数理解析研究所講究録
第 725 巻 1990 年 81-89

STUDYING MANIFOLDS OVER SIMPLE DISCRIMINANTS

Mahito Kovayashi
Doctor Course, Department Of Math.
Tokyo Institute Of Technology

Motif.

The motif of this article is the study of C^∞ manifold by means of stable mappings.

For example, let $f : F \to M \to P$ be a fibration, we know $\chi(M) = \chi(F)\chi(P)$ and the monodromy or holonomy of f tells us some more fine properties about M. Let $f : M \to \mathbf{R}$ be a Morse function, then we have the Morse equality and it is usual in topology to show something using handlebody structure derived from Morse functions.

Here we assume the manifold M is simply connected and four-dimensional and the mapping $f : M \to \mathbf{R}^2$ is stable, mainly by the following reasons. First, if the target manifold is of high-dimension, then complicated singularities appear. Second, we want to do concrete argument, thus the trivial target is suitable and the difference of the source and the target dimension has to be small. Third, the differential topology of four dimensional manifolds is still interesting.

An expression of manifold.

Kushner-Levine-Porto [3] introduced the next space.

Definition.

For x, y in M, we define the relation $x \sim y$ as follows: $x \sim y$ if $f(x) = f(y)$ $(= a)$ and they are in the same connected component of $f^{-1}(a)$. We call the quotient space of M by this relation, as the quotient space associated to f.

We use the notations:

$$q_f : M \to W_f = M \backslash \sim .$$

We regard that the diffeomorphism class of the pair $D_f = (W_f, q(S(f)))$ gives an expression of M, and we aim at studying the source mainfold by means of these expressions.

Our program of this study is:

1 . Detect simple, in some sense, expressions of M;

2 . Derive fine properties on M, from these simple expressions.

3. A result.

On the first part of our program, the author got a result, restricting the source manifolds to a certain family of simply connected four manifolds, which is denoted by $\mathcal{M}_1$ (see [2], for the definition). That asserts, for a manifold in $\mathcal{M}_1$, one can show the followings:

1 . The existence of, in some sense, simple expressions which we call irreducible ones;

2 . The finiteness of the irreducible expressions;

3 . An inequality on the number of components of $S(f) = \coprod S^1$, which suggest the growth of the number of these expressions according to the growth of the Euler characteristic.

Precisely, we can show the theorem ([2]).

THEOREM.

a) For each Euler number constant family in $\mathcal{M}_1$, the diffeomorphism types of $D_f = (W_f, qS(f))$ of irreducible mappings are finite.

b) For an irreducible mapping $f \in W(M, \mathbf{R}^2)$, we have:

$$\sharp S(f) \leq \begin{cases} \frac{3}{2}\, b_2(M) + 1 & (\text{if } b_2(M) \text{ is even}) \\ \frac{3}{2}\,(b_2(M) + 1) & (\text{if } b_2(M) \text{ is odd}), \end{cases}$$

where $b_2(M)$ is the second Betti number of M and $\sharp S(f)$ is the number of connected component of $S(f) = \coprod S^1$ (disjoint union).

What can we derive from simple expressions?

Now we concern with the second part of the program. That is, what informations can we derive from simple expressions. I will show some examples.

Example I.

If $D_f = (D^2, \partial D^2)$, then the source manifold M_f is diffeomorphic to S^4. This fact is contained in the results of Furuya-Porto [1].

Example II.([2])

Suppose that D_f is such one as drawn in figure 1.

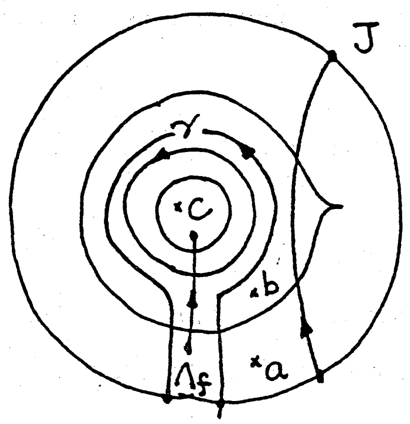


figure 1.

First, we know from the local properties of folds, the pull back image of regular values a,b taken as in figure 1, is diffeomorphic to S^2, T^2, respectively. That is, they are of genus 0 or 1, respectively (see [5] or [proposition 2.2 of 2]). That of c has to be 0 or 2. But 2 is no match for the assumption $\pi_1(M) = 1$. This means that M_f is in $\mathcal{M}_1$ and the theorem says that this is the (unique) simplest expression of M_f.

