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Maslov class of an isotropic map-germ arising from one dimensional symplectic reduction

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0. Introduction

Let (M^{2n}, ω) be a symplectic manifold of dimension $2n$ and $N^n \subset M^{2n}$ be a Lagrangian submanifold with singularities. For each regular point x of N, T_xN is a Lagrangian subspace of the sympletic vector space T_xM.

To investigate the local structure of N near a singular point x_0 of N , it is natural to study the behavior of the distribution $\{T_xN \mid x \text{ is a regular point of } N\}$ near x_0. Then we can grasp an invariant of the singularity, which is called the Maslov class in this paper.

Originary, the notion of Maslov class (or Keller-Maslov-Arnol'd class) is a global one, and it appears in context of asymptotic method of linear partial differential equation and symplectic topology ([A1],[GS],[Gr],[Hö],[Ma],[V],[W]).

Maslov classes represent obstruction for transversality of two Lagrangian subbundles. Applying this, we define the Maslov class of isotopic mapping as an obstruction for extendability of a partially defined Lagrangian subbundle.

Related to variational problems, Arnol'd introduced a Lagrangian variety, so called an open swallowtail, and invesigated it ([A3],[J]).

On the other hand, related to the problem of Lagrangian immersion of surfaces to four dimensional symplectic manifolds, Givental' introduced a Lagrangian variety, so called an open Whitney umbrella, and investigated some local and grobal problems. Especially he calculated the "Maslov index" of an open Whitney umbrella.

Our results generalize this result of Givental'.

The purpose of this paper is to develop a general theory on objects containing Lagrangian varieties mentioned above.

Singular Lagrangian varieties appear typically in the process of symplectic reduction.

Let $M^{2(n+k)}$ be a sympectic manifold of dimension $2(n+k)$, and K^{2n+k} a coisotropic submanifold of M of codimention K. Then naturally we have a symplectic manifold M'^{2n} of dimension $2n$, at least locally, and we call M' a k-dimensional reduction of M. Let L^{n+k} be a Lagrangian submanifold (without singularities) of M. Suppose $N = L \cap K$ is an n-dimensional submanifold of K. If L intersects transversely to K, then we have an immersed Lagrangian manifold L' in M'. Otherwise, we have a singular Lagrangian variety by reduction. (Open swallowtails are obtained in this situation; [A3],[G1],[J1]).

More generally, let N^n be an isotropic submanifold of M contained in K. If a characteristic direction of K is tangent to N, then, by reduction, we have a singular Lagrnagian variety.

Notice that singular Lagrangian varieties obtained by reduction are parametrized by isotropic mappings.

We consider, in this paper, Maslov classes of isotropic mappings obtained by one dimensional reduction process.

The first result is on vanishing of Maslov classes: The Maslov class of an isotropic map-germ obtained by 1-dimensional reduction of a Lagrangian manifold is zero (Theorem 6.1).

Thus, for a singularity of 1-dimensional reduction of an isotropic manifold, the Maslov class has a meaning of obstruction for representability as an intersection of a Lagrangian submanifold and a hypersurface.

In general, Maslov classes do not vanish. We give local model of singularities of isotropic mappings generically obtained by 1-dimensional reduction of isotropic submanifolds, up to local symplectic diffeomorphisms of the reduced symplectic manifold (Theorem 9.1). These models are open whitney umbrellas of arbitrary dimension and their suspensions, and their Maslov classes do not vanish (Theorem 10.3). Therefore we see that a generic isotropic submanifold in a hypersurface of a symplectic manifold is not an intersection of a Lagrnagian submanifold and the hypersurface, locally at each point, where the characteristic direction tangent to the hypersurface (Corollary 10.4).

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Throughout this paper, all manifolds and maps are of class C^∞.

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1. Classical Maslov class

Let (M^{2n}, ω) be a symplectic manifold, $N^n \subset M$ a Lagrnagian submanifold ($\omega|N = 0$), and $\pi : M \longrightarrow B$ a Lagrangian fibration. Then the symplectic vector bundle $\mathbf{E} = TM|N$ has two Lagrangian subbundles $\mathbf{L} = \operatorname{Ker}\pi_*|N$ and $\mathbf{L}' = TN$.

In general, for a symplectic vector bundle $\mathbf{E}$ of rank $2n$ over N and Lagrnagian subbundles $\mathbf{L}$ and $\mathbf{L}'$ of $\mathbf{E}$, the Maslov class $m(\mathbf{E}; \mathbf{L}, \mathbf{L}') \in H^1(N, \mathbf{Z})$ is defined as follows.

Consider the bundle $\Lambda(\mathbf{E})$ over N of Lagrangian subspaces of fibers of $\mathbf{E}$. The Lagrangian subbundle $\mathbf{L}'$ define a section $s(\mathbf{L}') : N \longrightarrow \Lambda(\mathbf{E})$ by $s(\mathbf{L}')(x) = \mathbf{L}'_x \in \Lambda(\mathbf{E}_x) \subset \Lambda(\mathbf{E}), (x \in N)$.

Let Ω denote the symplectic form of $\mathbf{E}$. Then there exist a complex structure J and a Hermitian form G on $\mathbf{E}$, unique up to homotopy, such that Ω is the imaginary part of G. Denote by g the real part of G.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal frame of $\mathbf{L}$ over an open subset U of N, with respect to g. Then $\{e_1, \ldots, e_n\}$ turns to be a unitary frame of the Hermitian vector bundle $(\mathbf{E}; J, G)$ over U. Then we have an isomorphism $\mathbf{E}|U \cong \mathbf{C}^n \times U$ as Hermitian vector bundle, which maps $\mathbf{L}|U$ to $\mathbf{R}^n \times U$.

Since $U(n)$ acts on the space $\Lambda(\mathbf{C}^n)$ of Lagrangian subspaces of $\mathbf{C}^n$ transitively, $\Lambda(\mathbf{C}^n)$ is identified with the homogeneous space $\Lambda(n) = U(n)/O(n)$, (see [A]).

Thus we have $\Lambda(\mathbf{E})|U \longrightarrow \Lambda(n)$, which is glued to a C^∞ mapping $\Phi(\mathbf{L}) : \Lambda(\mathbf{E}) \longrightarrow \Lambda(n)$.

Set $\Phi = \Phi(\mathbf{L}) \circ s(\mathbf{L}') : N \longrightarrow \Lambda(n) = U(n)/O(n)$. The homotopy type of Φ is independent of the choice of (J, G).

If $\{e'_1, \ldots, e'_n\}$ is an othonormal frame of $\mathbf{L}'$ over U, then, at $x \in U$, $e'_j = \sum_{i=1}^n a_{ij} e_i$, for some $A = (a_{ij}) \in U(n)$. Then $\Phi(x) = [A] \in \Lambda(n)$. Remark that $G(e'_j, e_i) = a_{ij}$.

Define $m(\mathbf{E}; \mathbf{L}, \mathbf{L}')$ to be the image of the generator of $H^1(S^1, \mathbf{Z}) \cong \mathbf{Z}$, relatively to counterclockwise orientation, under

$$(\det{}^2 \circ \Phi)^* : H^1(S^1, \mathbf{Z}) \longrightarrow H^1(N, \mathbf{Z}),$$

where $\det^2 : U(n)/O(n) \longrightarrow S^1$ is defined by $\det^2([A]) = (\det A)^2, A \in U(n)$.

LEMMA 1.1. *The Maslov class satisfies following properties:*

(0) $m(\mathbf{E}; \mathbf{L}, \mathbf{L}) = 0$.

(1) $m(\mathbf{E}; \mathbf{L}, \mathbf{L}') + m(\mathbf{E}; \mathbf{L}'\mathbf{L}'') = m(\mathbf{E}; \mathbf{L}, \mathbf{L}'')$.

(2) $\mathbf{L}$ *and* $\mathbf{L}'$ *are transverse in* $\mathbf{E}$,*then* $m(\mathbf{E}; \mathbf{L}, \mathbf{L}') = 0$.

(3) If there is an isomorphism between symplectic vector bundles $\mathbf{E}_1$ *and* $\mathbf{E}_2$ *mapping* $\mathbf{L}_1, \mathbf{L}'_1$ *to* $\mathbf{L}_2, \mathbf{L}'_2$, *then* $m(\mathbf{E}_1; \mathbf{L}_1, \mathbf{L}_1) = m(\mathbf{E}_2; \mathbf{L}_2, \mathbf{L}'_2)$.

(4) $m(h^*\mathbf{E}; h^*\mathbf{L}, h^*\mathbf{L}') = h^* m(\mathbf{E}; \mathbf{L}, \mathbf{L}') \in H^1(P, \mathbf{Z})$.

Here $\mathbf{L}, \mathbf{L}', \mathbf{L}''$ *are Lagrangian subbundles of* $\mathbf{E}$ *and* $h : P \longrightarrow N$.

Remark 1.2. Vaisman [V] defines the Maslov classes $\mu_h(\mathbf{E}; \mathbf{L}, \mathbf{L}') \in H^{4h-3}(N, \mathbf{R}), (h = 1, 2, 3, \ldots)$, for two Lagrangian subbundles $\mathbf{L}, \mathbf{L}'$ of a symplectic vector bundle $\mathbf{E}$ over N , such that

(i) $\mu_1(\mathbf{E};\mathbf{L},\mathbf{L}') = (1/2)m(\mathbf{E};\mathbf{L},\mathbf{L}') \in H^1(N,\mathbf{R})$

(ii) μ_h satisfies the properties of Lemma 1.1.

Though Lemma 1.1 is well known (see[V]), we give here an elementary proof of this fundamental lemma in this paper.

PROOF OF LEMMA 1.1: Since $\Phi = \Phi(\mathbf{L}) \circ s(\mathbf{L})$ is constant, we have (0).

