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Noncommutative Binomial Expansion*

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1. Introduction

In the present paper we shall be concerned with a binomial expansion in *noncommutative* indeterminates under certain commutation rules (2.1), which we study using a version of lattice path method in combinatorics (cf. [2]).

In Section 2 we first set up the notation and definitions; in particular, the bijective correspondence between the lattice paths and the noncommutative monomials (Proposition 2.1) is the key observation to the proof of the main identity (2.3) of this section, which is the noncommutative binomial expansion to positive integer powers under the commutation rules (2.1). The *complete homogeneous symmetric polynomials* $h_{p-i}(u_0, u_1, \ldots, u_i)$ appear as a set of generic binomial coefficients ((2.3), (2.6)). The specialization $u_i := q^i u_0$ $(i \in \mathbb{N})$ gives the q-binomial coefficients and the q-Chu-Vandermonde convolution (Example 2.4, (ii)).

Section 3 deals with the *general power* version of (2.3). We combine the usual general power binomial expansion with (2.3) to obtain the definition (3.3) of $h_{w-i}(1+v_0, \ldots, 1+v_i)$ with w and v's mutually commuting indeterminates, which has a justification using the determinantal expression for $h_{p-i}(u_0, u_1, \ldots, u_i)$ (Remark 3.1). The corresponding general power version of (2.6) etc. are presented.

2. Noncommutative binomial expansion to positive integer powers

Let u_i $(i \in \mathbb{N} := \{0, 1, 2, \ldots\})$ be mutually commuting indeterminates and x an indeterminate with commutation rules

$$u_i u_j = u_j u_i \quad (i, j \in \mathbb{N}),$$

$$x u_i = u_{i+1} x \quad (i \in \mathbb{N}). \tag{2.1}$$

We consider the set E of all noncommutative polynomial expressions

$$\sum_i f_i x^i \quad (f_i \in A),$$

where the summation is finite and A denotes the commutative polynomial algebra in u_i $(i \in \mathbb{N})$ over the rational integer ring $\mathbb{Z}$, i.e.,

$$A = \bigcup_{i \in \mathbb{N}} \mathbb{Z}[u_0, u_1, \ldots, u_i].$$

By (2.1) we see that E is a ring.

From E we take the expression

$$(u_0 + x)^p$$

which has an expansion of the form

$$\sum_{i=0}^{p} f_i x^i \quad (f_i \in A).$$

In the following we will show that

$$f_i = h_{p-i}(u_0, u_1, \ldots, u_i)$$

where $h_j(u_0, u_1, \ldots, u_k)$ is the j-th *complete homogeneous symmetric polynomial* in $u_0, u_1, \ldots, u_k$ with generating series

$$\sum_{j \in \mathbb{N}} h_j(u_0, u_1, \ldots, u_k) t^j = \prod_{i=0}^{k} (1 - u_i t)^{-1},$$

t being an indeterminate commuting with u's.

Consider the lattice points $L := \mathbb{N}^2 \subset \mathbb{R}^2$ on the plane. A *path* of L is defined to be a sequence $s = (s_0, s_1, \ldots, s_p)$ of points in L such that (i) $s_0 = (0, 0)$ and (ii) if $s_i = (a, b)$, then s_{i+1} is either $(a + 1, b)$ (a *horizontal step*) or $(a, b + 1)$ (a *vertical step*). p is the *length* of $s = (s_0, s_1, \ldots, s_p)$.

Let P_p be the set of all paths of L of length p and let $s = (s_0, s_1, \ldots, s_p) \in P_p$. s is identified with the ordered p steps whose i-th step is from s_{i-1} to s_i. If the i-th step is horizontal (resp. vertical), then we assign u_0 (resp. x); thus we obtain a monomial s_E in E. For example, the monomial

$$u_0{}^2 x u_0{}^3 x^2 u_0 x = u_0{}^2 u_1{}^3 u_3 x^4 \in E$$

corresponds to a path belonging to P_{10}.

Proposition 2.1. *With the notation above, we have*

$$\sum_{s \in P_p} s_E = (u_0 + x)^p. \qquad (2.2)$$

Proof. For $s \in P_p$, we have $s_E = s'_E u_0$ or $s_E = s'_E x$ with $s' \in P_{p-1}$; the converse also holds. Thus we see that

$$\sum_{s\in P_p} s_E = \sum_{s'\in P_{p-1}} s'_E\cdot(u_0 + x).$$

Since $\sum_{s\in P_1} s_E = u_0 + x$, induction gives (2.2).