Let's observe this expression more precisely.

Take arcs $\Lambda_f, J \cong [-1, 1]$ which are 'transverse' to the discriminant, and a closed curve γ, as in figure 1.

1. Set $q^{-1}(\Lambda_f) = \tilde{\Lambda}_f$, then $q|\tilde{\Lambda}_f$ is a Morese function (see [5] or [proposition 2.2 of 2]). Thus,

$\tilde{\Lambda}_f =$ (0-handle) $\cup$ (1-handle) $\cup$ (2-handle)

$= \mathbf{T}_1$ (solid torus) $\underset{\varphi}{\cup}$ $\mathbf{T}_2$ (solid torus) $\backslash D^3$

$= L(p,q)$ (lens space) $\backslash D^3$,

where φ is the diffeomorphism from $\partial\mathbf{T}_2$ to $\partial\mathbf{T}_1$.

We denote the isotopy class of φ by

$$[\varphi] = \begin{pmatrix} s & p \\ t & q \end{pmatrix} = A \in SL(2,\mathbf{Z}) : H_1(\partial\mathbf{T}_2,\mathbf{Z}) \to H_1(\partial\mathbf{T}_1,\mathbf{Z}).$$

2. Note that q^{-1}(interior of ) is a torus bundle over an annulus and the holonomy induced by γ is of the form (see [proposition 3.6 of 2]):

$$\Gamma = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}, a \in \mathbf{Z} : H_1(\partial\mathbf{T}_2,\mathbf{Z}) \to H_1(\partial\mathbf{T}_2,\mathbf{Z}).$$

3. By the same argument as in 1, $q^{-1}(J)$ is obtained by gluing two solid tori by a diffeomorphism on its boundary. That is, $q^{-1}(J) = \mathbf{T} \underset{\psi}{\cup} \mathbf{T}$, for some ψ.

As it is diffeomorphic to $q^{-1}(J')$, where J' is an arc taken as in figure 1,

$$[\psi] = A \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} A^{-1} = \begin{pmatrix} 1+apq & -ap^2 \\ aq^2 & 1-apq \end{pmatrix}.$$

Thus $q^{-1}(J) \cong L(-ap^2, 1-apq)$.

4. From the local properties of cusps, $q^{-1}(J)$ is diffeomorphic to S^3 (see [5] or [proposition 2.1 of 2]). This means $-ap^2 = \pm 1$, hence,

$$[\varphi] = \begin{pmatrix} s & \pm 1 \\ t & q \end{pmatrix}.$$

In other words, the 1- and the 2-handle of $\tilde{\Lambda}_f$ is a cancelling pair. Thus we can 'reduce' f to a stable mapping g which has the discriminant as in figure 2 (see [2],for the reduction). We will observe the new expression D_f.

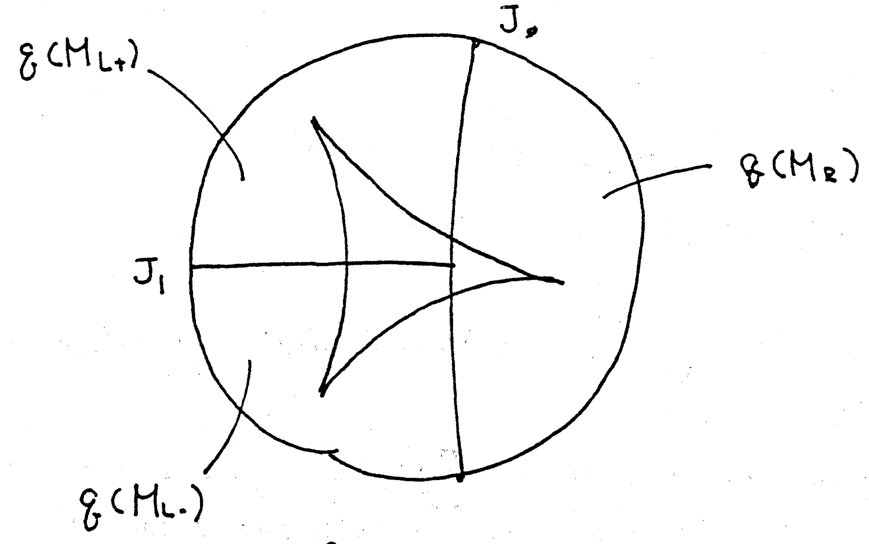


figure 2.

