

On the regularity of a germ of analytic mapping

J. Cl. TOUGERON

Let  $(X, o)$  be a germ of analytic space (reduced and of pure dimension  $n$ ) at the origin of  $\mathbb{C}^N$ ; let  $F: (\mathbb{C}^N, o) \rightarrow (\mathbb{C}^p, o)$  a germ of analytic mapping and  $f = F|_X$  the restriction of  $F$  to  $X$ . We denote  $\text{sing } F$  the singular set of  $F$ , i.e. the germ of points  $x \in \mathbb{C}^N$  such that  $dF(x)$  has a rank  $< r(F)$ ,  $r(F)$  meaning the generic rank of  $F$ . Many results on  $F$  or  $f$  are true and well known when  $\text{sing } F = \emptyset$  or when  $F$  is flat. In this paper, we give examples where these results can be extended with an hypothesis on the codimension of  $\text{sing } F$ .

1) If the rank of  $F$  is constant ( $= r$ ),  $F$  admits a factorisation  $(\mathbb{C}^N, o) \xrightarrow{h} (\mathbb{C}^r, o) \xrightarrow{g} (\mathbb{C}^p, o)$ , where  $h$  is a submersion and  $g$  an immersion. In the general situation, we associate to  $F$  a differential form  $\Omega_F$  of degree  $r$ ; if the codimension of  $\text{sing } \Omega_F$  in  $\mathbb{C}^N$  is  $\geq 3$  and if  $\Omega_F$  is decomposable, there exists a factorisation by a generic submersion and a generic immersion. If  $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 2$  and if there exists a formal factorisation  $F = \bar{g} \circ \bar{h}$ , then there exists an analytic factorisation which approximates the formal one. These results are an easy consequence of Malgrange's Frobenius theorem.

2) If  $s$  is the generic rank of  $f$ , there does not exist in general a factorisation  $(X, o) \rightarrow (Y, o) \xrightarrow{i} (\mathbb{C}^p, o)$ , where  $(Y, o)$  is an analytic germ, reduced and of pure dimension  $s$  at the origin of  $\mathbb{C}^p$  and  $i$  is the canonical injection. Nevertheless, this is true if  $F$  is a flat morphism and if  $\text{codim}_{\mathbb{C}^N} X = \text{codim}_{\mathbb{C}^p} f(X)$ . We prove analogous results when  $(X, o)$  is a complete intersection, an hypothesis about the codimension of  $\text{sing } F$  taking the place of the flatness.

3) At last, let  $y = (y_1, \dots, y_p)$  (resp.  $x = (x_1, \dots, x_N)$ ) a system of coordinates at the origin of  $\mathbb{C}^p$  (resp.  $\mathbb{C}^N$ ) and let  $\bar{N}$  a sub-modulus of  $\mathbb{C}[[y]]^q$ . Let us suppose that  $(\bar{N} \circ F) \mathbb{C}[[x]]$  is generated on  $\mathbb{C}[[x]]$  by elements of  $\mathbb{C}\langle x \rangle^q$  ( $\mathbb{C}\langle x \rangle$  is the ring of convergent series in  $x$ ); then, if  $F$  is flat,  $\bar{N}$  is also analytic, i.e. is generated on  $\mathbb{C}[[y]]$  by convergent series. The same is true when hypothesis about the codimension of  $\text{sing } F$  take the place of the flatness.

### 1 - A factorisation theorem.

Let  $r = r(F)$  be the generic rank of  $F : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^p, 0)$  and let  $\Lambda^r(x)$  be the free modulus on  $\mathbb{C}\{x\}$  composed with germs at  $0 \in \mathbb{C}^N$  of holomorphic differential forms of degree  $r$ .

**Lemma 1.1 :** *There exists a differential form  $\Omega_F \in \Lambda^r(x)$ ,  $r = r(F)$ , unic modulo multiplication by invertible elements of  $\mathbb{C}\{x\}$ , such that :*

(1)  $\text{codim}_{\mathbb{C}^N} \text{sing } \Omega_F \geq 2$ .

(2)  $\forall I = (i_1, \dots, i_r)$ ,  $1 \leq i_1 < \dots < i_r \leq p$ , there exists  $\theta_I \in \mathbb{C}\{x\}$  such that  $dF_I = \theta_I \cdot \Omega_F$ . ( $dF_I = dF_{i_1} \wedge \dots \wedge dF_{i_r}$  and  $\text{sing } \Omega_F = \{x; \Omega_F(x) = 0\}$ ).

**Proof :** For every  $I$  such that  $dF_I \neq 0$ , we can write  $dF_I = \theta'_I \cdot \Omega_I$  where  $\theta'_I \in \mathbb{C}\{x\}$  and  $\Omega_I$  is a form such that  $\text{codim}_{\mathbb{C}^N} \text{sing } \Omega_I \geq 2$ . Let  $I, J$  be such that  $dF_I \neq 0, dF_J \neq 0$ ; the generic rank of  $F$  being  $r$ , we have  $\Omega_J = \alpha \cdot \Omega_I$  with  $\alpha$  meromorphic at the origin of  $\mathbb{C}^N$ ; but  $\alpha$  is holomorphic in  $\mathbb{C}^N \setminus \text{sing } \Omega_I$ , so  $\alpha \in \mathbb{C}\{x\}$ . Permuting  $I$  and  $J$ , we see that  $\alpha$  is invertible and the lemma follows.

Let  $\Theta_F$  be the ideal generated by all the  $\theta_I$  in  $\mathbb{C}\{x\}$  and let us denote  $V(\Theta_F)$  the germ of zeros of  $\Theta_F$ ; obviously :

$$\text{sing } F = V(\Theta_F) \cup \text{sing } \Omega_F.$$

If  $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 2$  and if  $F = g \circ h$ , where  $h : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^r, 0)$  is a generic submersion and  $g : (\mathbb{C}^r, 0) \rightarrow (\mathbb{C}^p, 0)$  is a generic immersion, then we may choose  $\Omega_F = d h_1 \wedge \dots \wedge d h_r$ ;  $\Theta_F$  is then the ideal of  $\mathbb{C}\{x\}$  generated by all the determinants of order  $r$  of the matrix  $(dg) \circ h$ .