Proposition 2.2. *We have*

$$(u_0 + x)^p = \sum_{i=0}^{p} h_{p-i}(u_0, u_1, \ldots, u_i)x^i. \tag{2.3}$$

Proof. Putting

$$(u_0 + x)^p = \sum_{i=0}^{p} f_i x^i \quad (f_i \in A),$$

we see by (2.1) and (2.2) that

$$f_i x^i = \sum_{s\in P_p(i)} s_E,$$

where $P_p(i) := \{\ s \in P_p \mid s \text{ has } i \text{ vertical steps }\}$; in other words, $P_p(i)$ is the set of all paths $s = (s_0, s_1, \ldots, s_p)$ with $s_p = (p - i, i)$. For $s \in P_p(i)$, we have

$$\begin{aligned} s_E &= u_0^{j_0} x u_0^{j_1} x u_0^{j_2} x \cdots u_0^{j_{i-1}} x u_0^{j_i} \\ &= u_0^{j_0} u_1^{j_1} u_2^{j_2} \cdots u_{i-1}^{j_{i-1}} u_i^{j_i} x^i \end{aligned} \tag{2.4}$$

with

$$j_0 + j_1 + \cdots + j_i = p - i, \quad j_k \geq 0 \quad (k = 0, 1, \ldots, i); \tag{2.5}$$

conversely, for such integers $j_0, j_1, \ldots, j_i$, (2.4) equals s_E for a unique $s \in P_p(i)$. Hence

$$f_i x^i = \sum_C u_0^{j_0} u_1^{j_1} \cdots u_i^{j_i} x^i,$$

where the sum $\sum_C$ is taken under the condition (2.5), so that by the definition of h's,

$$f_i x^i = h_{p-i}(u_0, u_1, \ldots, u_i)x^i,$$

which completes the proof.

Proposition 2.2 gives a noncommutative binomial expansion under the commutation rules (2.1). Comparing the coefficients of like powers of x of both sides of

$$(u_0 + x)^{p+r} = (u_0 + x)^p(u_0 + x)^r,$$

we obtain

Proposition 2.3. *We have*

$$h_{p+r-i}(u_0, u_1, \ldots, u_i)$$

$$= \sum_{j=0}^{i} h_{p-j}(u_0, u_1, \ldots, u_j)h_{r-i+j}(u_j, \ldots, u_i) \qquad (2.6)$$

for $p, r, i \in \mathbb{N}$.

Proof. The left-hand side of (2.6) is the coefficient of x^i in $(u_0 + x)^{p+r}$; expanding $(u_0 + x)^p(u_0 + x)^r$ by Proposition 2.2 and noting (2.1), we see that the right-hand side of (2.6) equals the coefficient of x^i in $(u_0 + x)^p(u_0 + x)^r$. (See also (3.8).)

Propositions 2.2 and 2.3 show that $h_{p-i}(u_0, u_1, \ldots, u_i)$ are considered a set of generic binomial coefficients and (2.6) is a noncommutative version of the Chu-Vandermonde convolution

$$\binom{p+r}{i} = \sum_{j=0}^{i} \binom{p}{j} \binom{r}{i-j}.$$

Example 2.4. (i) The specialization $u_i := 1$ $(i \in \mathbb{N})$ does not contradict (2.1); in this case, (2.3) reduces to the usual binomial expansion

$$(1 + x)^p = \sum_{i=0}^{p} h_{p-i}(1, \ldots, 1)x^i = \sum_{i=0}^{p} \binom{p}{i} x^i.$$

(ii) The specialization $u_i := q^i u_0$ $(i \in \mathbb{N})$ with q an indeterminate commuting with both u_0 and x is consistent with (2.1); in this case the commutation rules reduce to

$$xu = qux$$

with $u := u_0$. Proposition 2.3 gives

$$h_{p+r-i}(1, q, \ldots, q^i)$$

$$= \sum_{j=0}^{i} h_{p-j}(1, q, \ldots, q^j) h_{r-i+j}(q^j, q^{j+1}, \ldots, q^i);$$

this can be rewritten

$$\begin{bmatrix} p+r \\ i \end{bmatrix}_q = \sum_{j=0}^{i} \begin{bmatrix} p \\ j \end{bmatrix}_q \begin{bmatrix} r \\ i-j \end{bmatrix}_q q^{j(r-i+j)}, \qquad (2.7)$$

where $\begin{bmatrix} p \\ i \end{bmatrix}_q$ are the q-binomial coefficients

$$\begin{bmatrix} p \\ i \end{bmatrix}_q := \frac{(1 - q^p)(1 - q^{p-1}) \cdots (1 - q^{p-i+1})}{(1 - q^i)(1 - q^{i-1}) \cdots (1 - q)}, \qquad (2.8)$$

since $\begin{bmatrix} p \\ i \end{bmatrix}_q$ equals $h_{p-i}(1, q, \ldots, q^i)$ (see [3, pp. 18-19]).