To show (1), we will show the followings:

(5) For Langrangian subbundles $\mathbf{L}_1, \mathbf{L}_1'$ of $\mathbf{E}_1$ and $\mathbf{L}_2, \mathbf{L}_2'$ of $\mathbf{E}_2$ of symplectic bundles $\mathbf{E}_1$ and $\mathbf{E}_2$ over N respectively, $m(\mathbf{E}_1 \oplus \mathbf{E}_2; \mathbf{L}_1 \oplus \mathbf{L}_2, \mathbf{L}_1' \oplus \mathbf{L}_2') = m(\mathbf{E}_1, \mathbf{L}_1, \mathbf{L}_1') + m(\mathbf{E}_2, \mathbf{L}_2, \mathbf{L}_2')$.

(6) For Lagrangian subbundles $\mathbf{L}_1, \mathbf{L}_2$ of $\mathbf{E}$ and $\mathbf{L}$ of $\mathbf{E} \oplus \mathbf{E}$ respectively, $m(\mathbf{E}; \mathbf{L}, \mathbf{L}_1 \oplus \mathbf{L}_2) = m(\mathbf{E}; \mathbf{L}, \mathbf{L}_2 \oplus \mathbf{L}_1)$.

If we have (5) and (6), then

$$\begin{aligned} m(\mathbf{E};\mathbf{L},\mathbf{L}') + m(\mathbf{E};\mathbf{L}',\mathbf{L}'') &= m(\mathbf{E}\oplus\mathbf{E}',\mathbf{L}\oplus\mathbf{L}',\mathbf{L}'\oplus\mathbf{L}''), \text{ (by(5))} \\ &= m(\mathbf{E}\oplus\mathbf{E}',\mathbf{L}\oplus\mathbf{L}',\mathbf{L}''\oplus\mathbf{L}'), \text{ (by(6))} \\ &= m(\mathbf{E};\mathbf{L},\mathbf{L}'') + m(\mathbf{E};\mathbf{L}',\mathbf{L}'), \text{ (by(5))} \\ &= m(\mathbf{E};\mathbf{L},\mathbf{L}''), \text{ (by(0))}. \end{aligned}$$

Thus we have (1).

To show (5), compare three maps

$$\Phi_1 = \Phi(\mathbf{L}_1) \circ s(\mathbf{L}_1') : N \longrightarrow \Lambda(n),$$

$$\Phi_2 = \Phi(\mathbf{L}_2) \circ s(\mathbf{L}_2') : N \longrightarrow \Lambda(m),$$

and

$$\Phi_3 = \Phi(\mathbf{L}_1 \oplus \mathbf{L}_2) \circ s(\mathbf{L}_1' \oplus \mathbf{L}_2') : N \longrightarrow \Lambda(n+m),$$

where we set rank $\mathbf{E}_1 = 2n$ and rank $\mathbf{E}_2 = 2m$.

Define

$$\Delta : \Lambda(n) \times \Lambda(m) \longrightarrow \Lambda(n+m)$$

by

$$\Delta([A],[B]) = \left[\begin{pmatrix} A & O \\ O & B \end{pmatrix} \right],$$

where we denote by [A] the class of $A \in U(n)$ in $\Lambda(n) = U(n)/O(n)$.

Then $\Phi_3 = \Lambda \circ (\Phi_1, \Phi_2)$ and $\det^2 \circ \Lambda([A],[B]) = \det^2([A]) \cdot \det^2([B])$. Thus $\det^2 \circ \Phi_3 = (\det^2 \circ \Phi_1) \cdot (\det^2 \circ \Phi_2) : N \longrightarrow S^1$.

For the generator $1 \in H^1(S^1, \mathbf{Z})$,

$$\begin{aligned} m(\mathbf{E}_1 \oplus \mathbf{E}_2; \mathbf{L}_1 \oplus \mathbf{L}_2, \mathbf{L}_1' \oplus \mathbf{L}_2') &= (\det^2 \circ \Phi_3)^* 1 \\ &= ((\det^2 \circ \Phi_1) \cdot (\det^2 \circ \Phi_2))^* 1 \\ &= (\det^2 \circ \Phi_1)^* 1 + (\det^2 \circ \Phi_2)^* 1 \\ &= m(\mathbf{E}_1; \mathbf{L}_1, \mathbf{L}_1') + m(\mathbf{E}_2; \mathbf{L}_2, \mathbf{L}_2'). \end{aligned}$$

Therefore we have (5).

To see (6), compare two maps

$$\Phi = \Phi(\mathbf{L}) \circ s(\mathbf{L}_1 \oplus \mathbf{L}_2) : N \longrightarrow \Lambda(2n),$$

$$\Phi' = \Phi(\mathbf{L}) \circ s(\mathbf{L}_2 \oplus \mathbf{L}_1) : N \longrightarrow \Lambda(2n),$$

Define $\kappa : \Lambda(2n) \longrightarrow \Lambda(2n)$ by

$$\kappa\left(\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix} \right]\right) = \left[\begin{pmatrix} B & A \\ D & C \end{pmatrix} \right],$$

for

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in U(2n),$$

with $A, B, C, D \in M_n(\mathbf{C})$. Then $\kappa \circ \Phi' = \Phi$ and $\det^2 \circ \kappa = \det^2$. Thus

$$\begin{aligned} m(\mathbf{E}; \mathbf{L}, \mathbf{L}_1 \oplus \mathbf{L}_2) &= (\det^2 \circ \Phi)^* 1 \\ &= (\det^2 \circ \kappa \circ \Phi')^* 1 \\ &= (\det^2 \circ \Phi')^* 1 \\ &= m(\mathbf{E}; \mathbf{L}, \mathbf{L}_2 \oplus \mathbf{L}_1). \end{aligned}$$

Therefore we have(6).

To see (2), remark that, if $\mathbf{L}$ and $\mathbf{L}'$ are transverse, then we can choose (J, G) such that $J\mathbf{L} = \mathbf{L}'$. Then $\Phi = \Phi(\mathbf{L}) \circ s(\mathbf{L}')$ is constant, and $m(\mathbf{E}; \mathbf{L}, \mathbf{L}') = 0$.

For (3), suppose $\alpha : (\mathbf{E}_1; \mathbf{L}_1, \mathbf{L}_1') \longrightarrow (\mathbf{E}_2; \mathbf{L}_2, \mathbf{L}_2')$ is an isomorphism. Choose a Hermitian structure $(\alpha \circ J \circ \alpha^{-1}, G \circ (\alpha^{-1}, \alpha^{-1}))$ on $\mathbf{E}_2$. Furthermore α induces a diffeomorphism $\tilde{\alpha} : \Lambda(\mathbf{E}_1) \longrightarrow \Lambda(\mathbf{E}_2)$. Then we have $s(\mathbf{L}_2') = \tilde{\alpha} \circ s(\mathbf{L}_1')$ and $\Phi(\mathbf{L}_2) \circ \tilde{\alpha} = \Phi(\mathbf{L}_1)$. Therefore $\Phi(\mathbf{L}_1) \circ s(\mathbf{L}_1') = \Phi(\mathbf{L}_2) \circ s(\mathbf{L}_2')$, and $m(\mathbf{E}_1; \mathbf{L}_1, \mathbf{L}_1') = m(\mathbf{E}_2; \mathbf{L}_2, \mathbf{L}_2')$.

Let $h : P \longrightarrow N$ be a C^∞ map. Consider $s(h^*\mathbf{L}') : P \longrightarrow \Lambda(h^*\mathbf{E})$ and $\Phi(h^*\mathbf{L}) : \Lambda(h^*\mathbf{E}) \longrightarrow \Lambda(n)$. Then we have $\Phi(h^*\mathbf{L}) \circ s(h^*\mathbf{L}') = \Phi(\mathbf{L}) \circ s(\mathbf{L}') \circ h$. Thus

$$\begin{aligned} m(h^*\mathbf{E}; h^*\mathbf{L}, h^*\mathbf{L}') &= (\det{}^2 \circ \Phi(h^*\mathbf{L}) \circ s(h^*\mathbf{L}'))^*1 \\ &= (\det{}^2 \circ \Phi(\mathbf{L}) \circ s(\mathbf{L}') \circ h)^*1 \\ &= h^* m(\mathbf{E}; \mathbf{L}, \mathbf{L}'). \end{aligned}$$

Thus we have (4). Q.E.D.

Corollary 1.3. $m(\mathbf{E}; \mathbf{L}, \mathbf{L}') = -m(\mathbf{E}; \mathbf{L}', \mathbf{L})$

Proof: By Lemma1.1.(1),(2),

$$m(\mathbf{E}; \mathbf{L}, \mathbf{L}') + m(\mathbf{E}, \mathbf{L}', \mathbf{L}) = m(\mathbf{E}; \mathbf{L}, \mathbf{L}) = 0.$$

Returning to the first situation, we define the Maslov class $m(N)$ by

$$m(N) = m(TM|N,\ \operatorname{Ker} \pi_*|N, TN) \in H^1(N, \mathbf{Z}).$$

2. Maslov class of an isotropic mapping

Let (M^{2n}, ω) be a symplectic manifold of dimension $2n$, and N^n be a C^∞ manifold of dimension n.

A C^∞ mapping $F : N \longrightarrow M$ is called an isotropic mapping if, for each $x \in N$, the image of $T_x f : T_x N \longrightarrow T_x M$ is an isotropic subspace of the symplectic vector space $T_x M$, that is, if $f^*\omega = 0$.

For an isotropic mapping f, set

$$\Sigma = \Sigma(f) = \{x \in N \mid T_x f \text{ is not injective}\}.$$

Then the restriction $f|N - \Sigma : N - \Sigma \longrightarrow M$ is a Lagrangian immersion.

Set $\Lambda(M) = \Lambda(TM)$, and denote by $\pi : \Lambda(M) \longrightarrow M$ the canonical projection. In the symplectic vector bundle π^*TM over $\Lambda(M)$, define the tautological bundle $\mathcal{L}$ by

$$\mathcal{L}_{(y,\lambda)} = \lambda \subset T_y M,$$

$(y, \lambda) \in \Lambda(M)$.