Our result is a corollary of the singular Frobenius's theorem :

**Theorem 1.2** (Malgrange, [3]) : Let  $\omega_1, \dots, \omega_r$  be in  $\Lambda^1(x)$  and let us put  $\Omega = \omega_1 \wedge \dots \wedge \omega_r$ . We suppose that for  $i = 1, \dots, r$ ,  $d\omega_i \wedge \Omega = 0$ . Then :

(1) If  $\text{codim}_{\mathbb{C}^N} \text{sing } \Omega \geq 3$ , the system  $\{\omega_1, \dots, \omega_r\}$  is integrable, i.e. there exist  $f_1, \dots, f_r \in \mathbb{C}(x)$  such that :

$$(\omega_1, \dots, \omega_r) \cdot \mathbb{C}(x) = (df_1, \dots, df_r) \cdot \mathbb{C}(x).$$

(2) If  $\text{codim}_{\mathbb{C}^N} \text{sing } \Omega \geq 2$  and if the system  $\{\omega_1, \dots, \omega_r\}$  is formally integrable (i.e. there are formal series  $\bar{f}_1, \dots, \bar{f}_r \in \mathbb{C}[[x]]$  such that  $(\omega_1, \dots, \omega_r) \cdot \mathbb{C}[[x]] = (d\bar{f}_1, \dots, d\bar{f}_r) \cdot \mathbb{C}[[x]]$ ), then the system is integrable.

We use also the following result (cf [3] or [4]) :

**Lemma 1.3** : Let  $h : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^r, 0)$  be a germ of holomorphic mapping such that  $r = r(h)$  and such that  $\text{codim}_{\mathbb{C}^N} \text{sing } h \geq 2$ . Then, if  $f : (\mathbb{C}^N, 0) \rightarrow \mathbb{C}$  verifies  $df \wedge dh_1 \wedge \dots \wedge dh_r = 0$ , we have  $f = g \circ h$ , with  $g : (\mathbb{C}^r, 0) \rightarrow \mathbb{C}$  analytic.

**Proposition 1.4** :

(1) Let us suppose that  $\text{codim}_{\mathbb{C}^N} \text{sing } \Omega_F \geq 3$  and let us suppose that  $\Omega_F$  is decomposable, i.e.  $\Omega_F = \omega_1 \wedge \dots \wedge \omega_r$ , with  $\omega_i \in \Lambda^1(x)$ . Then, there exists a factorisation  $F = g \circ h$ , where  $h : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^r, 0)$  and  $g : (\mathbb{C}^r, 0) \rightarrow (\mathbb{C}^p, 0)$  are analytic.

(2) Conversely, if  $F$  admits such a factorisation and if  $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 2$ ,  $\Omega_F$  is decomposable.

**Proof** : The system  $(\omega_1, \dots, \omega_r)$  is locally integrable in  $\mathbb{C}^N \setminus V(\Theta_F)$ , because  $dF_I = \theta_I \cdot \Omega_F$  and so  $d\omega_i \wedge \omega_1 \wedge \dots \wedge \omega_r = 0$  for  $i = 1, \dots, r$ . By theorem 1.2, we may suppose that  $\omega_i = dh_i$ ,  $i = 1, \dots, r$ , with  $h_i \in \mathbb{C}(x)$ ,  $h_i(0) = 0$ . At last, for every  $g = 1, \dots, p$ ,  $dF_j \wedge dh_1 \wedge \dots \wedge dh_r = 0$  and so  $F_j = g_j(h_1, \dots, h_r)$  with  $g_j$  analytic, by lemma 1.3. The converse (2) is obvious.

**Proposition 1.5 :** *Let us suppose that  $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 2$  and let us suppose that  $F$  admits a formal factorisation  $F = \bar{g} \circ \bar{h}$  ( $\bar{h} : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^r, 0)$  and  $\bar{g} : (\mathbb{C}^r, 0) \rightarrow (\mathbb{C}^p, 0)$ ). Then  $F$  admits an analytic factorisation  $F = g \circ h$  and we may choose  $g$  and  $h$  as closely as we wish to  $\bar{g}$  and  $\bar{h}$ .*

**Proof :** From the hypothesis,  $\Omega_F$  admits a formal decomposition :

$\Omega_F = \bar{\lambda} \cdot d\bar{h}_1 \wedge \dots \wedge d\bar{h}_r$ , with  $\bar{\lambda} \in \mathbb{C}[[x]]$  and  $\bar{\lambda}(0) \neq 0$ . By Artin's approximation theorem [1],  $\Omega_F$  is decomposable, i.e.  $\Omega_F = \omega_1 \wedge \dots \wedge \omega_r$  with  $\omega_i \in \Lambda^1(x)$  and the system  $\{\omega_1, \dots, \omega_r\}$  is formally integrable. By the part (2) of theorem 1.2, the system is integrable and we conclude as in the proof of 1.4.

**Proposition 1.6 :** *Let  $\underline{F} : \mathbb{C}^N \supset U \rightarrow \mathbb{C}^p$  be an holomorphic mapping with generic rank  $r$ ; we suppose that the set of singular points of  $\underline{F}$  has codimension  $\geq 3$ . Then, the set  $\Gamma$  of points  $x \in U$  such that the germ  $\underline{F}_x : (U, x) \rightarrow (\mathbb{C}^p, \underline{F}(x))$  is factorisable in the sense of 1.4, is the compliment of a closed analytic subset of  $U$ .*

**Proof :** The result being of local nature, we may suppose that there exists  $\underline{\Omega} \in \Lambda^r(U)$  such that  $\forall x \in U$ , the germ  $\underline{\Omega}_x$  induced by  $\underline{\Omega}$  in  $x$ , is a differential form  $\Omega_{\underline{F}_x}$ . By 1.4, the point  $x$  belongs to  $\Gamma$  if and only if the equation :  $\underline{\Omega}_x = \omega_1 \wedge \dots \wedge \omega_r$  admits an holomorphic solution. The proposition results from a general theorem about the solutions of a system of analytic equations depending analytically of a parameter (cf [6]).

**Remark 1.7 :** Let us suppose that  $F : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^p, 0)$  admits a factorisation by  $(\mathbb{C}^r, 0)$ , with  $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 2$ . Then this factorisation is unic, in the following sense : if  $F = g \circ h$ ,  $F = g' \circ h'$  are two factorisations, there is a unic analytic

difféomorphism  $\tau : (\mathbb{C}^r, 0) \xrightarrow{\sim} (\mathbb{C}^r, 0)$  such that the following diagram is commutative :

$$\begin{array}{ccccc}
 & & (\mathbb{C}^r, 0) & & \\
 & \nearrow h & \downarrow \lambda_\tau & \nwarrow g & \\
 (\mathbb{C}^N, 0) & & & & (\mathbb{C}^p, 0) \\
 & \searrow h' & & \nearrow g' & \\
 & & (\mathbb{C}^r, 0) & & 
 \end{array}$$

### 1.8. Special Cases

(1.8.1.) Let us suppose that  $V(\Theta_F) = \emptyset$  ; for instance, let us suppose that  $\theta_{(1, \dots, r)}(0) = 0$ . Then we may choose  $\Omega_F = dF_1 \wedge \dots \wedge dF_r$  and if  $j > r$ , we get  $F_j = g_j$  ( $F_1, \dots, F_r$ ), with  $g_j$  analytic. So  $F = g \circ h$ , where  $g$  is the immersion  $\mathbb{C}^r \ni (z_1, \dots, z_r) \rightarrow (z_1, \dots, z_r ; g_{r+1}(z), \dots, g_p(z))$ . The converse is obvious and we get an equivalence :

$(V(\Theta_F) = \emptyset) \Leftrightarrow \text{codim}_{\mathbb{C}^N} \text{sing } F \geq 2$  and there exists a factorisation  $F = g \circ h$ , where  $g : (\mathbb{C}^r, 0) \rightarrow (\mathbb{C}^p, 0)$  is an immersion.