(2.7) is the q-Chu-Vandermonde convolution (cf. [1, pp.469]). If we put $q := 1$, then we recover the usual binomial coefficients.

3. General power expansion

Let v_i $(i \in \mathbb{N})$ be mutually commuting indeterminates and x an indeterminate with the same commutation rules as (2.1):

$$v_i v_j = v_j v_i \quad (i, j \in \mathbb{N}),$$

$$x v_i = v_{i+1} x \quad (i \in \mathbb{N}). \tag{3.1}$$

We consider the set $\hat{E}_w$ of all noncommutative powerseries expressions

$$\sum_{i \in \mathbb{N}} f_i x^i \quad (f_i \in \hat{A}_w)$$

where $\hat{A}_w$ denotes the commutative powerseries algebra in v_i $(i \in \mathbb{N})$ over the polynomial algebra $\mathbb{Q}[w]$ with an indeterminate w commuting with all the other indeterminates, i.e.,

$$\hat{A}_w = \{ \sum_j C_j v^j \mid C_j \in \mathbb{Q}[w] \}$$

where $j = (j_0, j_1, \ldots)$ $(j_i \in \mathbb{N})$ are multi-indices such that all but finite j_i are zero, and the summation $\sum_j$ is taken over them. By (3.1) we see that $\hat{E}_w$ is a ring.

From $\hat{E}_w$ we take $((1 + v_0) + x)^w = (1 + (v_0 + x))^w$ which is defined to be equal to

$$\sum_{p \in \mathbb{N}} \binom{w}{p} (v_0 + x)^p. \tag{3.2}$$

By Proposition 2.2 we compute:

$$(3.2) = \sum_{p \in \mathbb{N}} \binom{w}{p} \sum_{i=0}^{p} h_{p-i}(v_0, v_1, \ldots, v_i) x^i$$

$$= \sum_{i\in\mathbb{N}}\left[\sum_{p\geq i}\binom{w}{p}h_{p-i}(v_0,\ v_1,\ \ldots,\ v_i)\right]x^i$$

$$= \sum_{i\in\mathbb{N}}\left[\sum_{j\in\mathbb{N}}\binom{w}{i+j}h_j(v_0,\ v_1,\ \ldots,\ v_i)\right]x^i.$$

Referring back to Proposition 2.2, we put

$$h_{w-i}(1 + v_0,\ \ldots,\ 1 + v_i)$$

$$:= \sum_{j\in\mathbb{N}}\binom{w}{i+j}h_j(v_0,\ \ldots,\ v_i) \in \hat{A}_w; \tag{3.3}$$

thus we can write

$$(1 + v_0 + x)^w = \sum_{i\in\mathbb{N}} h_{w-i}(1 + v_0,\ \ldots,\ 1 + v_i)x^i \tag{3.4}$$

which is considered general power noncommutative binomial expansion under the commutation rules (3.1).

Remark 3.1. There is a justification for the definition (3.3): we have the identity

$$h_{p-i}(u_0,\ \ldots,\ u_i) = \sum_{j=0}^{i}\frac{u_j^{\,p}}{\prod\limits_{\substack{0\leq k\leq i\\ k\neq j}}(u_j - u_k)}, \tag{3.5}$$

the right-hand side of which follows from the determinantal expression for $h_{p-i}(u_0,\ \ldots,\ u_i)$, i.e.,

$$\det(u_j^{\lambda_k+i-k})/\det(u_j^{i-k}) \tag{3.6}$$

where $0 \leq j,\ k \leq i$ and $\lambda_0 = p - i$, $\lambda_k = 0$ $(1 \leq k \leq i)$; cf. [3, pp. 23-26]. Replacing u_j by $1 + v_j$ and p by w in the right-hand side of (3.5), we have

$$\sum_{j=0}^{i} \frac{(1+v_j)^w}{\prod_{\substack{0\le k\le i\\ k\ne j}} (1+v_j-(1+v_k))}$$

$$= \sum_{r\in\mathbb{N}} \binom{w}{r} \sum_{j=0}^{i} \frac{v_j{}^r}{\prod_{\substack{0\le k\le i\\ k\ne j}} (v_j - v_k)}$$

$$= \sum_{r\in\mathbb{N}} \binom{w}{r} h_{r-i}(v_0, \ldots, v_i)$$

$$= \sum_{r\ge i} \binom{w}{r} h_{r-i}(v_0, \ldots, v_i)$$

$$= \sum_{j\in\mathbb{N}} \binom{w}{i+j} h_j(v_0, \ldots, v_i), \tag{3.7}$$

since it follows from (3.6) that

$$h_{r-i}(v_0, \ldots, v_i) = 0 \quad (0 \le r \le i-1). \tag{3.8}$$

(3.7) is exactly the right-hand side of (3.3).

