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A New Series of Δ_2^p-Complete Problems

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Abstract

We prove that the lexicographically first maximal connected subgraph problem for a graph property π is Δ_2^p-complete if π is hereditary, determined by the blocks, and nontrivial on connected graphs.

1. Introduction

The class Δ_2^p consists of problems solvable in polynomial time using oracles in NP. Recently some Δ_2^p-complete problems have been reported [6, 13, 14, 17]. In [13] we have shown that the lexicographically first maximal induced path problem is Δ_2^p-complete. This paper gives a very general theorem that derives a new series of Δ_2^p-complete problems related to lexicographically first maximal subgraph problems.

For a given hereditary property π on graphs, we consider the problem of finding the lexicographically first maximal (abbreviated to lfm) subset U of vertices of a graph $G = (V, E)$ such that U induces a *connected* subgraph satisfying π, where we assume that V is linearly ordered as $V = \{1, ..., n\}$. Problems of this kind have been extensively studied in [1, 2, 5, 8, 9, 10, 11, 12, 13, 15, 16]. In particular, without the connectedness restriction, the P-completeness of the lfm subgraph problem for *any* nontrivial polynomial time testable hereditary property is proved in [11] as an analogue of the results in Lewis and Yannakakis [7], Yannakakis [19], Yannakakis [20], Asano and Hirata [3], Watanabe et al. [18]. However, since the connectedness is not necessarily inherited by subgraphs, a new analysis is required.

Some of the lfm connected subgraph problems for hereditary properties are polynomial time solvable. For example, the lfm clique problem is obviously in P. We prove a general theorem asserting that the lfm connected subgraph problem for a graph property π is Δ_2^p-complete if π is hereditary, determined by blocks, and nontrivial on connected graphs. Hence the connectedness makes the problem drastically harder.

2. Δ_2^p-Completeness Theorem

For any graph property π, the lexicographically first maximal subgraph satisfying π is computed by the following greedy algorithm:

begin
  $U \leftarrow \emptyset$;
  for $j = 1$ **to** n **do**
    if $U \cup \{j\}$ can be extended to subgraph of G satisfying π
      then $U \leftarrow U \cup \{j\}$
end

It is clear from the algorithm that the lfm subgraph problem for π is in Δ_2^p if π is polynomial time testable. We consider the following decision problem:

Definition 2.1.

LFMCSP(π) (the lfm connected subgraph problem for π)
Instance: A graph $G = (V, E)$ and a vertex $v \in V$, where $V = \{1, ..., n\}$.
Question: Let U be the lfm subset of V whose induced subgraph, denoted $\langle U \rangle$, is a connected subgraph satisfying π. Then $v \in U$?

Papadimitriou [14] defined the *deterministic satisfiability problem* and showed that it is Δ_2^p-complete. We use this problem as a basis of reduction. The problem is described as follows:

Definition 2.2. Let $x_1, \ldots, x_{k-1}$ be $k-1$ variables and $Y_1, \ldots, Y_k$ be k sets of variables. A boolean formula $F_0(x_1, ..., x_{k-1}, Y_1, ..., Y_k)$ in conjunctive normal form is said to be *deterministic* if F_0 consists of the following clauses:

1. Either (y) or $(\bar{y})$ is a clause of F_0 for each y in $Y_1 \cup Y_k$.
2. For each $i = 1, ..., k-1$ and each y in Y_{i+1}, there are sets C_y^i and D_y^i of conjunctions of literals from $Y_i \cup \{x_i\}$ with the following properties:
   (a) Exactly one of the conjunctions in $C_y^i \cup D_y^i$ is true for any truth assignment (this can be checked in polynomial time).
   (b) F contains clauses $(\alpha \rightarrow y)$ and $(\beta \rightarrow \bar{y})$ for each $\alpha \in C_y^i$ and each $\beta \in D_y^i$.

Definition 2.3.