(1.8.2.) Let us suppose that  $\text{sing } \Omega_F = \emptyset$  ; the form  $\Omega_F$  is generically decomposable and non singular and so, by remark 1.9, it is decomposable, and we may apply 1.4. We get that  $F = g \circ h$  where  $h$  is a submersion and the converse is obvious :

$(\text{sing } \Omega_F = \emptyset) \Leftrightarrow$  There exists a factorisation  $F = g \circ h$  where  $h : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^r, 0)$  is a submersion.

(1.8.3.) Let us suppose that the rank of  $F$  at 0 is  $r-1$ . Then  $\Omega_F$  is decomposable ; indeed, with a convenient choice of coordinates, we may suppose that  $F_1 = x_1, \dots, F_{r-1} = x_{r-1}$  and so  $\Omega_F = dx_1 \wedge \dots \wedge dx_{r-1} \wedge \omega$ , and we may apply 1.4.

1.9 A decomposable form must verify obvious conditions. Let  $E$  be a vector space of dimension  $N$  on  $\mathbb{C}$  and let  $e_1, \dots, e_N$  be a basis of  $E$ . Let us consider the mapping :

$$\mathbb{C}^{Nr} \simeq E^r \ni (\omega_1, \dots, \omega_r) \rightarrow \Omega = \omega_1 \wedge \dots \wedge \omega_r \in \Lambda^r E \simeq \mathbb{C}^{\binom{N}{r}}.$$

Obviously,  $\omega_1 \wedge \dots \wedge \omega_r = \omega'_1 \wedge \dots \wedge \omega'_r$  if and only if there exists a matrix  $M \in GL(N, \mathbb{C})$  with determinant 1 such that

$$(\omega_1, \dots, \omega_r) M = (\omega'_1, \dots, \omega'_r).$$

Outside  $\phi^{-1}(0)$  the mapping  $\phi$  is a fibering with fiber of dimension  $r^2-1$  and  $\phi^{-1}(0)$  is the set of matrices  $(\omega_1, \dots, \omega_r)$  with rank  $< r$  and so  $\phi^{-1}(0)$  is an algebraic variety in  $E^r$  of codimension  $N-r+1$ . The image of  $\phi$  is an algebraic variety in  $\Lambda^r E$ , of dimension  $Nr-r^2+1$ , with an isolated singularity at the origin.

If  $\Sigma \theta_I e_I$  ( $e_I = e_{i_1} \wedge \dots \wedge e_{i_r}$ ) is the generic point of  $\Lambda^r E$  and if  $U_I$  is the open set  $\theta_I \neq 0$ , then  $\text{Im } \phi \cap U_I$  is regular and is the transverse intersection of  $\binom{N}{r} - (Nr-r^2+1)$  algebraic hypersurfaces  $F_{I,\alpha} = 0$ , where  $F_{I,\alpha}$  is homogeneous of degree  $r$  (if  $r=2$ ,  $\text{Im } \phi = \{\Omega; \Omega \wedge \Omega = 0\}$ ).

So, a decomposable differential form  $\Omega$  must verify  $\binom{N}{r} - (Nr-r^2+1)$  independent conditions. Conversely, if these conditions are full filled and if  $\Omega$  is non singular, then  $\Omega$  is decomposable. If  $\Omega$  is singular and decomposable ( $\Omega = \omega_1 \wedge \dots \wedge \omega_r$ ), every irreducible component of  $\text{sing } \Omega = \{x; \text{rank } (\omega_1 \wedge \dots \wedge \omega_r) < r\}$  has codimension  $\leq N-r+1$ .

Another condition is the following one. Let  $v(\Omega) = \inf_I v(\theta_I)$  be the infimum of the multiplicities at the origin of the  $\theta_I$  ( $\Omega = \sum_I \theta_I dx_I$ ) and let  $\mathfrak{J}(\Omega)$  be the ideal generated in  $\mathbb{C}\{x\}$  by  $\theta_I$ 's. If  $v(\Omega) = s < r$  and if  $\Omega = \omega_1 \wedge \dots \wedge \omega_r$ , then  $r-s$  forms  $\omega_i$  are linearly independent at the origin and by choosing suitable coordinates :

$$\Omega = \left( \sum_I \theta_I^* dx_I \right) \wedge dx_{N-(r-s)+1} \wedge \dots \wedge dx_N$$

with  $I = (i_1, \dots, i_s)$ ,  $1 \leq i_1 < \dots < i_s \leq N-(r-s)$ ; so the minimal number of generators of  $\mathcal{F}(\Omega)$  is  $\leq \binom{N-(r-s)}{s}$ .

For instance, if  $N = 3$ ,  $r = 2$ , the conditions  $F_{I,\alpha}(\Omega) = 0$  are vacant. A form  $\Omega$  with an isolated singularity at the origin is not decomposable; the form  $\Omega = xy dx dy + y^2 dy dz + z dx dz$ , with the  $x$ -axis as a line of singularities, is not decomposable. Nevertheless, it is decomposable at every point outside the origin (if  $x \neq 0$ ,  $\Omega = (x dx - y dz) \wedge (y dy + \frac{z}{x} dz)$ ).

If  $N = 4$ ,  $r = 2$ , there is one condition  $F_{i,\alpha} = 0 : \theta_{12}\theta_{34} - \theta_{13}\theta_{24} + \theta_{14}\theta_{23} = 0$ .

This condition is not sufficient, but I do not know if the hypotheses that  $\Omega$  is decomposable at every point outside the origin, implies that  $\Omega$  is decomposable.

