

モノドロミー群について

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超幾何微分方程式系

$$(1) \left\{ \begin{array}{l} (t-B) \frac{dX}{dt} = AX \quad A \in M_d(\mathbb{C}) \\ B = \text{diag}(\overbrace{\lambda_1, \dots, \lambda_1}^{m_1}, \overbrace{\lambda_2, \dots, \lambda_2}^{m_2}, \dots, \overbrace{\lambda_r, \dots, \lambda_r}^{m_r}), \\ A = (a_{ij}) \sim \text{diag}(\overbrace{\rho_1, \dots, \rho_1}^{n_1}, \overbrace{\rho_2, \dots, \rho_2}^{n_2}, \dots, \overbrace{\rho_q, \dots, \rho_q}^{n_q}) \end{array} \right.$$

の有限な確定特異点の位置の集合 B と A の固有値の集合とに、
 $L(B) = m_1 \cdot m_2 \cdots m_r$, $L(A) = n_1 \cdot n_2 \cdots n_q$ なるラベルを付ける。
 (1) のアクセサリー・パラメータの数は

$$N = d^2 - d + 2 - \sum_{j=1}^r m_j^2 - \sum_{k=1}^q n_k^2$$

であり、 $N \leq 0$ であればモノドロミー群は computable であることが知られている [1]。代表的な例は $L(B) = 1 \cdot 1 \cdots 1$, $L(A) = (d-1) \cdot 1$ なる Jordan-Pochhammer 方程式と、それと対をなすラベル $L(B) = (d-1) \cdot 1$, $L(A) = 1 \cdot 1 \cdots 1$ の Generalized Hypergeometric Equation である。

これ以外で、explicit にモノドロミー群が示されている例は $d=4$, $L(B) = 2 \cdot 1 \cdot 1$, $L(A) = 2 \cdot 2$ ([2]) だけのようなので、ここに $d \leq 5$ で、次のラベルに対応するモノドロミー群を示す。

$$\left\{ \begin{array}{l} d = 4, \quad L(B) = 2 \cdot 2, \quad L(A) = 2 \cdot 1 \cdot 1, \\ d = 5, \quad L(B) = 3 \cdot 2, \quad L(A) = 2 \cdot 2 \cdot 1, \\ d = 5, \quad L(B) = 2 \cdot 2 \cdot 1, \quad L(A) = 3 \cdot 2, \end{array} \right.$$

generic の場合のみであるが、non-generic の時の計算は [3] を参照。

以下、

$$\left\{ \begin{array}{l} e_j = \exp(2\pi i a_{jj}) \quad (j=1, 2, \dots, d) \\ f_k = \exp(2\pi i \rho_k) \quad (k=1, 2, \dots, q) \end{array} \right.$$

とし、次の記法を導入する：

$$\left\{ \begin{array}{l} \varphi(i) = e_i - f_1, \quad \psi(i) = e_i - f_2, \\ \varphi(p_1 \cdot p_2 \cdots p_r) = \varphi(p_1) \cdot \varphi(p_2) \cdots \varphi(p_r), \\ \psi(p_1 \cdot p_2 \cdots p_r) = \psi(p_1) \cdot \psi(p_2) \cdots \psi(p_r), \end{array} \right.$$

$$\underline{d=4, \quad L(B)=2 \cdot 2, \quad L(A)=2 \cdot 1 \cdot 1}$$

$$M_1 = \begin{pmatrix} e_1 & 0 & (e_1 - f_1)(e_3 - f_1) & (e_1 - f_1)(e_4 - f_1) \\ 0 & e_2 & (e_2 - f_1)(e_3 - f_1) & \alpha \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \beta_1 & \beta_2 & e_3 & 0 \\ \beta_3 & \beta_4 & 0 & e_4 \end{pmatrix}$$

$$\alpha = \frac{(e_2 - f_1)(e_4 - f_1)(f_1 f_3 + f_1 f_2 - e_1 e_4 - e_2 e_3)}{(f_1 f_3 + f_1 f_2 - e_1 e_3 - e_2 e_4)}$$