Deterministic Satisfiability (DSAT)

Instance: A deterministic formula

$F_0(x_1, ..., x_{k-1}, Y_1, ..., Y_k)$ and $k-1$ formulas in 3-conjunctive normal form $F_1(Y_1, Z_1)$, ... , $F_{k-1}(Y_{k-1}, Z_{k-1})$, where $\{x_1, ..., x_{k-1}\}, Y_1, ..., Y_k, Z_1, ..., Z_{k-1}$ are mutually disjoint sets of variables.

Question: Decide whether there is a truth assignment $\hat{x}_1, ..., \hat{x}_{k-1}, \hat{Y}_1, ..., \hat{Y}_k$ satisfying 1 and 2.

1. $F_0(\hat{x}_1, ..., \hat{x}_{k-1}, \hat{Y}_1, ..., \hat{Y}_k) = 1$.
2. $F_i(\hat{Y}_i, Z_i)$ is satisfiable $\Longleftrightarrow \hat{x}_i = 1$ for $i = 1, ..., k-1$.

Lemma 2.1 [14]. *DSAT is Δ_2^p-complete.*

Remark 2.1. For an instance $(F_0, ..., F_{k-1})$ of DSAT, we may assume that clauses in F_0 are of conjunctive normal form. For example, clause $(\alpha \rightarrow y)$ can be written in the form of disjunction of literals since α is a conjunction of literals.

Lemma 2.2. *Let $F_i(Y_i, Z_i)$ be a formula in 3-conjunctive normal form. Then there is a formula $F'(Y_i, Z_i')$ in 3-conjunctive normal form satisfying the following conditions:*

(i) *Each clause in $F'(Y_i, Z_i')$ contains at most one literal from Y_i.*

(ii) *For any truth assignment $\hat{Y}_i$, $F(\hat{Y}_i, Z_i)$ is satisfiable if and only if $F'(\hat{Y}_i, Z_i')$ is satisfiable.*

Proof. We just give an idea of construction. For a clause $(y_1 + y_2 + y_3)$ with $y_1, y_2, y_3 \in Y_i$, we replace it by $(y_1 + \bar{u})(y_2 + \bar{v})(y_3 + u + v)$ using new variables u, v which shall be put into Z_i'.□

A graph property π is said to be *hereditary* on induced subgraphs if, whenever a graph G satisfies π, all vertex-induced subgraphs of G also satisfy π. We say that π is *nontrivial* if π is satisfied by infinitely many graphs and there is a graph violating π. We say that π is *determined by the blocks* [18] if for any graphs G_1 and G_2 satisfying π the graph formed by identifying a vertex of G_1 and a vertex of G_2 also satisfies π.

A *block* is a connected graph with at least two vertices which contains no cutpoint. We use the following result (see [4]).

Lemma 2.3. *Let G be a block with at least three vertices and let v be a vertex of G. Then there is an edge $\{u, v\}$ such that the graph obtained by deleting vertices u and v together with adjacent edges is connected.*

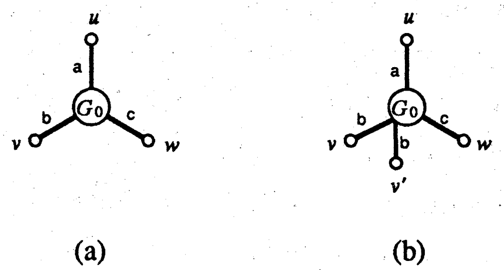


Fig. 1.

Our main result is the following theorem.

Theorem 2.4. *Let π be a hereditary property satisfying the following conditions:*

(i) *π is determined by the blocks.*

(ii) *π is nontrivial on connected graphs.*

Then LFMCSP(π) is Δ_2^p-complete.