**1.10** In this paragraph, we give some upper bounds for  $\text{codim}_{\mathbb{C}^N} \text{sing } F$

(1.10.1). First, every irreducible component of  $F^{-1}(0)$  has codimension  $\leq r = r(F)$  (indeed, if  $F$  is the germ at 0 of  $\underline{F} : U \rightarrow \mathbb{C}^p$ , the generic codimension of the fiber  $\underline{F}_\xi^{-1} \underline{F}(\xi)$  is  $r$  and this codimension is a lower semi-continuous function of  $\xi$ );

after,  $F^{-1}(0) \setminus \text{sing } F$  is a regular variety of codimension  $r$ . Accordingly :

$$r(F) \geq \text{codim}_{\mathbb{C}^N} F^{-1}(0) \geq \inf(r(F), \text{codim}_{\mathbb{C}^N} \text{sing } F)$$

(1.10.2) If  $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 2$  and if  $V(\Theta_F) \neq \emptyset$ , there is an inclusion :

$$F^{-1}(0) \subset V(\Theta_F).$$

Accordingly, if  $V(\Theta_F) \neq \emptyset$ , we get  $\text{codim}_{\mathbb{C}^N} V(\Theta_F) \leq r$ , and so :

$$\text{codim}_{\mathbb{C}^N} \text{sing } F \leq r \quad (\text{we suppose } r \geq 1).$$

Indeed, if  $F^{-1}(0) \not\subset V(\Theta_F)$  there exists an holomorphic curve  $\mathbb{C} \ni t \rightarrow x(t) \in \mathbb{C}^N$  such that  $x(0) = 0$  and  $x(t) \in F^{-1}(0) \setminus V(\Theta_F)$  if  $t \neq 0$ . From (1.8.1), the morphism  $\underline{F}_t : (\mathbb{C}^N, x(t)) \rightarrow (\mathbb{C}^p, 0)$  ( $t \neq 0$  small enough) admits a factorisation through a

germ  $\Sigma_t$  of analytic variety of dimension  $r$  at the origin of  $\mathbb{C}^p$ . All  $\Sigma_t$  are equal to a  $\Sigma$  and  $F = \underline{F}_0$  admits a factorisation through  $\Sigma$ . From (1.8.1),  $V(\Theta_F) = \emptyset$ , c.q.f.d.

(1.10. 3) Let us suppose that  $\Omega_F$  is decomposable. From 1.9, if  $\text{sing } \Omega_F \neq \emptyset$  :

$$\text{codim}_{\mathbb{C}^N} \text{sing } \Omega_F \leq N-r+1.$$

Nevertheless, if  $\text{sing } \Omega_F = \emptyset$ , there is from (1.8.2) a factorisation  $F = g \circ h$ , where  $h$  is a submersion, and  $V(\Theta_F)$  is the germ of zeros of the ideal  $h^* \mathfrak{J}$ , where  $\mathfrak{J}$  is the ideal generated by all determinants of order  $r$  of the matrix  $p \times r (dg_1, \dots, dg_p)$ .

So, if  $V(\Theta_F) \neq \emptyset$ ,  $\text{codim}_{\mathbb{C}^N} V(\Theta_F) \leq p-r+1$ .

Accordingly, if  $\text{sing } F \neq \emptyset$  and if  $\Omega_F$  is decomposable :

$$\text{codim}_{\mathbb{C}^N} \text{sing } F \leq \sup (p,n)-r+1.$$

The codimension being lower semi-continuous :

*Let us suppose that there exist points  $x \in \text{sing } F$ , as closely as we wish from the origin, such that  $\Omega_{\underline{F}_x}$  is decomposable (this is true if  $d_x F$  has rank  $r-1$ , cf*

(1.8.3)). Then :

$$\text{codim}_{\mathbb{C}^N} \text{sing } F \leq \sup (p,n)-r+1.$$

We have also the following remarks :

*Let us suppose there exist points  $x \in \text{sing } \Omega_F$ , as closely as we wish from the origin, such that  $\Omega_{\underline{F}_x}$  is decomposable ; then*

$$\text{codim}_{\mathbb{C}^N} \text{sing } \Omega_F \leq n-r+1$$

*Let us suppose there exist points  $x \in V(\Theta_F)$ , as closely as we wish from the origin, such that  $\Omega_{\underline{F}_x}$  is decomposable ; if  $\text{codim}_{\mathbb{C}^N} \text{sing } \Omega_F \geq 3$ , then :*

$$\text{codim}_{\mathbb{C}^N} V(\Theta_F) \leq p-r+1.$$

**Remarks 1.11 :** The proposition 1.5 is false in general if we suppose that the dimension of the space by which we factorise is not equal to  $r = r(F)$ . For example (cf [2]), there exists an analytic morphism  $F : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^4, 0)$



with  $F_1 = x_1$ ,  $F_2 = x_1 \cdot x_2$ ,  $F_3 = x_1 x_2 e^{x_2}$ ,  $F_4 = \bar{\phi}(F_1, F_2, F_3)$ , where  $\bar{\phi}$  is formal and cannot be chosen analytic. Then  $r(F) = 2$ ,  $\ker F^* = 0$  and  $\ker \hat{F}^*$  is generated by  $y_4 - \bar{\phi}(y_1, y_2, y_3)$ . So, there is a formal factorisation of  $F: (\mathbb{C}^2, 0) \xrightarrow{\bar{h}} (\mathbb{C}^3, 0) \xrightarrow{\bar{g}} (\mathbb{C}^4, 0)$ , where  $\mathbb{C}^3 = \{y \in \mathbb{C}^4; y_4 = 0\}$  and  $\bar{g}$  is the graph of  $\bar{\phi}$ ; if  $F: g \circ f$  is an analytic factorisation of  $F$  near of the preceding one,  $g$  is (as  $\bar{g}$ ) an immersion and  $\ker F^* \neq 0$ , but this is false. In this example,  $\text{codim}_{\mathbb{C}^2} \text{sing } F = 1$ , but we may modify it, in such a way that  $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 2$ .

## 2 - On the regularity of a germ of analytic mapping.

In this paragraph and the following one, we suppose that  $F$  is a generic submersion, i.e.  $r(F) = p$ .