Associated to f, define

$$\varphi(f) : N - \Sigma \longrightarrow \Lambda(M),$$

by $\varphi(f)(x) = (f(x),\ \mathrm{Im}(T_x f))$. Then $\pi \circ \varphi(f) = f$.

Set $\mathbf{L}_f = \varphi(f)^*\mathcal{L}$. Then $\mathbf{L}_f$ is a Lagrangian subbundle of $f^*TM = \varphi(f)^*\pi^*TM$ over $N - \Sigma$.

Definition 2.1. Assume f^*TM has a Lagrangian subbundle (over N). Then define the Maslov class of f by

$$m(f) = \delta(m(f^*TM; \mathbf{L}, \mathbf{L}_f)) \in H^2(N, N - \Sigma; \mathbf{Z}),$$

where $\mathbf{L}$ is a Lagrangian subbundle of f^*TM over N, and $\delta : H^1(N - \Sigma; \mathbf{Z}) \longrightarrow H^2(N, N - \Sigma; \mathbf{Z})$ is the coboundary map.

Remark that $m(f^*TM, \mathbf{L}, \mathbf{L}_f) \in H^1(N-\Sigma; \mathbf{Z})$. For an another Lagrangian subbundle $\mathbf{L}'$ of f^*TM over N, we have

$$m(f^*TM; \mathbf{L}', \mathbf{L}_f) = m(f^*TM; \mathbf{L}', \mathbf{L}) + m(f^*TM; \mathbf{L}, \mathbf{L}_f),$$

in $H^1(N - \Sigma; \mathbf{Z})$, by Lemma 1.1.(2).

Since $m(f^*TM; \mathbf{L}', \mathbf{L})$ comes from an element of $H^1(N; \mathbf{Z})$, we have

$$\delta(m(f^*TM; \mathbf{L}', \mathbf{L}_f)) = \delta(m(f^*TM; \mathbf{L}, \mathbf{L}_f)).$$

Therefore $m(f)$ is independent of the choise of $\mathbf{L}$.

Remark 2.2. By Remark 1.2 and the argument as above, we can define a class $\mu_h(f) \in H^{4h-2}(N, N - \Sigma; \mathbf{Z})$ for an isotropic mapping $f : N \longrightarrow M$ by

$$\mu_h(f) = \delta(\mu_h(f^*TM, \mathbf{L}, \mathbf{L}_f)),$$

$(h = 1, 2, \dots)$, provided f^*TM has a Lagrangian subbundle $\mathbf{L}$.

3. Symplectic equivalance

Let $f : N^n \longrightarrow (M^{2n}, \omega)$ and $f' : N'^n \longrightarrow (M'^{2n}, \omega')$ be isotropic mappings.

Defnition 3.1. A pair (σ, τ) of a diffeomorphism $\sigma : N \longrightarrow N'$ and a symplectic diffeomorphism $\tau : M \longrightarrow M', (\tau^*\omega' = \omega)$, is called a sympectic equivalance between f and f' if $\tau \circ f = f' \circ \sigma$. Then we call f is symplectically equivalent to f, and write $f \sim f'$.

If (σ, τ) is a symplectic equivalance between f and f', then σ incluces an isomorphism $\sigma^* : H^2(N', N' - \Sigma'; \mathbf{Z}) \longrightarrow H^2(N, N - \Sigma; \mathbf{Z})$, where $\Sigma' = \Sigma(f')$, and $\sigma^* m(f') = m(f)$, if f^*TM has a Lagrangian subbundle $\mathbf{L}$.

In fact, τ induces isomorphisims

$$\tau' : \tau^*TM' \longrightarrow TM \text{ and } \tau'' : \sigma^* f'^* TM' = f^*\tau^*TM' \longrightarrow f^*TM$$

of symplectic vector bundles over M and N respectively, and τ'' maps $\sigma^*\mathbf{L}_{f'}$ to $\mathbf{L}_f$. Thus

$$\begin{aligned}\sigma^* m(f'^*TM', \tau''^{-1}\mathbf{L}, \mathbf{L}_{f'}) &= m(\sigma^* f'^*TM'; \sigma^*\tau''^{-1}\mathbf{L}, \sigma^*\mathbf{L}_{f'}) \\ &= m(f^*TM, \mathbf{L}_1, \mathbf{L}_f).\end{aligned}$$

for some Lagrangian subbundle $\mathbf{L}_1$ of f^*TM, by Lemma1.1.(3) and (4). Therefore

$$\begin{aligned}\sigma^* m(f') &= \sigma^*\delta m(f'^*TM', \tau''^{-1}\mathbf{L}, \mathbf{L}_{f'}) \\ &= \delta(m(f^*TM; \mathbf{L}_1, \mathbf{L}_f)) \\ &= m(f)\end{aligned}$$

4. Maslov class of an isotropic map-germ

Let $f : N^n, x \longrightarrow (M^{2n}, \omega)$ be a germ of an isotropic mapping. For each representative (f, U) such that $f : U \longrightarrow M$ is isotropic and U is a contractible neighborhood of x, we have $m(f, U) \in H^2(U, U - \Sigma; \mathbf{Z})$, since f^*TM is trivial over U. If V is a contractible neighborhood of x, with $V \subset U$ and $\iota^* : H^2(U, U - \Sigma; \mathbf{Z}) \longrightarrow H^2(V, V - \Sigma; \mathbf{Z})$ is the restriction, then $\iota^*(m(f, U)) = m(f, V)$ by Lemma1.1.(3). Set

$$H^2(N, N - \Sigma, \mathbf{Z})_x = \varinjlim H^2(U, U - \Sigma; \mathbf{Z}),$$

where U runs over contractible neighborhoods of x. Then we have an element

$$m(f) \in H^2(N, N - \Sigma; \mathbf{Z})_x.$$

We call it the Maslov class of the isotropic map-germ f.

We can define the notin of symplectic equivalance between two isotropic map-germs similarly as §3.

If (σ, τ) is a symplectic equivalance between f and another $f' : N', x' \longrightarrow (M', \omega')$, then $\sigma^* : H^2(N', N' - \Sigma', \mathbf{Z}) \longrightarrow H^2(N, N - \Sigma; \mathbf{Z})$ maps $m(f')$ to $m(f)$.

5. Symplectic reduction

Let $(M^{2(n+k)}, \omega)$ be a symplectic manifold of dimension $2(n+k)$, and $K^{2n+k} \subset M$ be a coisotropic submanifold of codimension k. We denote by $(TK)^\perp$ the skewothogonal complement to TK in $TM|K$.

Remark that the rank of $(TK)^\perp$ is equal to k. Since K is coisotropic, $(TK)^\perp \subset TK$ and $(TK)^\perp$ is integrable ([AM]). We call $(TK)^\perp$ (resp. induced foliation on K) the characteristic distribution (resp. foliation) relatively to K.

Let $x \in K$. Then, in an open neighborhood U of x in K, a submersion $\pi : U \longrightarrow M'^{2n}$ induced, where M' is the leaf space. Then M' has the unique symplectic structure ω' up to symplectic diffeomorphisms of M' such that $\pi^*\omega' = \omega|K$ ([AM]).

(1) By this reduction procesure, Lagrangian submanifolds of M also reduced to "Lagrangian varieties".

Now, let $L^{n+k} \subset M$ be a Lagrangian submanifold and $x \in L$. If $N = L \cap K$ is an n-dimensional submanifold of K in a neighborhood of x, then

$$f = \pi|N : N, x \longrightarrow M'$$

is an isotropic map-germ. In fact

$$f^*\omega' = \pi^*\omega'|N = \omega \mid N = 0.$$

Remark that f is an immersion at x if and only if $T_xL \cap (T_xK)^\perp = 0$.

Especially, if L is transverse to K, then we have an immersed Lagrangian submanifold in the reduced symplectic manifold M'.

In fact, in this case, $T_xL \cap (T_xK)^\perp = T_xL \cap (T_xL + T_xK)^\perp = T_xL \cap (T_xM)^\perp = T_xL \cap 0 = 0$.

In general, f is not an immersion and has a singularity.

Defition 5.1. Let f be as in above. Then f is called an isotropic map-germ arising from a k-dimensional reduction of a Lagrangian manifold.

(2) More in general, let N^n be an isotropic submanifold of $M^{2(n+k)}$ contained in K^{2n+k} and containing x.

Then $f = \pi|N : N, x \longrightarrow M'$ is isotropic and f is immersive if and only if $T_xN \cap (T_xK)^{\perp} = 0$.

Defnition 5.2. Such germ f is simply called an isotropic map-germ arising from k-dimensional reduction.

In what follows, we concentrate to the case $k = 1$.

6. Reduction of a Lagrangian manifold and Maslov class

In §4, we have defined the Maslov class $m(f) \in H^2(N, N - \Sigma, \mathbf{Z})_x$, for an isotropic map germ $f : N^n, x \longrightarrow M^{2n}$, where Σ is the singular set of f.

THEOREM 6.1. *Let $f : N, x \longrightarrow M$ be an isotropic map-germ. If f is symplectically equivalent to an isotropic map-germ arising from a 1-dimensional reduction of a Lagrangian manifold, then $m(f) = 0 \in H^2(N, N - \Sigma, \mathbf{Z})_x$.*

Precisely, for any open neighborhood U of x, and for any respersentative $f : U \longrightarrow M$ of f, there exist a contractible open neighborhood V such that $x \in V \subset U$ and $m(f|V) = 0$ in $H^2(V, V - \Sigma; \mathbf{Z})$.