$$\beta_1 = \frac{f_1 f_3 + f_1 f_2 - e_1 e_4 - e_2 e_3}{f_1 (e_1 - e_2) (e_3 - e_4)}$$

$$\beta_2 = \frac{e_1 e_3 + e_2 e_4 - f_1 f_3 - f_1 f_2}{f_1 (e_1 - e_2) (e_3 - e_4)}$$

$$\beta_3 = \frac{e_1 e_3 + e_2 e_4 - f_1 f_3 - f_1 f_2}{f_1 (e_1 - e_2) (e_3 - e_4)}$$

$$\beta_4 = \frac{f_1 f_3 + f_1 f_2 - e_1 e_3 - e_2 e_4}{f_1 (e_1 - e_2) (e_3 - e_4)}$$

$$\underline{d=5, \quad L(B)=3 \cdot 2, \quad L(A)=2 \cdot 2 \cdot 1}$$

$$M_1 = \begin{pmatrix} e_1 & 0 & 0 & c_{14} & c_{15} \\ 0 & e_2 & 0 & c_{24} & c_{25} \\ 0 & 0 & e_3 & c_{34} & c_{35} \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ c_{41} & c_{42} & c_{43} & e_4 & 0 \\ c_{51} & c_{52} & c_{53} & 0 & e_5 \end{pmatrix},$$

$$C_{14} = C_{15} = C_{24} = C_{34} = 1$$

$$C_{25} = \frac{(e_1 e_5 - f_1 f_2)(e_2 e_4 - f_1 f_2)}{(e_1 e_4 - f_1 f_2)(e_2 e_5 - f_1 f_2)}$$

$$C_{35} = \frac{f_1^2 f_2 (e_2 - e_3)(e_4 - e_5)}{\psi(3)(e_1 e_4 - f_1 f_2)}$$

$$\times \frac{(e_2 e_3 e_5 - e_1 e_5 f_2 + e_1 f_1 f_2 - e_2 f_1 f_2 - e_3 f_1 f_2 + f_1 f_2^2)}{(e_2 e_5 - f_1 f_2)(e_3 e_5 - f_1 f_2)}$$

$$C_{41} = \frac{\varphi(1)\psi(1)(e_3 e_4 - f_1 f_2)(e_2 e_4 - f_1 f_2)(e_1 e_5 - f_1 f_2)}{f_1^2 f_2^2 (e_1 - e_3)(e_1 - e_2)(e_4 - e_5)}$$

$$C_{42} = \frac{-\varphi(2)\psi(2)(e_1 e_4 - f_1 f_2)(e_2 e_5 - f_1 f_2)(e_3 e_4 - f_1 f_2)}{f_1^2 f_2^2 (e_1 - e_2)(e_2 - e_3)(e_4 - e_5)}$$

$$C_{43} = \frac{-\varphi(3)\psi(33)(e_1 e_4 - f_1 f_2)(e_1 e_5 - f_1 f_2)(e_2 e_4 - f_1 f_2)}{f_1^3 f_2^2 (e_1 - e_3)(e_2 - e_3)^2 (e_4 - e_5)^2}$$

$$\times \frac{(e_2 e_5 - f_1 f_2)(e_3 e_4 - f_1 f_2)(e_3 e_5 - f_1 f_2)}{(e_2 e_3 e_5 - e_1 e_5 f_2 + e_1 f_1 f_2 - e_2 f_1 f_2 - e_3 f_1 f_2 + f_1 f_2^2)}$$

$$C_{51} = \frac{-\varphi(1)\psi(1)(e_1 e_4 - f_1 f_2)(e_2 e_5 - f_1 f_2)(e_3 e_5 - f_1 f_2)}{f_1^2 f_2^2 (e_1 - e_3)(e_1 - e_2)(e_4 - e_5)}$$