Proof. Let G_π be a connected graph with minimum number of vertices which violates π. Since π is nontrivial on connected graphs and hereditary, the complete graph K_2 satisfies π. Therefore G_π is a block with at least three vertices since π is determined by the blocks. We denote G_π as Fig. 1(a), where bold lines represent edges adjacent to vertices u, v, w, respectively. We put labels a, b, c to specify the correspondence with u, v, w. By Lemma 2.3 we can assume that three vertices u, v, w are chosen so that the graph remains connected after deletion of v, w. Fig. 1(b) shows a graph obtained by adding a new vertex v' and edges in the same way as v. We use such abbreviation in the following construction.

Before getting into the reduction, we start with the following lemma which gives a basic construction in the reduction.

Lemma 2.5. *For a formula $F(x_1, ..., x_n) = c_1 c_2 \cdots c_m$ in conjunctive normal form with variables $x_1, ..., x_n$, we can construct a graph G_F with specified vertices h_1, h_0 and an order on vertices such that F is satisfiable (resp. not satisfiable) if and only if $h_1 \in U$ (resp. $h_0 \in U$), where U is the lfm subset of vertices of G_F which induces a connected subgraph satisfying π.*

Proof. For variable x_i, we construct the variable graph $G[x_i]$ in Fig. 2(a) using G_π, where $s_i = |\{c_j \mid c_j \text{ contains } x_i\}|$ and $t_i = |\{c_j \mid c_j \text{ contains } \bar{x}_i\}|$. When $s_i = 0$ (resp. $t_i = 0$), we do not put edge $\{f_i, d_{i+1}\}$. We call vertices in gray circles which are copies of G_0 *gray vertices.* Let $V(x_i)$ (resp. $V(\bar{x}_i)$) be the set of vertices x_i^k, $k = 1, ..., t_i$ (resp. $\bar{x}_i^k$, $k = 1, ..., s_i$). Let S be the set of black and gray vertices of $G[x_i]$ and let $\tilde{S}$ be any maximal set containing S whose induced subgraph is connected and satisfies π. Then it can be easily checked that $\tilde{S}$ is either $S \cup V(x_i) \cup \{x_i\}$ or $S \cup V(\bar{x}_i) \cup \{\bar{x}_i\}$.

For simplicity we deal with clauses with three literals but the argument below can be extended to the general case by a slight modification. The clause graph $H[c_j]$ for $c_j = (\alpha_j + \beta_j + \gamma_j)$ is shown in Fig. 2(b). Let $V(c_j)$ be the set of three vertices labeled with literals $\alpha_j, \beta_j, \gamma_j$. These vertices shall be connected to vertices in variable graphs corresponding to the literals. Again let C be the set of black and gray vertices of $H[c_j]$ and $\tilde{C}$ be any maximal set containing C whose induced subgraph is connected and satisfies π. Then exactly one of α, β, γ can be put into $\tilde{C}$.

We also use the graph R in Fig. 2(c) called the *root graph.* We call vertex d_0 the *root.*

The graph G_F is constructed as follows: We connect graphs R, $G[x_1], ..., G[x_n]$ by identifying d_i for each $i = 1, ..., n-1$. We denote the resulting graph by T_F and call it the *trunk graph.* Consider clause $c_j = (\alpha_j + \beta_j + \gamma_j)$. Let the occurrence of literal α_j (resp. β_j, γ_j) in c_j be the k_1-th (resp. k_2-th, k_3-th) occurrence of α_j (resp. β_j, γ_j) counted from c_1 to c_m. Then we put edges $\{\alpha_j^{k_1}, \alpha_j\}$, $\{\beta_j^{k_2}, \beta_j\}$, $\{\gamma_j^{k_3}, \gamma_j\}$, where $\alpha_j^{k_1}, \beta_j^{k_2}, \gamma_j^{k_3}$ are vertices in variable graphs and $\alpha_j, \beta_j, \gamma_j$ are vertices in $V(c_j)$. The clause graphs $H[c_1], ..., H[c_m]$ are connected to T_F in this way.

Finally we put edges $\{h_0, v\}$ for all black vertices v except the root. Fig. 3 illustrates the whole construction of graph G_F focussed on $G[x_p]$ and $H[c_j]$, where $c_j = (x_p + x_q + \bar{x}_r)$.