Proposition 3.2. *We have*

$$h_{w+z-i}(1+v_0, \ldots, 1+v_i)$$

$$= \sum_{j=0}^{i} h_{w-j}(1+v_0, \ldots, 1+v_j) h_{z-i+j}(1+v_j, \ldots, 1+v_i)$$

$$(i \in \mathbb{N}) \tag{3.9}$$

in $\hat{A}_{w,z}$, *where* $\hat{A}_{w,z}$ *is defined by replacing* $\mathbb{Q}[w]$ *by* $\mathbb{Q}[w, z]$ *in the definition of* $\hat{A}_w$ *with another indeterminate* z *commuting with all the other indeterminates.*

Proof. By the definition (3.2) we have

$$(1 + v_0 + x)^{w+z} = (1 + v_0 + x)^w (1 + v_0 + x)^z \qquad (3.10)$$

in $\hat{E}_{w,z}$, where $\hat{E}_{w,z}$ is defined by replacing $\hat{A}_w$ by $\hat{A}_{w,z}$ in the definition of $\hat{E}_w$. By (3.4) the left-hand side of (3.10) is equal to

$$\sum_{i\in\mathbb{N}} h_{w+z-i}(1 + v_0, \ldots, 1 + v_i)x^i. \qquad (3.11)$$

By (3.4), (3.3), and (3.1) the right-hand side of (3.10) is equal to

$$\sum_{j\in\mathbb{N}} h_{w-j}(1 + v_0, \ldots, 1 + v_j)x^j \cdot \sum_{k\in\mathbb{N}} h_{z-k}(1 + v_0, \ldots, 1 + v_k)x^k$$

$$= \sum_{i\in\mathbb{N}} \left[\sum_{j=0}^{i} h_{w-j}(1 + v_0, \ldots, 1 + v_j) \cdot \right.$$

$$\left. \cdot h_{z-i+j}(1 + v_j, \ldots, 1 + v_i) \right] x^i. \qquad (3.12)$$

Comparing the coefficients of x^i in (3.11) and (3.12), we obtain (3.9).

The identity (3.4) and Proposition 3.2 show that $h_{w-i}(1 + v_0, \ldots, 1 + v_i)$ are considered a set of generic general power binomial coefficients, and that (3.9) is a noncommutative version of the polynomial Chu-Vandermonde convolution

$$\binom{w + z}{i} = \sum_{j=0}^{i} \binom{w}{j}\binom{z}{i - j} \in \mathbb{Q}[w, z].$$

Remark 3.3. The specialization $w := p \in \mathbb{N}$ reduces (3.4) to (2.3): substituting $w := p$ in the right-hand side of (3.3), we have

$$\sum_{j\in\mathbb{N}} \binom{p}{i + j} h_j(v_0, \ldots, v_i)$$

$$= \sum_{j=0}^{p-i} \binom{p}{i + j} h_j(v_0, \ldots, v_i)$$

$$= \sum_{j=i}^{p} \binom{p}{j} h_{j-i}(v_0, \ldots, v_i)$$

$$= \sum_{j=0}^{p} \binom{p}{j} h_{j-i}(v_0, \ldots, v_i) \qquad \text{(by (3.8))}$$

$$= \sum_{j=0}^{p} \binom{p}{j} \sum_{k=0}^{i} \frac{v_k{}^j}{\prod_{\substack{0 \le r \le i \\ r \ne k}} (v_k - v_r)} \qquad \text{(by (3.5))}$$

$$= \sum_{k=0}^{i} \frac{(1 + v_k))^p}{\prod_{\substack{0 \le r \le i \\ r \ne k}} ((1 + v_k) - (1 + v_r))}$$

$$= h_{p-i}(1 + v_0, \ldots, 1 + v_i) \qquad \text{(by (3.5));}$$

thus, substituting $v_j := u_j - 1$ $(j \in \mathbb{N})$ (which contradicts neither the commutation rules (2.1) nor (3.1)), we are back to (2.3).