$$C_{52} = \frac{\varphi(2)\psi(2)(e_3e_5 - f_1f_2)(e_1e_4 - f_1f_2)(e_2e_5 - f_1f_2)}{f_1^2 f_2^2 (e_1 - e_2)(e_2 - e_3)(e_4 - e_5)}$$

$$C_{53} = \frac{-\varphi(3)\psi(3)(e_1e_4 - f_1f_2)(e_2e_5 - f_1f_2)(e_3e_5 - f_1f_2)}{f_1^2 f_2^2 (e_1 - e_2)(e_2 - e_3)(e_4 - e_5)}$$

$$\underline{d=5, \quad L(B)=2 \cdot 2 \cdot 1, \quad L(A)=3 \cdot 2}$$

$$M_1 = \begin{pmatrix} e_1 & 0 & C_{13} & C_{14} & C_{15} \\ 0 & e_2 & C_{23} & C_{24} & C_{25} \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ C_{31} & C_{32} & e_3 & 0 & C_{35} \\ C_{41} & C_{42} & 0 & e_4 & C_{45} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ C_{51} & C_{52} & C_{53} & C_{54} & e_5 \end{pmatrix},$$

$$c_{13} = \frac{\varphi(1)}{f_1}, \quad c_{14} = \frac{\varphi(1)}{f_1}, \quad c_{15} = \frac{\varphi(1)}{f_1}, \quad c_{23} = \frac{\varphi(2)}{f_1}$$

$$c_{24} = \frac{\varphi(2)(e_1 e_4 - f_1 f_2)(e_2 e_3 - f_1 f_2)}{f_1(e_1 e_3 - f_1 f_2)(e_2 e_4 - f_1 f_2)}$$

$$c_{25} = \frac{\varphi(2)(e_2 e_3 - f_1 f_2)}{f_1(e_1 e_3 - f_1 f_2)}$$

$$c_{31} = \frac{\varphi(3)(e_1 e_4 - f_1 f_2)(e_2 e_3 - f_1 f_2)}{(e_1 - e_2)(e_3 - e_4)}$$

$$c_{32} = \frac{-\varphi(3)(e_1 e_3 - f_1 f_2)(e_2 e_4 - f_1 f_2)}{(e_1 - e_2)(e_3 - e_4)}$$

$$c_{35} = \frac{e_4 \varphi(3)(e_2 e_3 - f_1 f_2)}{f_1^2 f_2 (e_3 - e_4)}$$

$$c_{41} = \frac{-\varphi(4)(e_1 e_3 - f_1 f_2)(e_2 e_4 - f_1 f_2)}{(e_1 - e_2)(e_3 - e_4)}$$

$$c_{42} = \frac{\varphi(4)(e_1 e_3 - f_1 f_2)(e_2 e_4 - f_1 f_2)}{(e_1 - e_2)(e_3 - e_4)}$$

$$c_{45} = \frac{-e_3 \varphi(4)(e_2 e_4 - f_1 f_2)}{f_1^2 f_2 (e_3 - e_4)}$$

$$C_{51} = \frac{-e_2 e_5 (e_1 e_4 - f_1 f_2) (e_1 e_3 - f_1 f_2)}{f_1^2 f_2^2 (e_1 - e_2)}$$

$$C_{52} = \frac{e_1 e_5 (e_1 e_3 - f_1 f_2) (e_1 e_3 - f_1 f_2)}{f_1^2 f_2^2 (e_1 - e_2)}$$

$$C_{53} = \frac{-e_5 (e_1 e_3 - f_1 f_2)}{f_1 f_2}$$

$$C_{54} = \frac{-e_5 (e_1 e_4 - f_1 f_2)}{f_1 f_2}$$

ただ、がむしゃらに計算したのみですが、数式処理の実験例にでもなればと思ひ、結果のみを書き下しました[4]。

参考文献

- [1] K. Okubo: On the group of Fuchsian equations, Seminar Report of Tokyo Metropolitan University, 1987
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