The vertices are ordered so that the following relations hold:

$$\begin{aligned} &B < h_1 < h_0 < x_1 < \bar{x}_1 < V(x_1) < V(\bar{x}_1) \\ &< \cdots < x_n < \bar{x}_n < V(x_n) < V(\bar{x}_n) \\ &< V(c_1) < \cdots < V(c_m), \end{aligned}$$

where B is the set of black and gray vertices.

Then it is clear from the definition of G_F that $B \subset U$ since h_0 is connected to all black vertices and π is determined by the blocks. It should be noticed that either $h_1 \in U$ or $h_0 \in U$ since G_π violates π. If $h_1 \in U$, then $h_0 \notin U$, and therefore $\langle U \rangle$ can have no edge with h_0 as an endpoint. For each variable x_i, either $V(x_i) \cup \{x_i\} \subset U$ or $V(\bar{x}_i) \cup \{\bar{x}_i\} \subset U$. Since for each clause c_j, U contains vertices in $H[c_j]$, one of the vertices in $V(c_j)$ must be in U and joined to a vertex in U which belongs to a variable graph. It is now obvious that F is satisfied by the truth assignment defined by $\hat{x}_i = 1$ (if $x_i \in U$), $\hat{x}_i = 0$ (if $x_i \notin U$).

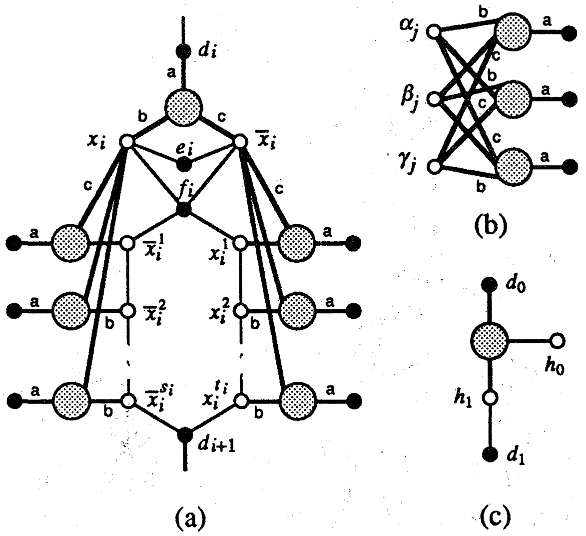
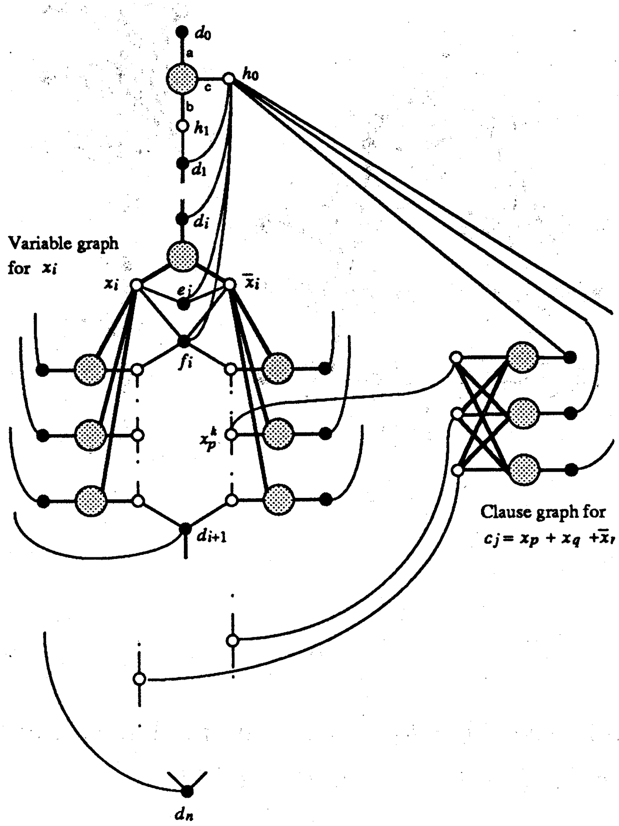


Fig. 2.