7. Reduction of symplectic vector bundles

(1) Let $\mathbf{E}$ be a symplectic vector bundle over a manifold X, and $\mathbf{K}$ be a coisotropic subbundles. Then the bundle $\mathbf{K}/\mathbf{K}^{\perp}$ has the induced symplectic structure, where $\mathbf{K}^{\perp}$ is the skew-orthogonal complement of $\mathbf{K}$ in $\mathbf{E}$,(see [AM],[W]).

Let $\mathbf{L}$ be a Lagrangian subbundle of $\mathbf{E}$. If $\mathbf{L} \subset \mathbf{K}$, then $\mathbf{K}^{\perp} \subset \mathbf{L}^{\perp} = \mathbf{L} \subset \mathbf{K}$, and $\mathbf{L}/\mathbf{K}^{\perp} \subset \mathbf{K}/\mathbf{K}^{\perp}$ is a Lagrangian subbundle.

LEMMA 7.1. *Let $\mathbf{E}$ be a symplectic vector bundle over X, $\mathbf{K}$ a coisotropic subbundle of $\mathbf{E}$, and $\mathbf{L}$ (resp. $\mathbf{L}'$) be a Lagrangian subbundle contained in $\mathbf{K}$. Then*

$$m(\mathbf{K}/\mathbf{K}^{\perp}; \mathbf{L}/\mathbf{K}^{\perp}, \mathbf{L}'/\mathbf{K}^{\perp}) = m(\mathbf{E}; \mathbf{L}, \mathbf{L}')$$

in $H^1(X; \mathbf{Z})$, (cf. §1).

PROOF: Set rank$\mathbf{E} = 2(n+k)$ and rank$\mathbf{K} = 2n+k$. Then rank$\mathbf{K}^{\perp} = k$.

Compare

$$\Phi_1 = \Phi(\mathbf{L}) \circ s(\mathbf{L}') : X \longrightarrow \Lambda(n+k),$$

and

$$\Phi_2 = \Phi(\mathbf{L}/\mathbf{K}^{\perp}) \circ s(\mathbf{L}'/\mathbf{K}^{\perp}) : X \longrightarrow \Lambda(n).$$

Set $\Lambda(n+k,k) = \{\lambda \in \Lambda(\mathbf{C}^{n+k}) \mid \lambda \subset \mathbf{C}^n \times \mathbf{R}^k\}$. Then we can choose a Hermitian structure on $\mathbf{E}$ such that $\Phi_1(X) \subset \Lambda(n+k,k)$ and $\tilde{\pi} \circ \Phi_1 = \Phi_2$, where $\pi : \mathbf{C}^n \times \mathbf{C}^k \longrightarrow \mathbf{C}^n$ is the projection and $\tilde{\pi} : \Lambda(n+k,k) \longrightarrow \Lambda(n)$ is defined by $\tilde{\pi}(\lambda) = \pi(\lambda) \subset \mathbf{C}^n$, $(\lambda \in \Lambda(n+k,k))$. Remark that $\det^2 \circ \tilde{\pi} = \det^2 : \Lambda(n+k,k) \longrightarrow S^1$. Then $\det^2 \circ \Phi_2 = \det^2 \circ \tilde{\pi} \circ \Phi_1 = \det^2 \circ \Phi_1$.

Thus we have required result.

(2) We apply Lemma 7.1 to the situation of §5,(1).

Shrinking K, around x if necessary, we assume that the characteristic foliation of K comes from a submisssion $\pi : K \longrightarrow M'$, $N = L \cap K$ is an n-dimential submanifold in K, and that K and N are contractible.

Set $\mathbf{E} = TM|N - \Sigma, \mathbf{K} = TK|N - \Sigma, \mathbf{K}' = (TN)^{\perp}|N - \Sigma, \mathbf{L} = TL|N - \Sigma$, and $\mathbf{L}' = TN + (TK)^{\perp}|N - \Sigma$.

Notice that $TN + (TK)^{\perp}$ is a direct sum in TM over $N - \Sigma$. therefore $\mathbf{L}'$ is a subbundle of $\mathbf{E}$ of rank $n + k$. Furthermore $\mathbf{L}'^{\perp} = \mathbf{K} \cap \mathbf{K} \supset \mathbf{L}'$. Therefore $\mathbf{L}'$ is Lagrangian.

Thus we have a symplectic vector bundle $\mathbf{E}$, coisotropic subbundles $\mathbf{K}$ and $\mathbf{K}'$ of rank $2n + k$ and $n + 2k$ respectively, and Lagrnagian subbundles $\mathbf{L}$ and $\mathbf{L}'$ with $\mathbf{L} \subset \mathbf{K}', \mathbf{L}' \subset \mathbf{K}$ and $\mathbf{L}' \subset \mathbf{K}'$.

Since M' is a symplectic reduction of M relatively to K, we have anisomorphism

$$\alpha : TK/(TK)^{\perp} \longrightarrow \pi^*TM',$$

which induces an isomorphism

$$\beta : TK/(TK)^{\perp}|N \longrightarrow f^*TM'.$$

For each $y \in N - \Sigma$, $\beta(\mathbf{L}'_y/\mathbf{K}_y^{\perp}) = T_y f(T_y N) = (\mathbf{L}_f)_y$ in the fiber (f^*TM') over y. By restriction, β induces an isomorphism

$$\gamma : \mathbf{K}/\mathbf{K}^{\perp} \longrightarrow f^*TM'|N - \Sigma,$$

such that $\beta(\mathbf{L}'/\mathbf{K}^{\perp}) = \mathbf{L}_f$.

Therefore, for a Lagrangian subbundle $\mathbf{L}_1$ of f^*TM' over N, we have

$$m(f^*TM'; \mathbf{L}_1, \mathbf{L}_f) = m(\mathbf{K}/\mathbf{K}^{\perp}; \beta^{-1}(\mathbf{L}_1), \mathbf{L}'/\mathbf{K}^{\perp}),$$

in $H^1(N - \Sigma; \mathbf{Z})$, by Lemma 1.1.(3).

Take the Lagrangian subbundle L_2 of $TM|N$ contained in $TK|N$ which projects to $\beta^{-1}(\mathbf{L}_1) \subset (TK/TK^{\perp})|N$. Then, by Lemma 7.1,

$$m(\mathbf{K}/\mathbf{K}^{\perp}; \beta(\mathbf{L}_1), \mathbf{L}'/\mathbf{K}^{\perp}) = m(\mathbf{E}; \mathbf{L}_2, \mathbf{L}').$$

By Lemma 1.1.(1),

$$m(\mathbf{E};\mathbf{L}_2,\mathbf{L}') = m(\mathbf{E};\mathbf{L}_2,\mathbf{L}) + m(\mathbf{E};\mathbf{L},\mathbf{L}').$$

Since $\mathbf{L} = TL|N - \Sigma$ is a restriction of the Lagrangian subbundle $TL|N$ over N, $m(\mathbf{E};\mathbf{L}_2,\mathbf{L})$ is the restriction of an element in $H^1(N,\mathbf{Z})$.

For these arguments are valid over any contractible neighborhood V of x in N, we have Theorem 6.1 if $m(\mathbf{E};\mathbf{L},\mathbf{L}') = 0$ in $H^1(N-\Sigma,\mathbf{Z})_x$.

Furthermore, by Lemma 7.1 again,

$$m(\mathbf{E};\mathbf{L},\mathbf{L}') = m(\mathbf{K}'/\mathbf{K}'^{\perp};\mathbf{L}/\mathbf{K}'^{\perp},\mathbf{L}'/\mathbf{K}'^{\perp}).$$

In the next section, we will show the right hand side is equal to zero in $H^1(N-\Sigma,\mathbf{Z})_x$, at least if $k=1$.

8. Proof of Theorem 6.1

It is sufficient to show $m(\mathbf{K}'/\mathbf{K}^{\perp};\mathbf{L}/\mathbf{K}^{\perp},\mathbf{L}'/\mathbf{K}'^{\perp}) = 0$ in $H^1(V-\Sigma,\mathbf{Z})$ for any sufficiently small contractible neighborhood V of x, under the notations in §7,(2).

Let $h=0$ be a local equation of K in M, where $h \in C^\infty(M)$. By the sign of h, $L-N$ is devided into two part: $L - N = L_+ \cup L_-, L_\pm = \{y \in N \mid \pm h(y) > 0\}$.

Take a vector field v tangent to L tawarding to L_+ over N. Then $dh(v) \geq 0$.

Let W be the Hamiltonian vector field with Hamilton h. Then the imaginary part of $G(w,v)$ is equal to $\Omega(w,v) = (w \rfloor \Omega)(v) = -dh(v) \leq 0$.

Remark that normalized v (resp. w) turns into an orthomormal frame of $\mathbf{L}/\mathbf{K}^{\perp}$ (resp. $\mathbf{L}'/\mathbf{K}'^{\perp}$), Therefore, for $\Phi : V-\Sigma \longrightarrow \Lambda(1) = U(1)/O(1)$ in the definition of Maslov class, we see $\det\circ\Phi : V-\Sigma \longrightarrow S^1 \subset \mathbf{C}$ has non-positive imaginary part. Therefore $\det\circ\Phi$ is homotopically zero, and so is $\det^2\circ\Phi$. Thus

$$m(\mathbf{K}'/\mathbf{K}'^{\perp};\mathbf{L}/\mathbf{K}'^{\perp},\mathbf{L}'/\mathbf{K}'^{\perp}) = (\det^2\circ\Phi)^*1 = 0$$

in $H^1(V-\Sigma;\mathbf{Z})$ for any sufficiently small contractible neighborhood V of x in N. Q.E.D.

9. Local models for generic 1-dimensional reductions

We consider a generic local classification by symplectic equivalances of isotropic mapping arising from a symplectic reduction relatively to hypersurface (i.e. 1-dimentioanl reduction),(see Definition 5.2).

Let (M^{2n+2}, ω) be a symplectic manifold,K^{2n+1} be a hypersurface if M, and N^n be an n-dimensional manifold.