2.1. Let us suppose that  $(X, 0)$  is irreducible and let us suppose that  $f: F|X: (X, 0) \rightarrow \mathbb{C}^p$  has generic rank  $s$ . If  $\mathcal{O}_X$  is the ring of germs of holomorphic functions on  $(X, 0)$  and if  $f^*: \mathbb{C}\{y\} \rightarrow \mathcal{O}_X$  is the homomorphism induced by  $f$ , there is inequalities :

$$s \leq s' = \dim(\widehat{\mathcal{O}_X} / \ker \hat{f}^*) \leq s'' = \dim(\mathcal{O}_X / \ker f^*).$$

Let us recall the following result (Gabrielov, [2]) :

**Theorem 2.2** : *If  $s = s'$ , we get  $s = s' = s''$ , i.e if the topological dimension of the image  $f(X)$  is equal to its formal dimension, then it is also equal to its analytical dimension ; therefore :  $\ker \hat{f}^* = \widehat{\ker f^*}$ .*

The morphism  $f$  is regular (Gabrelov's definition) if  $s = s' = s''$  ; if  $(X, 0)$  is reduced, the morphism  $f$  is regular if it is regular in restriction to each irreducible component of  $(X, 0)$  . The morphism  $f$  is regular if  $f$  is finished or if  $(X, 0)$  and  $f$  are algebraic ; here is another condition :

**Proposition 2.3 :** *Let us suppose that  $(X,0)$  is irreducible ; the morphism  $f$  is regular under every following hypothesis :*

- (1)  $r(f) = \text{codim}_X f^{-1}(0)$  (the inequality  $r(f) \geq \text{codim}_X f^{-1}(0)$  is always true)  
 (2)  $F$  is a flat morphism and  $\text{codim}_{\mathbb{C}^N} X = \text{codim}_{\mathbb{C}^p} f(X)$ .

**Proof :** (1) Let  $\Sigma \subset \mathbb{C}^N$  be a generic plane of codimension  $n-s$  ( $s = r(f)$ ) passing through the origin ; then, every irreducible component  $X_{\Sigma,i}$  of  $X \cap \Sigma$  has codimension  $(n-s)+(N-n) = N-s$  in  $\mathbb{C}^N$ , so has dimension  $s$  and  $X_{\Sigma,i} \cap f^{-1}(0) = (0)$ .

If  $g = f|_{X_{\Sigma,i}}$ ,  $g : X_{\Sigma,i} \rightarrow \mathbb{C}^p$  is a finite morphism, and its rank is  $s$ .

The kernel of  $g^* : \mathbb{C}\{y\} \rightarrow \mathcal{O}_{X_{\Sigma,i}}$  is a prime ideal  $\mathfrak{p}_{\Sigma,i}$  such that  $\mathbb{C}\{y\}/\mathfrak{p}_{\Sigma,i}$  has dimension  $s$ . Generically,  $\underline{X}_{\Sigma,i}$  contains points  $x$  as closely as we wish to the origin, which are regular for  $\underline{X}_{\Sigma,i}$  and  $\underline{X}$  with :

$$\text{rank } d_x \underline{f} = \text{rank } d_x \underline{g} = s$$

(the notations  $\underline{X}$ ,  $\underline{g}$  etc... mean sets, functions etc., the germs of which at the origin being  $X$ ,  $g$ ...). If  $\varphi \in \mathfrak{p}_{\Sigma,i}$ ,  $\varphi \circ f$  is null on  $X_{\Sigma,i}$  and  $\varphi \circ g$  is null on  $\underline{X}$  in the neighborhood of every  $x$ . Therefore,  $\varphi \circ f = 0$  and  $\mathfrak{p}_{\Sigma,i} \subset \ker f^*$ . The inverse inclusion is obvious because  $\mathbb{C}\{y\}/\mathfrak{p}_{\Sigma,i}$  has dimension  $s$ , and the morphism  $f$  is regular.

(2) The morphism  $F$  being flat,  $\text{codim}_{\mathbb{C}^N} F^{-1}(0) = p$ , so  $\text{codim}_{\mathbb{C}^N} f^{-1}(0) \geq p$  and  $\text{codim}_X f^{-1}(0) = \text{codim}_{\mathbb{C}^N} f^{-1}(0) - \text{codim}_{\mathbb{C}^N} X \geq p - \text{codim}_{\mathbb{C}^p} f(X) = r(f)$ .

Therefore  $r(f) = \text{codim}_X f^{-1}(0)$  and the result is a consequence of (1).

**Example 2.4 :** Let  $\varphi_1(x_n), \dots, \varphi_{n-1}(x_n) \in \mathbb{C}\{x_n\}$  be germs, algebraically independent on  $\mathbb{C}$ , such that  $\varphi'_1(0) = \dots = \varphi'_{n-1}(0) = 1$ . Let us consider the morphism  $f$  :

$$(\mathbb{C}^n, 0) \ni (x_1, \dots, x_n) \rightarrow (x_1, \dots, x_{n-1}, x_1 \varphi_1(x_n), \dots, x_{n-1} \varphi_{n-1}(x_n)) \in \mathbb{C}^{2n-2}.$$

Then  $r(f) = n$  and  $\text{sing } f = f^{-1}(0)$  is the  $x_n$ -axis ; so  $\text{codim}_{\mathbb{C}^N} f^{-1}(0) = n-1$  ;

besides, the rank of  $df$  at 0 is  $n-1$ . The morphism  $f$  is not regular if  $n > 2$ ; more precisely,  $\ker \hat{f}^* = 0$ . Indeed, let  $g \in \mathbb{C}[[y]]$  be such that :

$$g(x_1, \dots, x_{n-1}, x_1 \varphi_1(x_n), \dots, x_{n-1} \varphi_{n-1}(x_n)) = 0.$$

If  $g = \sum_{v=1}^{\infty} g_v$  is the decomposition of  $g$  in homogeneous polynomials, and if

$$x_1 = t \xi_1, \dots, x_{n-1} = t \xi_{n-1} :$$

$$\sum_{v=1}^{\infty} t^v g_v(\xi_1, \dots, \xi_{n-1}, \xi_1 \varphi_1(x_n), \dots, \xi_{n-1} \varphi_{n-1}(x_n)) = 0$$

so  $g_v(\xi_1, \dots, \xi_{n-1}, \xi_1 \varphi_1(x_n), \dots, \xi_{n-1} \varphi_{n-1}(x_n)) = 0$ , i.e.  $g_v = 0, \forall v$ .

This example, a variant of Osgood's example, shows that it is difficult to improve 2.3. Nevertheless, in 2.3 (2), we may replace the hypothesis of flatness on  $F$  by a condition of regularity on  $F$ .

**Remark 2.5** : If  $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq p$ , the morphism  $F$  is flat. Indeed, by (1.10.1),  $p = r(F) = \text{codim}_{\mathbb{C}^N} F^{-1}(0)$  and this means exactly that  $F$  is flat. Here is an example where  $\text{codim}_{\mathbb{C}^N} \text{sing } F = p-1$  and  $F$  is not flat ;  $F : \mathbb{C}^{2p-2} \rightarrow \mathbb{C}^p$  is defined by

$$F_1(x) = x_1 ; \dots ; F_{p-1}(x) = x_{p-1} ; F_p(x) = x_1 x_p + x_2 x_{p+1} + \dots + x_{p-1} x_{2p-2}.$$

**Proposition 2.6** : Let us suppose that  $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 3$ , where  $F : (\mathbb{C}^N, 0) \rightarrow (\mathbb{C}^p, 0)$  is a generic submersion. If  $(X, 0)$  is a germ of hypersurface at the origin of  $\mathbb{C}^N$  such that  $\text{codim}_{\mathbb{C}^p} f(X) = 1$  ( $= \text{codim}_{\mathbb{C}^N} X$ ), then  $f = F|_X$  is regular ( $X = F^{-1}(Y)$ , where  $Y$  is a germ of hypersurface at the origin of  $\mathbb{C}^p$ ).