Conversely, it can also be seen that $h_1 \in U$ if F is satisfiable. Hence F is satisfiable (resp. not satisfiable) if and only if $h_1 \in U$ (resp. $h_0 \in U$).□

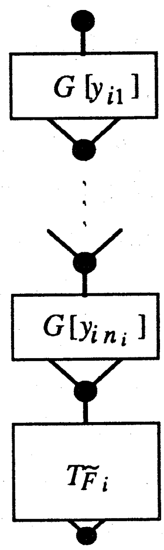
Proof continued. We shall give a reduction from DSAT. Let $F_0(x_1, ..., x_{k-1}, Y_1, ..., Y_k)$ be a deterministic formula and let $F_1(Y_1, Z_1), ..., F_{k-1}(Y_{k-1}, Z_{k-1})$ be formulas in 3-conjunctive normal form. We construct a graph $G(F_0, ..., F_{k-1})$ and an order on it as follows.

For each $i = 1, ..., k-1$, we first construct a graph $\tilde{G}_{F_i}$ in the following way: We denote by $\tilde{F}_i(Z_i)$ be a formula obtained from $F_i(Y_i, Z_i)$ by deleting all occurrences of literals from Y_i. Let $Y_i = \{y_{i1}, ..., y_{in_i}\}$. For each y_{ip} in Y_i we use the variable graph $G[y_{ip}]$, where the occurrences of literals y_{ip} and $\bar{y}_{ip}$ are counted for F_0 and F_i. We connect these variable graphs $G[y_{i1}], ..., G[y_{in_i}]$ and the trunk graph $T_{\tilde{F}_i}$ consecutively as shown in Fig. 4. We denote by h_0^i, h_1^i the vertices corresponding to h_0, h_1 in the construction in Lemma 2.5, respectively. We put an edge between h_0^i and each black vertex in the trunk graph $T_{\tilde{F}_i}$ except the root. By Lemma 2.2 we can assume that each clause in $F_i(Y_i, Z_i)$ contains at most one literal from Y_i. Let c_j^i be a clause in $F_i(Y_i, Z_i)$. If c_j^i contains only literals from Z_i, the clause graph $H[c_j^i]$ is connected to the trunk graph $T_{\tilde{F}_i(Z_i)}$ and we put edges between h_0^i and black vertices in $H[c_j^i]$ in the same way as Lemma 2.5. If c_j^i contains a literal from Y_i, let $c_j^i = (\alpha_j^i + \beta_j^i + \gamma_j^i)$, where α_j^i is a literal from Y_i and β_j^i, γ_j^i are literals from Z_i. For such clause, we use the graph $\tilde{H}[c_j^i]$ shown in Fig. 5 instead of $H[c_j^i]$. Vertices β_j^i and γ_j^i are connected to the trunk graph in the same way and we put edges $\{h_0^i, \beta_j^i\}$, $\{h_0^i, \gamma_j^i\}$. For literal α_j^i, let $\alpha_j^i = y_{ip}$ for simplicity. Then we connect $\tilde{H}[c_j^i]$ to some vertex in $V(y_{ip})$ of



d_0
a
h_0
c
b
h_1
d_1
d_i
Variable graph
for x_i
x_i
e_i
$\overline{x}_i$
f_i
x_p^k
d_{i+1}
Clause graph for
$c_j = x_p + x_q + \overline{x}_r$
d_n

Fig. 3.



$G[y_{i1}]$
$G[y_{i\,n_i}]$
$T\widetilde{F}_i$

Fig. 4.