Denote by $\mathcal{I}$ the set of isotropic embeddings $i : N \longrightarrow M$ with $i(N) \subset K$, endowed with the Whitney C^∞ topology.

Next we prepare special isotropic map-germs $f_{n,k}$ as local modes for singularities of isotropic mappings. Consider the cotangent bundle $T^*\mathbf{R}^n$ with canonical coordinates $q_1, \dots, q_n; p_1 \dots, p_n$ and with the symplectic form $\omega = \sum_i dp_i \wedge dq_i$. Besides, consider the space $\mathbf{R}^n$ with coordinates $x_1, \dots, x_n$. Then

$$f_{n,k} : \mathbf{R}^n, 0 \longrightarrow T^*\mathbf{R}^n, (0 \leq k \leq [n/2])$$

is defined by

$$q_i \circ f_{n,k} = x_i, (1 \leq i \leq n-1),$$

$$u = q_n \circ f_{n,k} = \frac{x_n^{k+1}}{(k+1)!} + \sum_{i=1}^{k-1} x_i \frac{x_n^{k-i}}{(k-i)!},$$

$$v = p_n \circ f_{n,k} = \sum_{i=0}^{k-1} x_{k+i} \frac{x_n^{k-i}}{(k-i)!},$$

and

$$p_j \circ f_{n,k} = \int_0^{x_n} \left(\frac{\partial v}{\partial x_j} \frac{\partial u}{\partial x_n} - \frac{\partial v}{\partial x_n} \frac{\partial u}{\partial x_j} \right) dx_n, (l \leq j \leq n-1),$$

that is,

$$p_j = \begin{cases} -\sum_{i=0}^{k-1} x_{k+i} \frac{x_n^{2k-i-j}}{(k-i-1)!(k-j)!(2k-i-j)}, & (1 \leq j \leq k-1), \\ \frac{x_n^{3k-j+1}}{k!(2k-j)!(3k-j+1)} + \sum_{i=1}^{k-1} x_i \frac{x_n^{3k-i-j}}{(k-i-1)!(2k-j)!(3k-i-j)}, & (k \leq j \leq 2k-1), \\ 0, & (2k \leq j \leq n). \end{cases}$$

Remark that each $f_{n,k}$ is a polynomial mapping of kernel rank one and of very simple form.

Theorem 9.1. *There exists an open dense subset $\mathcal{G}$ in $\mathcal{I}$ such that, for each $i \in \mathcal{G}$ and for each $x \in N$, the isotropic map-germ $f : N^n, x \longrightarrow M'^{2n}$ arising from 1-dimensional reduction relatively to K is symplectically equivalent to some $f_{n,k}, (0 \leq k \leq [n/2])$.*

10. Maslov class of an open Whitney umbrella

Let us study properties of local models $f_nk : \mathbf{R}^n, o \longrightarrow T^*\mathbf{R}^n, (0 \leq k \leq [n/2])$, in §9,

For $k = 0$, $f_{n,0}$ is just the zero-section $\zeta_n : \mathbf{R}^n, 0 \longrightarrow T^*\mathbf{R}^n$ and is an immersion.

For $k \neq 0$, easily we verify $\Sigma = \Sigma(f_{n,k}) = \{\partial u/\partial x_n = \partial v/\partial x_n = 0\}$, is a submanifold of codimension 2 in $\mathbf{R}^n$. Thus we have

$$(*) : H^2(\mathbf{R}^n, \mathbf{R}^n - \Sigma, \mathbf{Z})_0 \cong H^1(\mathbf{R}^n - \Sigma, \mathbf{Z})_0 \cong \mathbf{Z}.$$

By defnition, we can write $f_{n,k} = f_{2k,k} \times \zeta_{n-2k}$. Then $f_{n,k}$ is a "suspension" of $f_{2k,k}$.

Definition 10.1. $f_{2n,n}$ is called the n-dimensional open Whitney umbrella.

Remark 10.2. $f_{4,2}$ is just the (2-dimensional) open Whitney umbrella inroduced by Givental'[G].

For Maslov classes, we have

Theorem 10.3. *Under the identification (*),*

$$m(f_{n,k}) = \begin{cases} 0, & k = 0, \\ \pm 2, & 0 < k \leq \left[\frac{n}{2}\right]. \end{cases}$$

Corollary 10.4. *For a generic, that is belonging to $\mathcal{G}$ in Theorem 9.1, isotropic submanifold $i : N^n \longrightarrow K^{2n+1} \subset M^{2n+2}$, if T_xN contains the characteristic direction of K at a point $x \in N$, then N is never representable as an intersection of any Lagrangian submanifold and K, as germ at x*

Proof: If N is an intersection of a Lagrangian submanifold and K, then the Maslov class of isotropic map-germ arising from reduction relatively to K necessarily vanishes, by Theorem 6.1.

By Theorem 9.1, that map-germ is symplectically equivalent to some $f_{n,k}, (k \neq 0)$. By Theorem 10.3, $m(f_{n,k}), (k \neq 0)$, does not vanish. Combined with the argument in §4, this leads a contradiction. Q.E.D.

11. Generating functions

Let $g = (g_1, \ldots, g_n) : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^n, 0$ be a map-germ, and $f : \mathbf{R}^n, 0 \longrightarrow T^*\mathbf{R}^n$ be an isotropic map-germ covering g, that is, $\pi \circ f = g$, where $\pi : T^*\mathbf{R}^n \longrightarrow \mathbf{R}^n$ is the projection. Set $\theta = \sum_{i=1}^n p_i dq_i$, (Louiville form on $T^*\mathbf{R}^n$). Then $f^*\theta$ is closed. Thus, by Pioncaré lemma, $de = f^*\theta$, for some $e \in E_n$, where E_n denotes the $\mathbf{R}$-algebra of function-germs of $\mathbf{R}^n, 0$. The function-germ (unique up to additive constant) e is called a generating function of f.

Denote by H_g the set of generating functions of isotropic map-germs covering g.

LEMMA 11.1. $H_g = \{e \in E_n \mid de \in \langle dg_1, \ldots, dg_n \rangle E_n\}$.

PROOF: Let $e \in H_g$. Then $de = f^*\theta$, for some isotropic $f : \mathbf{R}^n \longrightarrow T^*\mathbf{R}^n$. Then

$$de = \sum_{i=1}^{n} p_i \circ f dg_i.$$

Conversely, suppose $de = \sum_{i=1}^n a_i dg_i$, for some $a_i \in E_n$. Then define f by $p_i \circ f = a_i$ and $q_i \circ f = g_i, (1 \leq i \leq n)$. then f is isotropic and covers g.

Definition 11.2. Two isotropic map-germs f and $f' : \mathbf{R}^n, 0 \longrightarrow T^*\mathbf{R}^n$ are called Lagrangian equivalent if there exist a diffeomorphism-germ $\sigma : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^n, 0$ and a π-fiber-preserving symplectic diffeomorphism-germ $T : T^*\mathbf{R}^n, f(0) \longrightarrow T^*\mathbf{R}^n, f'(0)$ such that $f' \circ \sigma = T \circ f$.

Definition 11.3. Let g and $g' : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^n, 0$ be map-germs and e and $e' : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}$ be function-germs. Then two pairs (e, g) and (e', g') are called R^+-equivalent if there exist diffeomorphism-germs σ and $\tau : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^n, 0$ are function-germ $\alpha : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}$ such that

$$e = e' \circ \sigma + \alpha \circ g, \text{ and } g' \circ \sigma = \tau \circ g.$$

(See [AVG].)

PROPOSITION 11.4. *Let f and $f' : \mathbf{R}^n, 0 \longrightarrow T^*\mathbf{R}^n$ be isotropic map-germs covering g and $g' : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^n, 0$ with generating functions e and e' respectively. Then, if (e, g) and (e', g') are R^+-equivalent, then f and f' are Lagrangian equivalent.*

Furthermore, assume the critical point set-germ of g' is nowhere dense. Then, if f and f' are Lagrangian equivalent, then (e, g) and (e', g') are R^+-equivalent.

PROOF: First suppose f and f' are Lagrangian equivalent, and (σ, T) realizes this equivalence. Write T in the form

$$(p, q) \mapsto (P(p, q), \tau(q)).$$

Then $\sum_i P_i d\tau_i - \sum_i p_i dq_i = dA(p, q)$, for some $A : T^*\mathbf{R}^n, f(0) \longrightarrow \mathbf{R}$. Since $\partial A / \partial p_i = 0$, $(1 \leq i \leq n)$, we have $A = \pi^* \tilde{\alpha}$ for some $\tilde{\alpha} : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}$. Thus $T^*\theta = \theta + \pi^* d\tilde{\alpha}$.

Since $f' \circ \sigma = T \circ f$, we see

$$\begin{aligned} d(\sigma^* e') = \sigma^* f'^* \theta = f^* T^* \theta = f^*(\theta + \pi^* d\tilde{\alpha}) \\ = d(e + g^* \tilde{\alpha}), \end{aligned}$$

Thus $e = e \circ \sigma - \tilde{\alpha} \circ g + c$ for some $c \in \mathbf{R}$. then it suffices to set $\alpha = -\tilde{\alpha} + c$ to see (e, g) and (e', g') are R^+-equivalent.

Next suppose (e, g) and (e', g') are R^+-equivalent, and (σ, τ, α) realizes this equivalence. Define a fiber-preserving symplectic diffeomorphism-germ $T : T^*\mathbf{R}^n, T_0^*\mathbf{R}^n \longrightarrow$

$T^*\mathbf{R}^n, T_0^*\mathbf{R}^n$ covering τ by $T^*\theta - \theta = -d\pi^*\alpha$. Then,

$$\begin{aligned}(T \circ f \circ \sigma^{-1})^*\theta &= \sigma^{-1*} f^*(\theta - d\pi^*\alpha) = \sigma^{-1*}(de - dg^*\alpha)\\ &= \sigma^{-1*}(d\sigma^* e') = de'\\ &= f'^*\theta.\end{aligned}$$

Both $T \circ f \circ \sigma^{-1}$ and f' cover g'. Since the critical point set of g' is nowhere dense, as easily verfied, each map-germ covering g' is uniquely determined by the pull-back of θ. Hence $T \circ f \circ \sigma^{-1} = f'$. Q.E.D.