**Proof** : Let  $\varphi = 0$  be a reduced equation of  $X$ ; the condition on the generic rank of  $f$  means that at each regular point of  $X$ ,  $d_x \varphi$  is a linear combination of  $d_x F_i$ , i.e. :

$$(*) \quad d\varphi = \varphi \cdot \omega + \sum_{i=1}^p \varphi_i dF_i$$

with  $\omega \in \Lambda^1(x)$  and  $\varphi_i \in \mathbb{C}(x)$ .

$$\text{So} \quad d\omega = - \sum_{i=1}^p d\left(\frac{\varphi_i}{\varphi}\right) \wedge dF_i;$$

and  $d\omega \wedge dF_1 \wedge \dots \wedge dF_p = 0$ . By 2.9, the hypothesis  $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 3$  implies that  $\omega = d\Psi \text{ mod}(dF)$ , where  $(dF)$  is the submodule of  $\Lambda^1(x)$  generated by  $dF_1, \dots, dF_p$ . From (\*):

$$d(\varphi e^{-\Psi}) \in (dF)$$

and from lemma 1.3 :  $\varphi = e^\Psi \cdot (\theta \circ F)$ , with  $\theta \in \mathbb{C}(y)$ .

If  $Y$  is the hypersurface with reduced equation  $\theta = 0$ , then  $X = f^{-1}(Y)$ .

**Corollary 2.7 :** *If  $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 3$  and if  $Y$  is a germ of irreducible hypersurface at the origin of  $\mathbb{C}^p$ ,  $X = F^{-1}(Y)$  is also irreducible (indeed, if  $X = X' \cup X''$  is a proper decomposition of  $X$ , we may apply to  $X'$  and  $X''$  the previous reasoning, and  $X' = F^{-1}(Y')$ ,  $X'' = F^{-1}(Y'')$ ; so  $Y = Y' \cup Y''$  is a proper decomposition of  $Y$  which is not irreducible).*

**Corollary 2.8 :** *Let us suppose that  $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 3$  and that  $\Omega_F$  is decomposable (we do not suppose that  $F$  is a generic submersion). If  $\text{codim}_{\mathbb{C}^N} V(\Theta_F) \geq r = r(F)$ ,  $F$  and the restriction of  $F$  to every hypersurface  $(X, 0)$  of  $(\mathbb{C}^N, 0)$ , are regular morphisms.*

**Proof :** By 1.4, there is a factorisation,  $F = g \circ h$ , where  $h$  is a generic submersion and  $g$  is a generic finite immersion. So  $F$  is regular ; if  $X$  is a germ of irreducible hypersurface at the origin of  $\mathbb{C}^N$ , either the rank of  $h|_X$  is equal to  $r$  and  $f = F|_X$  is regular ; or this rank is  $r-1$ , but then we may apply 2.6 and again  $f$  is regular.

2.9. In the proof of 2.6, we used a very particular case of the following result (cf [3] or [4]). Let us suppose that  $\text{codim}_{\mathbb{C}^N} \text{sing } F > q$

( $F : (\mathbb{C}^v, 0) \rightarrow (\mathbb{C}^p, 0)$  is a generic submersion) and let  $1 \leq s \leq r \leq q$  be integers.

We put :

$$\Lambda_F^{r,s}\{x\} = \{\omega \in \Lambda^r\{x\} ; \omega \wedge dF_{i_1} \wedge \dots \wedge dF_{i_{p-s+1}} = 0\}$$

for every  $1 \leq i_1 < i_2 < \dots < i_{p-s+1} \leq p$  =

$$\{\omega \in \Lambda^r\{x\} ; \omega = \sum_{j_1 < \dots < j_s} \theta_{j_1} \dots \theta_{j_s} \wedge dF_{j_1} \wedge \dots \wedge dF_{j_s}\}$$

with  $\theta_{j_1} \dots \theta_{j_s} \in \Lambda^{r-s}\{x\}$ .

This last equality is an easy consequence of the division lemma (Saito, [8]) stated below. If we write  $\Lambda^{r,s}(F) = \Lambda^r\{x\} / \Lambda_F^{r,s}\{x\}$  then  $d$  induces a morphism :

$$\Lambda^{r,s}(F) \rightarrow \Lambda^{r+1,s}(F) ;$$

there is an exact sequence :

$$\Lambda^{s-1}\{x\} \xrightarrow{d} \Lambda^{s,s}(F) \xrightarrow{d} \Lambda^{s+1,s}(F) \xrightarrow{d} \dots \xrightarrow{d} \Lambda^{q,s}(F)$$

and the kernel of the first  $d$  is the submodule of  $\Lambda^{s-1}\{x\}$  generated by the images of  $F^* : \Lambda^{s-1}\{y\} \rightarrow \Lambda^{s-1}\{x\}$  and  $d : \Lambda^{s-2}\{x\} \rightarrow \Lambda^{s-1}\{x\}$ . In particular, if  $q = 2$ , there is an exact sequence  $0 \rightarrow \mathbb{C}\{y\} \xrightarrow{F^*} \mathbb{C}\{x\} \xrightarrow{d} \Lambda^{1,1}(F) \xrightarrow{d} \Lambda^{2,1}(F)$  ; this sequence is used in the proof of 2.6.

The division lemma says that, if  $\text{codim}_{\mathbb{C}^N} \text{sing } F > q$  and if  $\omega \in \Lambda^q\{x\}$  is such that  $\omega \wedge dF_1 \wedge \dots \wedge dF_p = 0$ , then  $\omega = \sum_{i=1}^p \theta_i \wedge dF_i$ , with  $\theta_i \in \Lambda^{q-1}\{x\}$ .