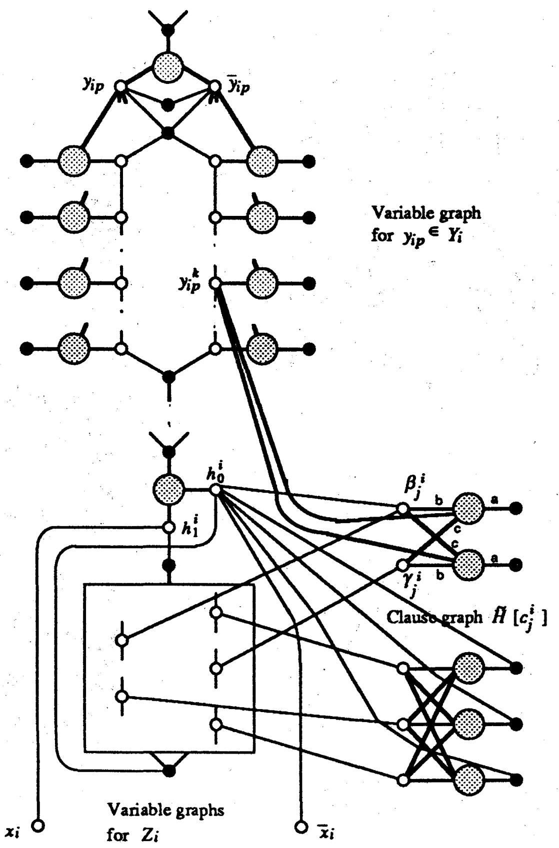


Fig. 5.

the variable graph $G[y_{ip}]$ as shown in Fig. 5. We denote by $\hat{G}_{\tilde{F}_i(Z_i)}$ the part consisting of the trunk graph and the clause graphs for $\tilde{F}_i(Z_i)$. Finally we add two vertices x_i and $\bar{x}_i$ which are connected to h_1^i and h_0^i, respectively. This is the end of the construction of $\tilde{G}_{F_i}$.

Let B_i be the set of all black and gray vertices of $\tilde{G}_{F_i}$. Let $\hat{Y}_i$ be a truth assignment for variables in Y_i. Then if $\hat{y}_{ip} = 1$, then let $\hat{V}(y_{ip}) = V(y_{ip})$ else $\hat{V}(y_{ip}) = V(\bar{y}_{ip})$. Let $B_i(\hat{Y}_i) = B_i \cup \bigcup_{p=1}^{n_i} \hat{V}(y_{ip})$. Assume that the order on white vertices on variable graphs for Z_i and clause graphs follows Lemma 2.5. In Fig. 5, it should be noticed that if $y_{ip}^k \in B_i(\hat{Y}_i)$ then the black and gray vertices in $\tilde{H}[c_j^i]$ are connected to $G[y_{ip}]$ and none of β_j^i, γ_j^i can be selected. Then it can be seen that

Fact. $F(\hat{Y}_i, Z_i)$ is satisfiable if and only if the lfm set containing $B_i(\hat{Y}_i)$ whose induced subgraph is connected and satisfies π contains h_1^i.

