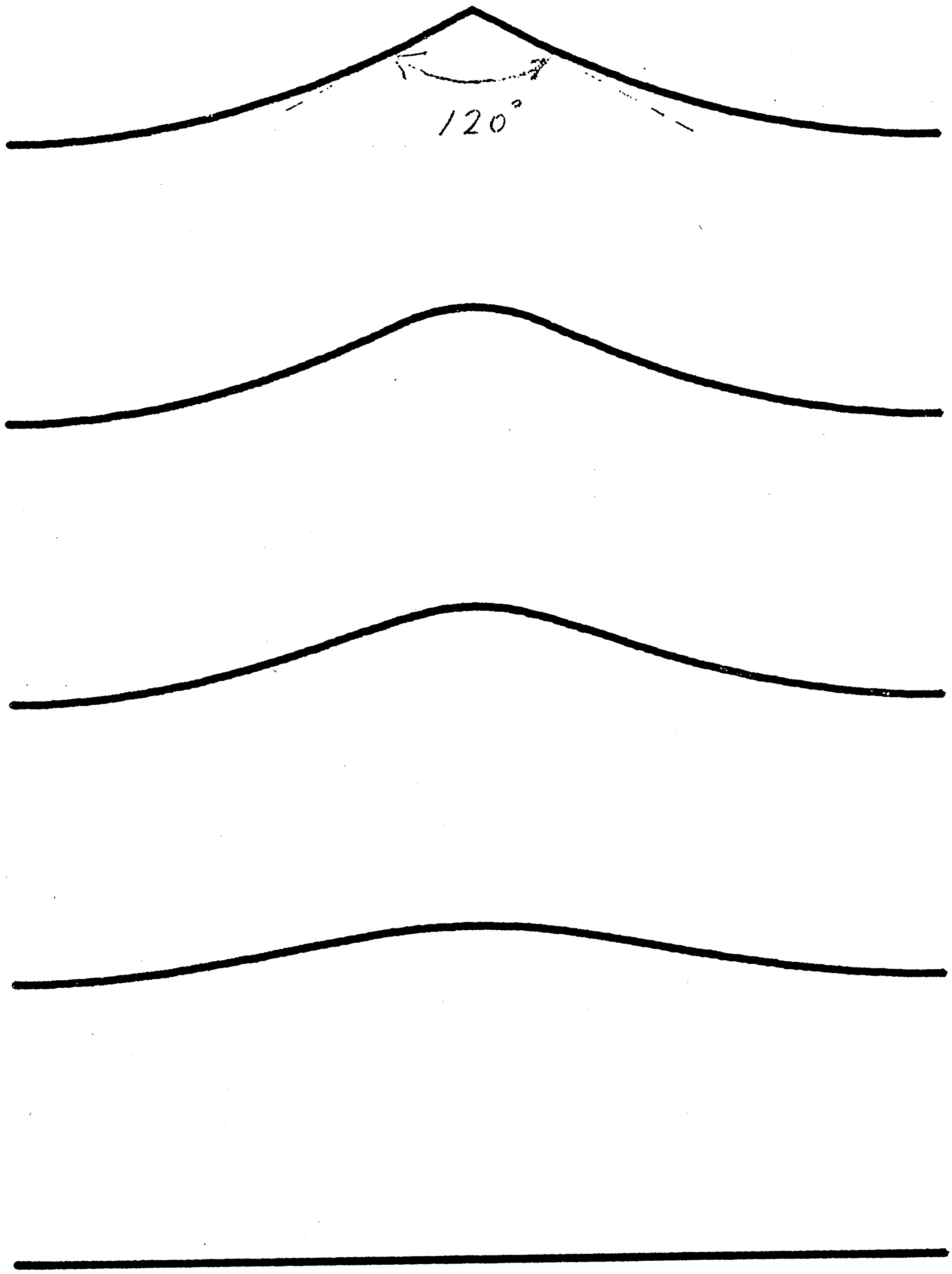

Fig. 5