Example 3.4. (i) The specialization $v_i = 0$ $(i \in \mathbb{N})$ does not contradict (3.1); in this case, (3.4) reduces to the usual general power binomial expansion

$$(1 + x)^w = \sum_{i \in \mathbb{N}} h_{w-i}(1, \ldots, 1)x^i$$

$$= \sum_{i \in \mathbb{N}} \left[\sum_{j \in \mathbb{N}} \binom{w}{i + j} h_j(0, \ldots, 0) \right] x^i$$

$$= \sum_{i \in \mathbb{N}} \binom{w}{i} x^i .$$

(ii) The specialization $v_i := (1 + r)^i(1 + v_0) - 1$ with r an indeterminate commuting with all the other indeterminates is consistent with (3.1); in this case the commutation rules reduce to $x(1 + v) = (1 + r)(1 + v)x$ with $v := v_0$. Proposition 3.2 gives

$$h_{w+z-i}(1 + v, (1 + r)(1 + v), \ldots, (1 + r)^i(1 + v))$$

$$= \sum_{j=0}^{i} h_{w-j}(1 + v, (1 + r)(1 + v), \ldots, (1 + r)^j(1 + v)) \cdot$$

$$\cdot h_{z-i+j}((1 + r)^j(1 + v), \ldots, (1 + r)^i(1 + v)) \qquad (i \in \mathbb{N}). \tag{3.13}$$

By Remark 3.1 we have

$$h_{w-i}(1 + v, (1 + r)(1 + v), \ldots, (1 + r)^i(1 + v))$$

$$= \sum_{j=0}^{i} \frac{((1 + r)^j(1 + v))^w}{\prod\limits_{\substack{0\le k\le i\\ k\ne j}} ((1 + r)^j - (1 + r)^k)(1 + v)^i}$$

$$= \sum_{j=0}^{i} \frac{((1 + r)^w)^j}{\prod\limits_{\substack{0\le k\le i\\ k\ne j}} ((1 + r)^j - (1 + r)^k)} \cdot (1 + v)^{w-i}, \tag{3.14}$$

since the powerseries identity

$$((1 + r)^j(1 + v))^w = ((1 + r)^w)^j(1 + v)^w$$

holds for commuting indeterminates r, v, and w; for justification, see [2, pp. 4-7]. Putting

$$\begin{bmatrix} w \\ i \end{bmatrix}_{1+r} := \sum_{j=0}^{i} \frac{((1 + r)^w)^j}{\prod\limits_{\substack{0\le k\le i\\ k\ne j}} ((1 + r)^j - (1 + r)^k)},$$

we have from (3.13) and (3.14) that

$$\begin{bmatrix} w + z \\ i \end{bmatrix}_{1+r} = \sum_{j=0}^{i} \begin{bmatrix} w \\ j \end{bmatrix}_{1+r} \begin{bmatrix} z \\ i - j \end{bmatrix}_{1+r} (1 + r)^{j(z-i+j)},$$

which is a general power generalization of (2.7). We will show that

$$\begin{bmatrix} w \\ i \end{bmatrix}_{1+r} = \frac{(1 - (1 + r)^{w})(1 - (1 + r)^{w-1})\cdots(1 - (1 + r)^{w-i+1})}{(1 - (1 + r)^{i})(1 - (1 + r)^{i-1})\cdots(1 - (1 + r))}; \tag{3.15}$$

it suffices to prove that, for commutative indeterminates t and q,

$$\frac{(1 - t)(1 - tq^{-1})\cdots(1 - tq^{-i+1})}{(1 - q^{i})(1 - q^{i-1})\cdots(1 - q)} = \sum_{j=0}^{i} \frac{t^{j}}{\prod_{\substack{0\le k\le i \\ k\ne j}} (q^{j} - q^{k})}, \tag{3.16}$$

since the substitution $t := (1 + r)^{w}$ and $q := 1 + r$ into (3.16) yields (3.15). Both sides of (3.16) belong to $\mathbb{Q}(q)[t]$ and (3.16) with $t := q^{p}$ $(p \in \mathbb{N})$ is the identity $\begin{bmatrix} p \\ i \end{bmatrix}_{q} = h_{p-i}(1, q, \ldots, q^{i})$; see Example 2.4, (ii) and Remark 3.1. We thus have (3.16) in $\mathbb{Q}(q)[t]$. (3.15) is a general power version of (2.8). (3.15) with specialization $r := 0$ recovers the usual general power binomial coefficient $\binom{w}{i}$.

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