2.10. It would be interesting to extend 2.6 to complete intersections. If a complete intersection  $(X, 0)$  of codimension  $k$  at the origin of  $\mathbb{C}^N$  is defined by a reduced system of equations  $\varphi_1 = \dots = \varphi_k = 0$  and if  $\text{codim}_{\mathbb{C}^p} F(X) = \text{codim}_{\mathbb{C}^N} X$ ,

then

$$d\varphi = \Omega \cdot \varphi + \sum_{i=1}^p \theta_i \cdot dF_i$$

where  $\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_k \end{pmatrix}$ ,  $\Omega$  is a  $k \times k$  matrix with coefficients in  $\mathbb{C}\{x\}$ ,  $\theta_i$  is a column

vector with coefficients in  $\mathbb{C}\{x\}$ . So :

$$(d\Omega - \Omega \wedge \Omega) \varphi = 0 \quad \text{mod } (dF_i).$$

If we may choose  $\Omega$  such that  $d\Omega - \Omega \wedge \Omega = 0 \quad \text{mod } (dF_i)$  and if  $\text{codim}_{\mathbb{C}^N} \text{sing } F \geq 3$ , then by the arguments of the proof of 2.9 :

$$\Omega = dM \cdot M^{-1} \quad \text{mod } (dF_i)$$

where  $M$  is an invertible  $k \times k$  matrix with coefficients in  $\mathbb{C}\{x\}$ , so

$$d(M^{-1} \varphi) = 0 \quad \text{mod } (dF_i)$$

By 2.9,  $M^{-1} \varphi = \theta \circ F$ , where  $\theta = \begin{pmatrix} \theta_1 \\ \vdots \\ \theta_k \end{pmatrix}$ ,  $\theta_i \in \mathbb{C}\{y\}$ ,  $\theta_i(0) = 0$ , and  $X = F^{-1}(Y)$

where  $Y$  is a complete intersection.

Therefore, the main problem is finding conditions on  $X$  and  $F$  such that the integrability condition  $d\Omega - \Omega \wedge \Omega = 0 \quad \text{mod } (dF_i)$  is verified by a suitable  $\Omega$ .

### 3 - A criteria of analyticity for modulus.

If  $A$  is a (commutative and unitary) ring without divisors of zero, we denote by  $[A]$  the quotient field of  $A$ . A modulus  $\mathcal{M}$  on  $A$ , of finite type, is without torsion if  $a \in A \setminus \{0\}$ ,  $m \in \mathcal{M} \setminus \{0\}$  implies  $a \cdot m \neq 0$ . This means also that  $\mathcal{M}$  is isomorphic to a submodule of  $A^r$ , where  $r = \dim_{[A]} \mathcal{M} \otimes_A [A]$  is the generic rank of  $\mathcal{M}$ .

If  $(X, 0)$  is a germ of analytic space, we denote by  $\hat{\mathcal{O}}_X$  the completion of the ring  $\mathcal{O}_X$  of analytic germs on  $X$ . We shall use the following result (Tougeron, [7]):

**Theorem 3.1** : Let  $f : (X,0) \rightarrow (Y,0)$  be a generic analytic submersion between two irreducible germs of analytic spaces (so  $f^* : \hat{\mathcal{O}}_Y \rightarrow \hat{\mathcal{O}}_X$  is injective). Then :

$$\{\varphi \in [\hat{\mathcal{O}}_Y] \text{ and } \varphi \circ f \in [\mathcal{O}_X]\} \Rightarrow \varphi \in [\mathcal{O}_Y].$$

A submodule  $\mathcal{N}$  of  $\hat{\mathcal{O}}_Y^q$  is analytic if it is generated on  $\hat{\mathcal{O}}_Y$  by elements of  $\mathcal{O}_Y^q$ .

**Corollary 3.2** : Under the hypothesis of 3.1, if  $\mathcal{M} = \hat{\mathcal{O}}_Y^q / \mathcal{N}$  is without torsion and if the vector space generated by  $\mathcal{N} \circ f$  in  $[\hat{\mathcal{O}}_X]^q$  is analytic (i.e. is generated by vectors with coefficients in  $\mathcal{O}_X$ ), then  $\mathcal{N}$  is analytic.

**Proof** : Let  $\varphi_1, \dots, \varphi_s \in \mathcal{N}$  be such that  $\varphi_1 \wedge \dots \wedge \varphi_s \neq 0$  and  $r =$  generic rank of  $\mathcal{M} = q-s$ . Then  $\varphi \in \mathcal{N} \Leftrightarrow \varphi \wedge \varphi_1 \wedge \dots \wedge \varphi_s = 0$  (because  $\mathcal{M}$  is without torsion). Let us put  $\varphi_1 \wedge \dots \wedge \varphi_s = \sum_I \theta_I e_I$  where  $e_I = e_{i_1} \wedge \dots \wedge e_{i_s}$  is the canonical basis of  $\hat{\mathcal{O}}_Y^q$ .

Let us suppose that  $\theta_{I_0} \neq 0$ ; the modulus generated by  $\mathcal{N} \circ f$  being analytic, each  $(\theta_I / \theta_{I_0}) \circ f$  is analytic, so by 3.1  $\theta_I / \theta_{I_0}$  is analytic for every  $I$ . Therefore  $\mathcal{N}$  is analytic, c.q.f.d.

This corollary admits the following extension :

**Proposition 3.3** : Let  $f : (X,0) \rightarrow (Y,0)$  be a morphism between two irreducible germs of analytic spaces and let us suppose that the germ of points  $x \in X$  such that  $f_x$  is not flat has codimension  $v$  in  $X$ . Let  $\mathcal{M} = \hat{\mathcal{O}}_Y^q / \mathcal{N}$  be a modulus such that the

two following conditions are full filled :

- every prime ideal associated to  $\mathcal{M}$  has height  $< v$ ,
- the submodule  $f^* \mathcal{N}$  generated by  $\mathcal{N} \circ f$  in  $\hat{\mathcal{O}}_X^q$  is analytic. Then  $\mathcal{N}$  is analytic.

Let us recall that if  $\mathcal{M}$  is a modulus on a ring  $A = \hat{\mathcal{O}}_Y$  and if  $\mathfrak{p}$  is a prime ideal of  $A$ , then  $\mathfrak{p}$  is associated to  $\mathcal{M}$  if there is an injective map  $A/\mathfrak{p} \hookrightarrow \mathcal{M}$ . The modulus  $\mathcal{M}$  is coprimary if  $a \in A \setminus \mathfrak{p} \Rightarrow (\mathcal{M} \ni m \rightarrow a.m \in \mathcal{M})$  is injective and  $a \in \mathfrak{p} \Rightarrow (\mathcal{M} \ni m \rightarrow a.m \in \mathcal{M})$  is nilpotent. If  $\mathfrak{p}_1, \dots, \mathfrak{p}_k$  are the prime ideals associated to  $\mathcal{M}$  then there exist submodules  $\mathcal{N}_i$  of  $\mathcal{M}$ , with  $\mathcal{M}/\mathcal{N}_i$   $\mathfrak{p}_i$ -coprimary, such that  $\bigcap_{i=1}^k \mathcal{N}_i = 0$  (cf [5]).