Now, assume g is of form $g = (x', u(x', x_n))$, where $x' = (x_1, \dots, x_{n-1})$.

Let $f : \mathbf{R}^n, 0 \longrightarrow T^*\mathbf{R}^n$ be an isotropic lifting of g. Set $v = p_n \circ f$. Then we have

LEMMA 11.5. *For a generating function e of f, there exists a function-germ $\alpha = \alpha(x')$ such that*

$$e(x', x_n) = \int_0^{x_n} v(x', t)\frac{\partial u}{\partial t}(x', t)dt + \alpha(x').$$

PROOF: Since $de = \sum_{i=1}^{n-1} p_i \circ f dx_i + v du$, we have $\partial e/\partial x_n = v(\partial u/\partial x_n)$. This leads the required result.

Given $v \in E_n$, with $v(x', 0) = 0$, define $f_v : \mathbf{R}^n, 0 \longrightarrow T^*\mathbf{R}^n$ by $\pi \circ f_v = g, p_n \circ f_v = v$ and

$$p_j \circ f = \int_0^{x_n} \left\{ \frac{\partial v}{\partial x_j}(x', t)\frac{\partial u}{\partial t}(x', t) - \frac{\partial v}{\partial t}(x', t)\frac{\partial u}{\partial x_j}(x', t)\right\} dt,$$

$(1 \le j \le n-1)$.

LEMMA 11.6. *Let $v \in E_n$ with $v(x', 0) = 0$. Then f_v is an isotropic lifting of g. Furthermore, any isotropic lifting f of g with $p_n \circ f = v$ is Lagrangian equivalent to f_v.*

PROOF: Set $e_v = \int_0^{x_n} v(x', t)(\partial u/\partial t)(x', t)dt$. Then we see $\partial e_v/\partial x_j = p_j \circ f_v + v(\partial u/\partial x_j)$. Thus we have $de_v = f^*\theta$. Hence we have the first half. By Proposition 11.4 and Lemma 11.5, we have the second half.

Example 11.7. Set $g = (x_1, \ldots, x_{n-1}, u(x', x_n)), u = x_n^{k+1}/(k+1)! + \sum_{i=1}^{k-1} x_i x^{k-i}/(k-i)!$. Then $f_{n,k} = f_v$, where $v = \sum_{i=0}^{k-1} x_{k+i} x_n^{k-i}/(k-i)!$. (See §9.)

12. A differential analysis

Consider the space $\mathbf{R}^n \times \mathbf{R}$ with coordinates $y_1, \ldots, y_n; t$. Write $\mathbf{R}^r = \{(y_1, \ldots, y_n) \in \mathbf{R}^n \mid y_{r+1} = \cdots = y_n = 0\}, (0 \leq r \leq n)$. Denote by $j_r : \mathbf{R}^{r-1} \times \mathbf{R} \longrightarrow \mathbf{R}^r \times \mathbf{R}$ the inclusion defined by $j_r(y_1, \ldots, y_{r-1}; t) = (y_1, \ldots, y_{r-1}, 0; t), (1 \leq r \leq n)$. Then, set $\phi_r = j_n \circ \cdots \circ j_{r+1} : \mathbf{R}^r \times \mathbf{R} \longrightarrow \mathbf{R}^n \times \mathbf{R}, (0 \leq r \leq n-1)$, and $\phi_n = \mathrm{id}_{\mathbf{R}^n \times \mathbf{R}}$.

Denote by E_{r+1} the $\mathbf{R}$-algebra of C^∞ function germs at $(0,0)$ in $\mathbf{R}^r \times \mathbf{R}$. Then E_{r+1} has unique maximal ideal m_{r+1} generated by $y_1, \ldots, y_r$ and t.

Now fix an element of E_{r+1},

$$P = t^k/k! + a_1(y)t^{k-1}/(k-1)! + \cdots + a_k(y),$$

which is a monic polynomial with respect to t, and assume each a_i is an analytic map-germ with $a_i(0) = 0, (1 \leq i \leq k)$.

Then, set

$$B_r = \{h \in E_{r+1} \mid \partial h/\partial t \text{ is a multiple of } \phi_r^* P \text{ in } E_{r+1}\}, (0 \leq r \leq n),$$

$$H_\ell(y,t) = \int_0^t \frac{s^\ell}{\ell!} P(y,s) ds \in E_{n+1}, (0 \leq \ell \leq k),$$

and

$$\Gamma_r = (y_1, \ldots, y_r, \phi_r^* H_0) : \mathbf{R}^r \times \mathbf{R} \longrightarrow \mathbf{R}^r \times \mathbf{R}, (0 \leq r \leq n).$$

PROPOSITION 12.1. *We have*

(a_r) $1, \phi_r^* H_1, \ldots, \phi_r^* H_k$ *generate* B_r *over* E_{r+1} *via* $\Gamma_r^* : E_{r+1} \longrightarrow E_{r+1}, (0 \leq r \leq n)$.

(b_r) Ker $j_{r+1}^* \subset \Gamma_{r+1}^* m_{r+1+1} B_{r+1}$, *where* $j_{r+1}^* : B_{r+1} \longrightarrow B_r$, $(0 \leq r \leq n-1)$.

(c_r) j_{r+1}^* *induces an isomorphism*

$$j : B_{r+1}/\Gamma_{r+1}^* m_{r+1+1} B_{r+1} \cong B_r/\Gamma_r^* m_{r+1} B_r,$$

of $\mathbf{R}$*-vector spaces,* $(0 \leq r \leq n-1)$.

PROOF: Step 1: (a_0).

It is easy to see that

$$B_0 = \{h(t) \mid \partial h/\partial t \text{ is a multiple of } t^k\}$$

is generated by $1, t^{k+2}, \ldots, t^{2k+1}$ over E_{0+1} via $(t^{k+1})^* : E_{0+1} \longrightarrow E_{0+1}$. Thus B_0 is generated by $1, \phi_0^* H_1, \ldots, \phi_0^* H_k$ over E_{0+1} via $\Gamma_0^* : E_{0+1} \longrightarrow E_{0+1}$. Therefore we have (a_0).

Step 2: (b_r), $(0 \leq r \leq n-1)$.

Let $h(y_1, \ldots, y_r, 0; t) = 0$. Then $h = y_{r+1} g(y_1, \ldots, y_{r+1}, t)$ for some $g \in E_{r+1+1}$ and $\partial h/\partial t = y_{r+1}(\partial g/\partial t)$. On the other hand, $\partial h/\partial t = w\phi_{r+1}^* P$ for some $w \in E_{r+1+1}$. Set $y_{r+1} = 0$. Then $w(y_1, \ldots, y_r, 0, t)\phi_r^* P = 0$. Since $\phi_r^* P$ is not flat, we see $w(y_1, \ldots, y_r, 0, t) = 0$. Thus $w = y_{r+1}\tilde{w}$ for some $\tilde{e} \in E_{r+1+1}$. Hence $y_{r+1}(\partial g/\partial t - \tilde{w}\phi_{r+1}^* P) = 0$. Therefore, $\partial g/\partial t = \tilde{w}\phi_{r+1}^* P$. This implies $g \in B_{r+1}$ and $h \in \Gamma_{r+1}^* m_{r+1+1} B_{r+1}$. We have (b_r), $(0 \leq r \leq n-1)$.

Step 3: $(\mathrm{a}_r)+(\mathrm{b}_r) \Longrightarrow (\mathrm{c}_r)$, $(0 \leq r \leq n-1)$.

For this, first we will show j_{r+1}^* is surjective. In fact, let $h \in B_r$. By (a_r),

$$h = \Gamma_r^* w_0 + \sum_{s=1}^{k} \Gamma_{r+1}^* w_s \phi_{r+1}^* H_s,$$

for some $w_0, w_1, \ldots, w_k \in m_{r+1}$. Set

$$\tilde{h} = \Gamma_{r+1}^* \tilde{w}_0 + \sum_{s=1}^{k} \Gamma_{r+1}^* \tilde{w}_s \phi_{r+1}^* H_s,$$

where $\tilde{w}_s \in m_{r+1+1}$ is defined by $\tilde{w}_s(y_1, \ldots, y_r, y_{r+1}; t) = w_s(y_1, \ldots, y_r; t)$. Then $j_{r+1}^* \tilde{h} = h$. Therefore j is surjective.

Next we will show j is injective. For this, let $h \in B_{r+1}$ with $j_{r+1}^* h \in \Gamma_r^* m_{r+1} B_r$. Then $j_{r+1}^* h = \sum_s \Gamma_r^* w_s \cdot u_s, (u_s \in B_r, w_s \in m_{r+1})$. By (a_r), there exists $\tilde{u}_s \in B_{r+1}$ such that $j_{r+1}^* \tilde{u}_s = u_s$. Set $\tilde{h} = \sum_s \Gamma_{r+1}^* \tilde{w}_s \tilde{u}_s \in \Gamma_{r+1}^* m_{r+1+1} B_{r+1}$. Then $j_{r+1}^* \tilde{h} = j_{r+1}^* h$. Thus

$j^*_{r+1}(h - \tilde{h}) = 0$, and $h - \tilde{h} \in \operatorname{Ker} j^*_{r+1}$. By (b$_r$), $h - \tilde{h} \in \Gamma^*_{r+1} m_{r+1+1} B_{r+1}$. Hence $h \in \Gamma^*_{r+1} m_{r+1+1} B_{r+1}$.

Step 4: (a$_r$)+(c$_r$)$\Longrightarrow$ (a$_{r+1}$), $(0 \leq r \leq n-1)$.