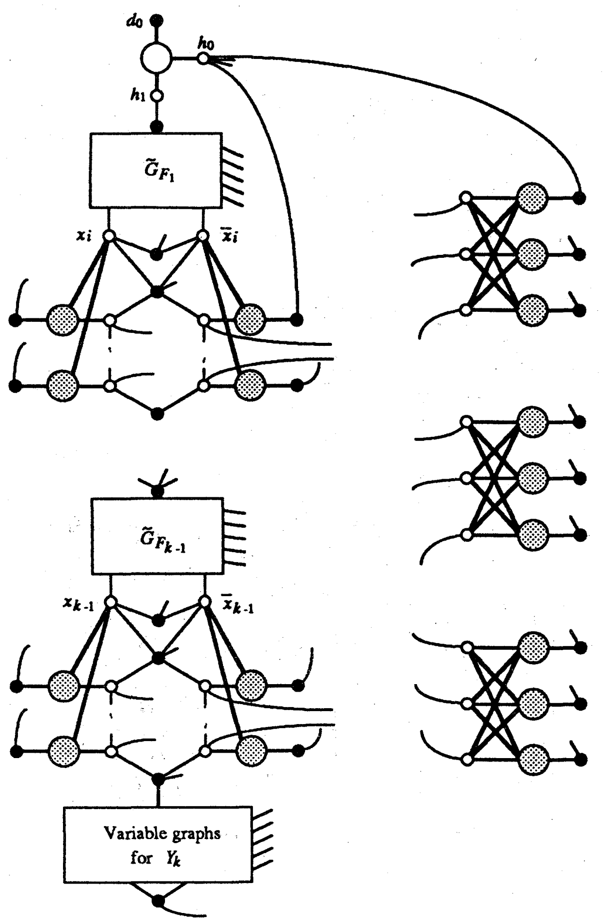


Fig. 6.

The graph $G(F_0, ..., F_{k-1})$ shown in Fig. 6 illustrates the whole graph for $(F_0, ..., F_{k-1})$. It is obtained by modifying the construction in Lemma 2.5. First we construct a trunk graph using the root graph R and the variable graphs $G[x_1], ..., G[x_{k-1}]$. Then for each x_i, the part consisting of $d_i, x_i, \bar{x}_i$ together with a copy of G_0 is replaced by $\tilde{G}_{F_i}$. Then the graphs for clauses in $F_0(x_1, ..., x_{k-1}, Y_1, ..., Y_k)$ are connected to the trunk in the same way as Lemma 2.5 using the variable graphs for $Y_1, ..., Y_k$ and modified variable graphs for $x_1, ..., x_{k-1}$. Finally we put edges connecting h_0 and all black vertices on the variable graphs for $Y_1, ..., Y_k$ and the clause graphs for F_0.

Let $\tilde{B}$ be the set of all black and gray vertices of $G(F_0, ..., F_{k-1})$. We denote by $W(Y_i)$ (resp. $W(x_i)$, $W(F_i)$, $C(F_0)$) the set of all white vertices in the variable graphs for Y_i (resp. the variable graph for x_i, the graph $\hat{G}_{\tilde{F}_i(Z_i)}$, the clause graphs for F_0). Then the vertices of $G(F_0, ..., F_{k-1})$ are ordered as follows:

$$\begin{aligned}&\tilde{B} < h_1 < h_0 < W(Y_1) < W(Y_k)\\&< W(x_1) < W(F_1) < \cdots < W(Y_{k-1})\\&< W(x_{k-1}) < W(F_{k-1}) < C(F_0),\end{aligned}$$

where the orders inside $W(Y_i)$, $W(x_i)$, $W(F_i)$ and $C(F_0)$ follow Lemma 2.5.

We shall show that $(F_0, ..., F_{k-1})$ is in DSAT if and only if h_1 is in $\tilde{U}$, where $\tilde{U}$ is the lfm subset of vertices such that $\langle\tilde{U}\rangle$ is connected and satisfies π. It can be seen from the construction that $\tilde{B} \subset \tilde{U}$.

If $h_1 \in \tilde{U}$, then $h_0 \notin \tilde{U}$. Hence there is no edge connecting h_0 and a black vertex in $\langle\tilde{U}\rangle$. First consider the variable graphs for Y_1 and Y_k. For each $y \in Y_1 \cup Y_k$, either $V(y) \subset \tilde{U}$ or $V(\bar{y}) \subset \tilde{U}$. Since for each $y \in Y_1 \cup Y_k$ either (y) or $(\bar{y})$ is a clause in F_0 and the corresponding clause graph contains black vertices, it follows that $V(y) \subset \tilde{U}$ (resp. $V(\bar{y}) \subset \tilde{U}$) if and only if (y) (resp. $(\bar{y})$) is a clause in F_0. Let $\hat{Y}_1$ and $\hat{Y}_k$ be a truth assignment defined by $\hat{y} = 1$ (resp. $\hat{y} = 0$) if (y) (resp. $(\bar{y})$) is in F_0 for $y \in Y_1 \cup Y_k$. From Fact, we see that $F_1(\hat{Y}_1, Z_1)$ is satisfiable if and only if vertex x_1 is in $\tilde{U}$. Therefore either $V(x_1) \subset \tilde{U}$ or $V(\bar{x}_1) \subset \tilde{U}$ holds according to the satisfiability of $F_1(\hat{Y}_1, Z_1)$. Let $\hat{x}_1 = 1$ if $x_1 \in \tilde{U}$ else $\hat{x}_1 = 0$.