If  $\mathcal{M} = A^q/\mathcal{N}$  is  $\mathfrak{p}$ -coprimary, we define a sequence  $\mathcal{N}_0 = \mathcal{N} \subset \mathcal{N}_1 \subset \dots \subset \mathcal{N}_s \subset \mathcal{N}_{s+1} = A^q$  of submodules of  $A^q$  such that for every  $i = 0, \dots, s$  :  $\mathcal{N}_{i+1} \setminus \{\xi \in A^q ; \mathfrak{p} \cdot \xi \subset \mathcal{N}_i\}$ . Then, for every  $i$  :  $\mathcal{N}_{i+1}/\mathcal{N}_i$  is a modulus on  $A/\mathfrak{p}$  without torsion. We prove first :

**Lemma 3.4** : *With the hypothesis of 3.3, let us suppose that  $\mathcal{M}$  is  $\mathfrak{p}$ -coprimary ; then 3.3 is true.*

**Proof** : First , we observe that  $\mathfrak{p}$  is analytic. In fact, as a consequence of the flatness, the prime ideals of height  $< v$  associated to  $\hat{\mathcal{O}}_X^q / f^*\mathcal{N}$  are exactly the prime ideals of height  $< v$  associated to  $\hat{\mathcal{O}}_X / f^*\mathfrak{p}$  ;  $f^*\mathcal{N}$  being analytic, these prime ideals are analytic, and there exist a prime ideal  $\mathfrak{p}'$  of  $\mathcal{O}_X$  such that  $\hat{\mathfrak{p}}'$  is a minimal prime ideal containing  $f^*\mathfrak{p}$ . If  $X'$  is the germ of analytic set ( $\subset X$ ) defined by  $\mathfrak{p}'$ , then  $\mathfrak{p}$  is the kernel of the morphism  $\hat{f}^* : \hat{\mathcal{O}}_Y \rightarrow \hat{\mathcal{O}}_X / \hat{\mathfrak{p}}'$  and by Gabrielov's theorem,  $\mathfrak{p}$  is analytic.

After, we prove by induction on  $i = s, s-1, \dots, 0$  that  $\mathcal{N}_i$  is analytic. If  $\mathcal{N}_{i+1}$  is analytic, let  $g_1, \dots, g_h$  be an analytic system of generators of  $\mathcal{N}_{i+1}$  and let us write  $\mathcal{N}_{i+1} = \hat{\mathcal{O}}_Y^h / \mathfrak{R}_{i+1}$  where  $\mathfrak{R}_{i+1}$  is the modulus of relations between  $g_1, \dots, g_h$ . Then  $\mathcal{N}_{i+1} / \mathcal{N}_i = \hat{\mathcal{O}}_Y^h / \mathfrak{R}'_{i+1}$ ,  $\mathfrak{R}'_{i+1} \supset \mathfrak{R}_{i+1}$  ; by the flatness of  $f$ ,  $f^*\mathfrak{R}'_{i+1}$  is analytic at the generic point of  $X'$  ; by 3.2,  $\mathfrak{R}'_{i+1}$  is analytic and so  $\mathcal{N}_i$  is analytic.



**Remark :** In the previous proof we don't use the complete assertion that  $f^*\mathcal{N}$  is analytic ; we only use that  $f^*\mathcal{N}$  is analytic at the generic point of  $X'$ .

**Proof of 3.3 :** Let  $\mathfrak{P}_1, \dots, \mathfrak{P}_k$  be the minimal prime ideals associated to  $\mathcal{N} = \hat{\mathcal{O}}_Y^q / \mathcal{N}$ . As in the proof of 3.4, let  $\mathfrak{P}'_i$  be a minimal prime ideal of  $\hat{\mathcal{O}}_X$  containing  $f^*\mathfrak{P}_i$  ; then  $\mathfrak{P}'_i$  is analytic. If  $X'_i$  is the germ of analytic set defined by  $\mathfrak{P}'_i$ , then  $\mathfrak{P}_i$  is the kernel of the morphism  $\hat{f}^* : \hat{\mathcal{O}}_Y \rightarrow \hat{\mathcal{O}}_X / \mathfrak{P}'_i$  and so, by Gabrielov's theorem,  $\mathfrak{P}_i$  is analytic. Let  $\mathcal{N}_i$  be a submodule of  $\hat{\mathcal{O}}_Y^q$  such

that  $\bigcap_{i=1}^k \mathcal{N}_i = \mathcal{N}$  and  $\hat{\mathcal{O}}_Y^q / \mathcal{N}_i$  is  $\mathfrak{P}_i$ -coprimary.

Let  $\mathfrak{P}_1, \dots, \mathfrak{P}_{k'}$  be the minimal prime ideals in the family  $\{\mathfrak{P}_1, \dots, \mathfrak{P}_k\}$  ; then, by flatness,  $f^*\mathcal{N}_i$  is analytic at the generic point of  $X'_i$ , for  $i = 1, \dots, k'$ . By 3.4 and the remark,  $\mathcal{N}_i$  is analytic if  $i \leq k'$ .

There is an injection :

$$(*) \quad \left( \bigcap_{i \leq k'} \mathcal{N}_i \right) / \left( \bigcap \mathcal{N}_i \right) \rightarrow \hat{\mathcal{O}}_Y^q / \bigcap_{i > k'} \mathcal{N}_i.$$

Let  $g_1, \dots, g_h$  be a system of analytic generators of  $\bigcap_{i \leq k'} \mathcal{N}_i$  and let us write

$\hat{\mathcal{O}}_Y^h / \mathfrak{R} = \bigcap_{i \leq k'} \mathcal{N}_i$  where  $\mathfrak{R}$  is the modulus of relations between the  $g_i$ . Then  $\left( \bigcap_{i \leq k'} \mathcal{N}_i \right) / \left( \bigcap \mathcal{N}_i \right) \sim \hat{\mathcal{O}}_Y^h / \mathfrak{R}'$  where  $\mathfrak{R}' \supset \mathfrak{R}$ . The prime ideal associated to  $\hat{\mathcal{O}}_Y^h / \mathfrak{R}'$

are among  $\mathfrak{P}_{k'+1}, \dots, \mathfrak{P}_k$ , because of the injection (\*) and by flatness  $f^*\mathfrak{R}'$  is analytic outside an analytic set of codimension  $v$ .

Therefore, we may prove the result by induction on the number  $k$  of prime ideals  $\mathfrak{P}_i$ . By the induction hypothesis,  $\mathfrak{R}'$  is analytic and so  $\mathcal{N}$  is analytic.

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