Take arcs $J_0, J_1 \cong [-1,1]$ as in figure 2. We denote the source manifold $M_f = M_g$ by M, and cut M along the arcs J_0 and J_1. That is,

$$M = M_L \cup M_R, \quad M_L = M_{L+} \cup M_{L-}.$$

5. Then by a technique of Levine [4], it is shown that these three peaces are diffeomorphic to D^4. Noticing that $q^{-1}(J_1)$ is a solid torus and $[\psi]$ is of the form

$$[\psi] = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$$

by the same argument as in 1,2, we can show that M_L is a D^2 bundle over S^2, which we denote by B_c.

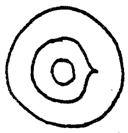
6. The boundary of B_c, that is, $q^{-1}(J_0)$ is diffeomorphic to S^3, by the same reason as in 4. It is known that ∂B_c is diffeomorphic to S^3 if and only if $c = \pm 1$.

Therefore,

$$M = B_{\pm 1} \cup (4 - ball) \cong \mathbf{C}^2P \text{ (or } \overline{\mathbf{C}^2P}).$$

Now we get the fact.

Fact.

If $D_f =$ , then M is diffeomprphic to $\mathbf{C}^2P$.

Together with the theorem, we have:

Fact.

If $b_2(M)$ is 1 and M is in $\mathcal{M}_1$, then M is diffeomorphic to $\mathbf{C}^2P$.

Example III.

If D_f is such as given in figure 3, like a pig nose. Then by the same argument as in Example II, the source manifold M_f is in $\mathcal{M}_1$ and the theorem says this is the (unique) simplest expression of the source manifold.

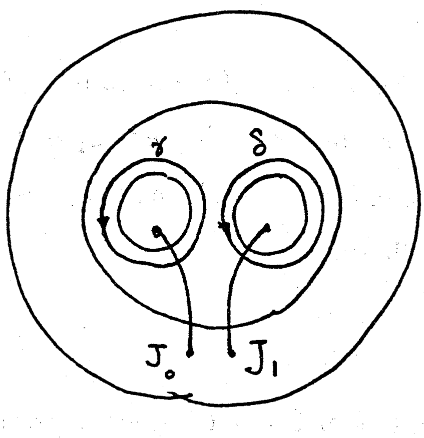


figure 3.

Let $J_0, J_1 \cong [-1, 1]$ be closed arcs that are 'transverse' to the discriminant, γ, δ be closed curves, taken as in figure 3. Then M_f is determined by the following data:

1 . The gluing data in $q^{-1}(J_i), i = 0, 1$, which are represented by two matrices A, B in $SL(2, \mathbf{Z})$;

2 . The holonomy data of the torus bundle q^{-1}() induced by γ, δ, which are determined by the two integers.

Using Levine's theorem in [4], we can know the homology of M_f. That is, $b_2(M_f) = 2$. Hence, from the theorem of Freedman,

$$M_f \approx \mathbf{C}^2P\sharp\mathbf{C}^2P \text{ or } \mathbf{C}^2P\sharp\overline{\mathbf{C}^2P} \text{ or } S^2 \times S^2.$$

Conversely, these three have this expression. Of course, as we see in Example II, these data are possibly dependent, but the problem is natural and makes sense.

Problem.

1 . Determine the homeomorphism type of M which shares this expression, by using these data.

2 . Find a diffeomorphism invariant of M.

Concluding assertion.

As we mentioned before, the author defined a family of simply connected four manifolds ([2]), which is denoted by $\mathcal{M}_1$. For example, the manifolds which have the expression appeared in the examples are in $\mathcal{M}_1$. Hence the problem stated in Example III is generalized as follows.

PROBLEM.

Do the concrete (and elementary I hope,) argument on M in $\mathcal{M}_1$ which have the "simple" expression and study the homeomorphism type and smooth structures of M.

Referrences

[1] Y. Furuya, P. Porto, Some remarks on generic maps from a closed manifold into the plane, preprint.

[2] M. Kovayashi, Simply connected 4-manifolds with simple sable mappings, preprint.

[3] L. Kushner, H. Levine, P. Porto, Mapping three manifolds into the plane I, Bol. Soc. Mat. Mex. **29**-1 (1984), 11–33.

[4] H. Levine, Mappings of manifolds into the plane, Amer. J. Math. **88** (1966), 357–365.

[5] H. Levine, *Classifying immersions into* $\mathbf{R}^4$ *over stable maps of 3-manifolds into* $\mathbf{R}^2$, Lec. Notes in Math **1157**, Springer ,1985