Remark that

$$j^*_{r+1}\phi^*_{r+1}H_s = (\phi_{r+1} \circ j_{r+1})^* H_s = \phi^*_r H_s, (1 \leq s \leq k).$$

By (a$_r$), $1, \phi^*_r H_1, \ldots, \phi^*_r H_k$ generate $B_r/\Gamma_r m_{r+1} B_r$ over $\mathbf{R}$. Thus $1, \phi^*_{r+1}H_1, \ldots, \phi^*_{r+1}H_k$ generate $B_{r+1}/\Gamma^*_{r+1} m_{r+1+1} B_{r+1}$ over $\mathbf{R}$, by (c$_r$).

By [I], we see B_{r+1} is a differentiable algebra in the sense of Malgrange. Then, we have (a$_{r+1}$) by Malgrange's preparation theorem ([M1]).

13. Generic condition for isotropic submanifolds

Let $\mathcal{I}$ be as in §9.

Define $\mathcal{G}_1 \subset \mathcal{I}$ by the following condition: $i \in \mathcal{G}_1$ if and only if, for any $x \in N$, there exist a chart $(V; x_1, \ldots, x_{n-1}, t)$ around x and a symplectic chart $(U; p_0, \ldots, p_n, q_0, \ldots, q_n)$ around $i(x)$ such that (†) $i(V) \subset U, K \cap U = \{p_0 = 0\}, \omega|U = \sum_{j=0}^n dp_j \wedge dq_j$ and

$$\begin{aligned} &q_j \circ i = x_j, \qquad 1 \leq j \leq n-1, \\ &q_n \circ i = u(x', t) = t^{k+1}/(k+1)! + \sum_{j=1}^{k-1} x_j t^{k-j}/(k-j)!, \\ &p_n \circ i = v(x', t), \end{aligned}$$

with $(\partial v/\partial t)(0,0) = \cdots = (\partial^k v/\partial t)(0,0) = 0$, for some k, $(0 \leq k \leq n)$, where $x' = (x_1, \ldots, x_{n-1})$.

It is easy to verify that $\mathcal{G}_1$ is open dense in $\mathcal{I}$.

Let $i \in \mathcal{G}_1$ and $x \in N$. Set $\theta = \sum_{j=1}^n p_j dq_j$ on U. Then $d\theta|K \cap U = \omega|K \cap U$. Thus $i^*\theta$ is a closed 1-form on V. Therefore, for some function-germ $e : N, x \longrightarrow \mathbf{R}$ (up to additive constant),

$$de = \sum_{i=1}^{n-1} p_j \circ i dx_i + v du.$$

Then $\partial e/\partial t = v(\partial u/\partial t)$. Thus e belongs to B_{n+1} introduced in §12, setting $P = \partial u/\partial t$, with respect to the coordinate (x, t). Remark that, in this situation, $H_0 = u$ and $\Gamma_n = (x_1, \ldots, x_{n-1}, u)$ is identified with $g = (q_1 \circ i, \ldots, q_n \circ i) : N, x \longrightarrow \mathbf{R}^n, 0$.

By (a_n) of Proposition 12.1, $e = a_0 \circ g + \sum_{j=1}^{k} a_j \circ gH_j$, for some function-germ $a_0, a_1, \ldots, a_k : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}$.

Define $\mathcal{G} \subset \mathcal{I}$ by the following condition: $i \in \mathcal{G}$ if and only if $i \in \mathcal{G}_1$ and, for each point x, there exist a chart $(U; x', t)$ around x and a symplectic chart (U, p, q) around $i(x)$ satisfying (†) and (††):

$$dq_1 \wedge \cdots \wedge dq_{k-1} \wedge da_1 \wedge \cdots \wedge da_k \wedge dq_n \neq 0,$$

at 0 in $\mathbf{R}^n$.

Remark 13.1. (1) The condition (††) is independent of the choice $a_0, a_1, \ldots, a_k$.

(2) The condition (††) is equivalent to that $a = (a_1, \ldots, a_k) : \mathbf{R}^n, 0 \to \mathbf{R}^k, 0$ ia a submersion and that $g(\Sigma^{1_k,0}(g)) = \{q_1 = \cdots = q_{k-1} = q_n = 0\}$ is transverse to $a^{-1}(0)$, where $\Sigma^{1_k,0} = \Sigma^{\overbrace{1,\ldots,1}^{k},0}$ is the Thom-Boardman symbol.

Lemma 13.2. *For any $x \in N$ and for any isotropic embedding $i : N \longrightarrow K$ belonging to $\mathcal{G}$, the isotropic map-germ f at x arising from symplectic reduction relatively to K is symplectically equivalent to $f_{n,k}$ for some $k, (0 \leq k \leq [n/2])$.*

Proof: Choose charts satisfying (†) and (††). Then define $\sigma : N, x \longrightarrow N, x$ by

$$x_i \circ \sigma = \begin{cases} x_i, & (1 \leq i \leq k-1, 2k \leq i \leq n), \\ a_{2k-i} \circ g, & (k \leq i \leq 2k-1). \end{cases}$$

Then $H_j \circ \sigma = H_j, (1 \leq j \leq k)$, and $g \circ \sigma = \tau \circ g$, for some diffeomorphism-germ $\tau : \mathbf{R}^n, 0 \longrightarrow \mathbf{R}^n, 0$.

Furthermore,

$$e = a_0 \circ g + \sum_{j=1}^{k} a_j \circ gH_j = a_0 \circ g + e' \circ \sigma,$$

where,

$$e' = \sum_{j=1}^{k} x_{2k-j} H_j = \int_0^t \{\sum_{j=1}^{k} x_{2k-j} s^j / j!\} \partial u / \partial s(x', s) ds$$

is a generating function of $f_{n,k}$. Thus (e, g) and (e', g) are R^+-equivalent, and f is Lagrangian, hence symplectically, equivalent to $f_{n,k}$ by Proposition 11.4.

14. Variety of singular isotropic jets

Let N be a manifold of dimension n and M be a symplectic manifold of dimension $2n$. In the k-jet bundle $J^k(N, M)$, we set

$$J_I^k(N, M) = \{j^k f(x) \in J^k(N, M) \mid f : N, x \longrightarrow M \text{ isotropic}\},$$

and

$$\widetilde{\Sigma} = \{j^1 f(x) \in J_I^1(N, M) \mid f : N, x \longrightarrow M \text{ is not an immersion}\}.$$

Further set

$$\widetilde{\Sigma}^i = \{j^1 f(x) \in J_I^1(N, M) \mid \text{ dim Ker } T_x f = i\}.$$

Then we have

$$\widetilde{\Sigma} = \bigcup_{i=1}^{n} \widetilde{\Sigma}^i.$$

Set

$$\bar{\Sigma}^j = \bigcup_{i=j}^{n} \widetilde{\Sigma}^i.$$

PROPOSITION 14.1. *The set $J_I^1(N, M) - \bar{\Sigma}^2$ of isotropic 1-jets with kernel dimension ≤ 2 is a submanifold of $J^1(N, M)$. Further, $\widetilde{\Sigma}^1$ is a submanifold of $J_I(N, M) - \bar{\Sigma}^2$ of codimension 2.*

Set $V = Hom_{\mathbf{R}}(\mathbf{R}^n, \mathbf{C}^n) \cong M_n(\mathbf{C})$.

Let $\langle \cdot, \cdot \rangle$ denote the standard Hermitian structure on $\mathbf{C}^n$. Define the symplectic structure $[\cdot, \cdot]$ on $\mathbf{C}^n$ by $[u, v] = \text{ Im} \langle u, v \rangle, u, v \in \mathbf{C}^n$. Let $X \subset V$ be the set of isotropic linear maps $\mathbf{R}^n \longrightarrow \mathbf{C}^n$, and $\Sigma^i \subset X$ be the set of isotropic linear maps $\mathbf{R}^n \longrightarrow \mathbf{C}^n$ with kernel dimension i. Set $S^j = \bigcup_{i=j}^n \Sigma^i$.

To prove Proposition 14.1, it is sufficient to show

LEMMA 14.2. *X is a real algebraic variety in V, with $Sing(X) \subset S^2$. Further, Σ^1 is a submanifold of codimension 2 in $X - S^2$.*

PROOF: Denote by $Alt(n)$ the set of skewsymmetric bilinear forms $a : \mathbf{R}^n \times \mathbf{R}^n \longrightarrow \mathbf{R}$ on $\mathbf{R}^n$ and by $Sp(n)$ the group of symplectic linear isomorphisms on $(\mathbf{C}^n, [\,,\,])$.

Set $G = GL(n, \mathbf{R}) \times Sp(n)$. Define G-actions on $V = \mathrm{Hom}_{\mathbf{R}}(\mathbf{R}^n, \mathbf{C}^n)$ and $Alt(n)$ by

$$(\sigma, \tau)\ell = \tau \circ \ell \circ \sigma^{-1},$$

$$(\sigma, \tau)a = a \circ (\sigma^{-1} \times \sigma^{-1}),$$

$((\sigma, \tau) \in G, \ell \in V, a \in Alt(n))$, respectively.

Consider the map $\rho : V \longrightarrow Alt(n)$ defined by $\rho(\ell)(u, v) = [\ell u, \ell v]$, $(\ell \in V, u, v \in \mathbf{R}^n)$. Then ρ is a G-equivariant polynomial map and $X = \rho^{-1}(O)$. Especially, X is a real analytic variety.

Let $\ell \in X$. Then $\mathrm{rank}(\ell) = i, (0 \leq i \leq n)$ if and only if there exists $g \in G$,

$$g \cdot \ell = \begin{pmatrix} E_i & O \\ O & O \end{pmatrix}.$$

In fact, if $\mathrm{rank}(\ell) = i$, then there exists $\tau \in U(n) \subset Sp(n)$, such that $\tau(\mathrm{image}\,\ell) = \mathbf{R}^i \times 0 \subset \mathbf{C}^n$. Thus, for some $\sigma \in GL(n, \mathbf{R})$, $\tau \circ \ell \circ \sigma^{-1} : \mathbf{R}^n \longrightarrow \mathbf{C}^n$ is the projection to $\mathbf{R}^i \times 0 \subset \mathbf{C}^n$. The converse is clear.