Since F_0 is deterministic, for each $y_{2p} \in Y_2$ there are sets $C^1_{y_{2p}}$ and $D^1_{y_{2p}}$ of conjunctions of literals from $Y_1 \cup \{x_1\}$ satisfying the conditions (a), (b) of Definition 2.2. For the truth assignment $\hat{Y}_1$, $\hat{x}_1$, there is exactly one conjunction $\gamma \in C^1_{y_{2p}} \cup D^1_{y_{2p}}$ which is true under this truth assignment. If $\gamma \in C^1_{y_{2p}}$, then $(\gamma \rightarrow y_{2p})$ is in F_0. By considering the clause graphs corresponding to $(\gamma \rightarrow y_{2p})$, we can see that $V(y_{ip}) \subset \tilde{U}$ must hold since otherwise the connectedness of $\langle\tilde{U}\rangle$ is violated. If $\gamma \in D^1_{y_{2p}}$, then $V(\bar{y}_{ip}) \subset \tilde{U}$ must hold. Let $\hat{y}_{2p} = 1$ (resp. $\hat{y}_{2p} = 0$) if $V(y_{ip}) \subset \tilde{U}$ (resp. $V(\bar{y}_{ip}) \subset \tilde{U}$). With this truth assignment we can see that clauses $(\alpha \rightarrow y_{2p})$ and $(\beta \rightarrow \bar{y}_{2p})$ are satisfied for each $\alpha \in C^1_{y_{2p}}$ and each $\beta \in D^1_{y_{2p}}$. In this way, we define $\hat{Y}_2$. Inductively we define $\hat{x}_2, \hat{Y}_3, ..., \hat{x}_{k-1}$. Finally we can see that the truth assignment given to Y_k together with Y_1 must coincide with the one determined from $\hat{Y}_{k-1}$ and $\hat{x}_{k-1}$ since the graph $\langle\tilde{U}\rangle$ is connected. Thus we have shown the conditions 1 and 2 of Definition 2.3 hold. Hence $(F_0, ..., F_{k-1})$ is in DSAT.

The converse can also be shown by repeating a similar argument.□

Examples of the properties on undirected graphs that satisfy the conditions of Theorem 2.4 are "planar", "outerplanar", "bipartite", "acyclic", etc. But the property "clique" is hereditary but not determined by blocks. In this case LFMCSP("clique") is P-complete [5].

3. Conclusion

We have shown a rather general Δ_2^p-completeness theorem for the lfm connected subgraph problems. This result does not cover the lfm induced path problem [13]. We believe that we could expect a more general result which inculde the results in [13]. As a candidate, we give the following conjecture.

We define the *diameter* $\delta(\pi)$ by $\sup\{\delta(G) \mid G$ is a connected graph satisfying $\pi\}$, where $\delta(G)$ is the diameter of a graph G. For example, δ("clique")=1 but δ("planar") = ∞. For the former property, LFMCSP(π) becomes P-complete [5] but from Theorem 2.4 LFMCSP("planar") is Δ_2^p-complete.

Conjecture. If a hereditary property π is nontrivial on connected graphs and satisfies $\delta(\pi) = \infty$, then LFMCSP(π) is Δ_2^p-complete.

It should be noticed that if a hereditary property π is nontrivial on connected graphs and satisfies $\delta(\pi) = \infty$ then all paths satisfy π.

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