Remark that the matrix representation of ρ is

$$A + \sqrt{-1}B \mapsto {}^tBA - {}^tAB \in Alt(n), \quad A, B \in M_n(\mathbf{R}).$$

Let ℓ be isotropic. Then ρ is submersion at ℓ if and only if kernel $\mathrm{rank}(\ell) \leq 1$.

To see this, we may assume

$$\ell = \begin{pmatrix} E_i & O \\ O & O \end{pmatrix},$$

without loss of generarity. The tangent map of ρ at ℓ,

$$T_\ell(\rho) : T_\ell V \longrightarrow T_{\rho(\ell)} Alt(n)$$

is described by

$$A' + \sqrt{-1}B' \mapsto {}^t B'\ell - {}^t \ell B' = \begin{pmatrix} {}^t B_{11} - B_{11} & -B_{12} \\ {}^t B_{12} & O \end{pmatrix},$$

where

$$B' = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

B_{11}, B_{12}, B_{21} and B_{22} are real matrices of type $(i,i), (i, n-i), (n-i, i)$ and $(n-i, n-i)$ respectively.

Therefore $T_\ell(\rho)$ is surjective if and only if $i = n$ or $i = n-1$. Q.E.D.

15. Transversality

Let $i \in \mathcal{G}_1$ in the situation of §13 and $x \in N$. Take a chart (V, x', t) around x and a symplectic chart (U, p, q) around $i(x)$ satisfying (†). Define

$$f' : V \longrightarrow \mathbf{R}^{n+1}$$

by $f' = (q_1 \circ i, \ldots, q_n \circ i; p_n \circ i)$. Then we have

Proposition 15.1. *Let $i \in \mathcal{G}_1$ and $r > [n/2] + 1$. Then $i \in \mathcal{G}$ if and only if the r-jet extension $j^r f' : V \longrightarrow J^r(N, \mathbf{R}^{n+1})$ of f' is transverse to $\Sigma^{1_\ell, 0}$, $(0 \leq \ell < r)$, for all $x \in N$. (See [B],[Mat1].)*

Proof: By assumption, $j^r f'(x) \in \Sigma^{1_k, 0}$ for some k. Under the notion of §13, $e = a_0 \circ g + \sum_{j=1}^k a_j \circ gH_j$, and $\partial e/\partial t = v \partial u/\partial t$. Therefore, we have

$$p_n \circ i = v = (\partial a_0/\partial y_n) \circ g + \sum_{j=1}^{n} (\partial a_j/\partial y_n) \circ gH_j + \sum_{j=1}^{k} a_j \circ g t^j / j!.$$

Since $j^{k+1}H_j(x) = 0$, where x is the origine of coordinates, we see $j^{k+1}f'$ is transverse to $\Sigma^{1_k,0}$ at x if and only if $k \le [n/2]$ and $dq_1 \wedge \cdots \wedge dq_{k-1} \wedge da_1 \wedge \cdots \wedge da_k \wedge dq_n \ne 0$, (see [B],[Mat1]), and this is equivalent to that $j^{r+1}f'$ is transverse to $\Sigma^{1_\ell,0}$, $(0 \le \ell \le r)$, near x.

This is valid for all $x \in V$. Thus we have required result. Q.E.D.

16. Proof of Theorem 9.1

By Proposition 15.1, $\mathcal{G}$ defined in §13 is open dense in $\mathcal{I}$ endowed with the Whitney C^∞ topology. Futher, by Lemma 13.2, we see $\mathcal{G}$ satisfies the required property. Q.E.D.

17. Universal Maslov class

The calculation of Maslov classes of isotropic map-germs can be reduced to that in jet spaces.

Define $\Psi : J^1_I(N, M) - \widetilde{\Sigma} \longrightarrow \Lambda(M)$ by

$$\Psi(j^1 f(x)) = T_x f(T_x N) \subset T_{f(x)} M,$$

$(j^1 f(x) \in J^1_I(N, M) - \widetilde{\Sigma})$.

Remark that $\Psi \circ j^1 f = \varphi(f)$, (see §2).

Definition 17.1. The universal Maslov class of an isotropic 1-jet $z = j^1 f(x)$ is defined by

$$m(z) = \delta(m(\Psi^* \pi^* TM; \mathbf{L}, \Psi^* \mathcal{L})) \in H^2(J^1_I(N, M), J^1_I(N, M) - \widetilde{\Sigma}, \mathbf{Z})_z,$$

where $\mathbf{L}$ is a Lagrnagian subbundle of $\Psi^* \pi^* TM|U$ over a contractible neighborhood of z in $J^1_I(N, M)$ and $\mathcal{L} \subset \pi^* TM$ is the tautological Lagrangian subbundle over $\Lambda(M)$.

Lemma 17.2. *Let $f : N, x \longrightarrow M$ be an isotropic map-germ. Then $j^1 f : N, x \longrightarrow J^1_I(N, M)$ induces*

$$(j^1 f)^* : H^2(J^1_I(N, M), J^1_I(N, M) - \widetilde{\Sigma}, \mathbf{Z})_{j^1 f(x)} \longrightarrow H^2(N, N - \Sigma, \mathbf{Z})_x,$$

which maps $m(j^1f(x))$ to $m(f)$.

PROOF: We have

$$\begin{aligned}(j^1f)^*m(j^1f(x)) &= (j^1f)^*\delta m(\mathbf{L},\Psi^*\mathcal{L})\\ &= \delta m((j^1f)^*\mathbf{L},(\Psi\circ j^1f)^*\mathcal{L})\\ &= \delta m((j^1f)^*\mathbf{L},(\varphi(f))^*\mathcal{L})\\ &= \delta m((j^1f)^*\mathbf{L},\mathbf{L}_f)\\ &= m(f).\end{aligned}$$

18. Calculation of an universal Maslov class

PROPOSITION 18.1. *Let $z\in\widetilde{\Sigma}^1$. Then*

$$H^2(J^1_I(N,M),J^1_I(N,M)-\widetilde{\Sigma};\mathbf{Z})_z\cong\mathbf{Z},$$

and $m(z)=\pm 2$.

PROOF: The first half is clear from Proposition 14.1.

Without loss of generarity, we may assume $z=j^1f(0)\in J^1_I(\mathbf{R}^n,T^*\mathbf{R}^n)$ with $q_i\circ f=x_i,(1\le i\le n-1),q_n\circ f=0,p_i\circ f=0,(1\le i\le n)$. Define $c:\mathbf{R}^2\longrightarrow J^1_I(\mathbf{R}^n,T^*\mathbf{R}^n)$ by

$$c(t,s)=j^1(x_1,\ldots,x_{n-1},tx_n;0,\ldots,0,sx_n)(0),$$

$(t,s)\in\mathbf{R}^2$. Then $c(0)=z$ and c is transverse to $\widetilde{\Sigma}^1$.

Take a small loop $\ell_\varepsilon:S^1\longrightarrow J^1_I(\mathbf{R}^n,T^*\mathbf{R}^n)$, where $\ell_\varepsilon(e^{i\theta})=c(\varepsilon\cos\theta,\varepsilon\sin\theta)$. Then ℓ_ε is a generator of $H_1(J^1_I(\mathbf{R}^n,T^*\mathbf{R}^n)-\widetilde{\Sigma},\mathbf{Z})_z$. Thus $|m(z)|$ is determined by the evaluation to ℓ_ε. Remark that $\Psi\circ\ell_\varepsilon$ is represented by

$$e^{i\theta}\mapsto\begin{pmatrix}E_{n-1} & O\\ O & e^{i\theta}\end{pmatrix}\in U(n)$$

Thus $\det^2\circ\Psi\circ\ell_\varepsilon:S^1\longrightarrow S^1$ is of degree 2. Therefore, $|m(z)|=2$.

19. Proof of Theorem 10.3

LEMMA 19.1. *Let $f : N, x \longrightarrow M$ be isotropic. If $j^1 f(x) \in \widetilde{\Sigma}^1$ and $j^1 f$ is transverse to $\widetilde{\Sigma}^1$ in $J^1_I(N, M)$. Then*

$$H^2(N, N - \Sigma; \mathbf{Z})_x \cong \mathbf{Z},$$

and $m(f) = \pm 2$, where $\Sigma = \Sigma(f) = (j^1 f)^{-1}(\widetilde{\Sigma})$.

PROOF: Since $j^1 f$ is transverse to $\widetilde{\Sigma}^1$, and $\Sigma = (j^1 f)^{-1}(\widetilde{\Sigma}^1)$, we see Σ is a submanifold of codimension 2 in N near x, and

$$\begin{aligned}(j^1 f)^* : H^2(N, N - \Sigma; \mathbf{Z}) &\cong H^2(J^1_I(N, M), J^1_I(N, M) - \widetilde{\Sigma}; \mathbf{Z})_{j^1 f(x)} \\ &\cong \mathbf{Z}.\end{aligned}$$

Since $m(j^1 f(x)) = \pm 2$ by Proposition 18.1, we have

$$m(f) = (j^1 f)^* m(j^1 f(x)) = \pm 2,$$

relatively to the above isomorphism.

LEMMA 19.2. *$j^1 f_{n,k}, (k \neq 0)$, is transverse to $\widetilde{\Sigma}^1$ in $J^1_I(\mathbf{R}^n, T^*\mathbf{R}^n)$ at $j^1 f_{n,k}(0)$.*

PROOF: It is clear, by checking 2-jets.

Now, Theorem 10.3 is clear from Lemmata 19.1 and 2.

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